Research Article

Gang Fang, Sibel Koparal, Neşe Ömür*, Ömer Duran, and Waseem Ahmad Khan

On three-dimensional *q*-Riordan arrays

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Abstract: In this article, we define three-dimensional q-Riordan arrays and q-Riordan representations for these arrays. Also, we give four cases of infinite multiplication three-dimensional matrices of these arrays. As applications, we obtain three-dimensional q-Pascal-like matrix and its inverse matrix by Heine's binomial formula, using combinatorial identities. Finally, we consider the generalization of three-dimensional q-Pascal-like matrix and give some identities involving q-binomial coefficients.

Keywords: q-Riordan arrays, three-dimensional matrices, q-binomial coefficients, generating functions

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1 Introduction

The harmonic numbers, denoted by H_m, are a sequence of numbers that hold great significance in various branches of mathematics, including number theory, combinatorics, and analysis. The harmonic numbers appear in a wide range of mathematical and scientific contexts, such as in the analysis of series, the study of prime numbers, and the analysis of algorithms. They have connections to diverse areas of mathematics, including calculus, number theory, and combinatorics. These numbers are defined as

$$H_0 = 0, \quad \text{and} \quad H_m = \sum_{k=1}^{m} \frac{1}{k}, \quad \text{for } m = 1, 2, \cdots .$$

One of the notable properties of the harmonic numbers is their generating function, which provides a powerful tool for understanding their behavior. The series expression corresponding to these numbers is expressed as

$$\sum_{j=1}^{\infty} H_j t^j = \frac{-\ln(1 - t)}{1 - t}. \tag{1}$$

This elegant formula captures the essence of the harmonic numbers and enables us to extract valuable information about their properties and relationships. By manipulating this generating function, one can derive various identities and explore the intricate interplay between the harmonic numbers and other mathematical objects.

The field of mathematics encompasses various branches where the q-calculus holds significance. Primarily explored by Euler [1], the q-calculus has established its indispensability in number theory, combinatorics, and

*** Corresponding author: Neşe Ömür,** Department of Mathematics, Kocaeli University, Kocaeli 41380, Türkiye, e-mail: neseomur@kocaeli.edu.tr
Gang Fang: Institute of Computing Science and Technology, Guangzhou University, Guangzhou 510006, China, e-mail: gangf@gzhu.edu.cn
Sibel Koparal: Department of Mathematics, Bursa Uludağ University, Bursa 16059, Türkiye, e-mail: sibelkoparal@uludag.edu.tr
Ömer Duran: Department of Mathematics, Kocaeli University, Kocaeli 41380, Türkiye, e-mail: omer20841@gmail.com
Waseem Ahmad Khan: Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O. Box 1664, Al Khobar 31952, Saudi Arabia, e-mail: wkhan1@pmu.edu.sa

numerous other domains. Over the years, substantial contributions and advancements have been made in the realm of q-calculus, leading to a deeper understanding of its intricacies and applications. Researchers continue to delve into this subject, uncovering new insights, novel techniques, and remarkable connections with other areas of mathematics. Consequently, the exploration of q-calculus remains an active and dynamic field, with ongoing investigations and important works contributing to its continuous growth and relevance in contemporary mathematical research. Let s be a non-negative integer. The notation $[s]_q$ signifies the q-analog of the non-negative integer s, which is determined by

$$[s]_q = \frac{q^s - 1}{q - 1}.$$

The q-factorial of $[s]_q$ is defined as

$$[s]_q! = \begin{cases} 1, & \text{if } s = 0, \\ [s]_q[s-1]_q\cdots[1]_q, & \text{if } s \geq 1. \end{cases}$$

The widely recognized q-binomial coefficient, also referred to as the Gaussian coefficient, is given by the expression

$$\begin{bmatrix} s \\ r \end{bmatrix}_q = \frac{[s]_q!}{[s-r]_q![r]_q!},$$

where $s \geq r$. This coefficient captures the essence of q-analog, extending the traditional binomial coefficient to encompass the parameter q. It is clear that $\lim_{q\to 1}\begin{bmatrix} s \\ r \end{bmatrix}_q = \binom{s}{r}$, where $\binom{s}{r}$ is the usual binomial coefficient. Also, it is known that

$$\begin{bmatrix} s \\ r \end{bmatrix}_{1/q} = q^{r(r-s)}\begin{bmatrix} s \\ r \end{bmatrix}_q. \tag{2}$$

The definition of the q-Pochhammer symbol, alternatively known as the q-shifted factorial, is given as follows:

$$(t;\, q)_s = \begin{cases} 1, & \text{if } s = 0, \\ (1 - q^{s-1}t)(1 - q^{s-2}t)\ldots(1 - t), & \text{if } s \geq 1. \end{cases}$$

Particularly, when $s = 1$, we write $(t;\, q)_s = (t;\, q)$.

The Heine's binomial formula provides a powerful expression that relates a certain summation involving q-binomial coefficients to a rational function. Specifically, for non-negative integer s, the formula is given as follows:

$$\sum_{j=0}^{\infty}\begin{bmatrix} s+j \\ j \end{bmatrix}_q t^j = \frac{1}{(t;\, q)_{s+1}}. \tag{3}$$

This formula allows us to evaluate the infinite sum involving q-binomial coefficients in terms of a rational function, providing a useful tool in varied areas of mathematics, such as combinatorics, number theory, and special functions. Heine's binomial formula has been widely studied and applied in different contexts because of its deep connections with q-series and its ability to simplify series expansions.

Carlitz [2] gave the following series as for $s, r \geq 0$,

$$\sum_{j=s}^{\infty}(-1)^{j-s}\begin{bmatrix} j+r \\ j-s \end{bmatrix}_q q^{\binom{j-s}{2}}\frac{t^j}{(t;\, q)_{j+r+1}} = t^s. \tag{4}$$

Consider the set F_n consisting of all generating functions of the form

$$f_n t^n + f_{n+1}t^{n+1} + f_{n+2}t^{n+2} + \ldots,$$

where $f_n \neq 0$. For $g(t) = \sum_{j=0}^{\infty} g_j t^j \in F_0$ and $f(t) = \sum_{j=0}^{\infty} f_j t^j \in F_1$, let $r_{n,k} = [t^n]gf^k$, where $[t^n]$ is coefficient operator, i.e., $[t^n]A(t) = a_n$ when $A(t) = \sum_{j=0}^{\infty} a_j t^j$. Then, $R = (r_{n,k})_{n,k\geq 0} = (g, f)$ is called the Riordan array. The set of these arrays forms a group known as the Riordan group. Considering (g, f) and (G, F) as Riordan arrays, the group operation is defined as

$$(g, f)(G, F) = (gG(f), F(f)),$$

with identity element $(1, t)$. The inverse element of (g, f), shown by $(g, f)^{-1}$, is given by

$$(g, f)^{-1} = \left(\frac{1}{g(\tilde{f})}, \tilde{f}\right),$$

where $\tilde{A} \circ A = A \circ \tilde{A} = I$ for any invertible function $A(t)$ and identity function I [3]. Several intriguing subgroups exist within the Riordan group, such as $\{(g, t) | g \in F_0\}$, which serves as a normal subgroup and called the Appell subgroup. Additionally, there are the Lagrange subgroup $\{(1, f) | f \in F_1\}$ and the derivative (or co-Lagrange) subgroup $\{(f', f) | f \in F_1\}$, each contributing unique properties and characteristics to the overall group structure. Riordan arrays, a dynamic and evolving field of study, have emerged as a powerful tool with broad applications in various branches of mathematics. With their intricate connections to combinatorics, group theory, matrix theory, and number theory, Riordan arrays serve as a bridge between these disciplines, fostering a rich interplay of ideas and techniques. They have not only benefited from the advancements in these fields but also significantly contributed to their progress. The exploration of Riordan arrays continues to unveil new insights, deepen our understanding, and inspire further investigations in mathematics and its diverse subfields. Also, Riordan arrays play a crucial role in showcasing combinatorial identities, investigating number sequences, and tackling problems in number theory [4–11]. To illustrate the Riordan matrix, taking the Riordan array (g, f) with the selecting functions $g(t)$ and $f(t)$ defined as $\frac{1}{1-t}$ and $\frac{t}{1-t}$, respectively, it is seen that

$$\left(\frac{1}{1-t}, \frac{t}{1-t}\right) = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

which is called Pascal's matrix. This matrix is usually used in linear algebra and combinatorics [12–16]. Pascal matrix is infinite matrix whose elements are formed by binomial coefficients. As analog to the Pascal matrix, q-Pascal matrix is obtained as entries included q-binomial coefficients using q-calculus. Ernst [15] considered some combinatorial properties of this matrix.

Recently, Solo [16] introduced the concept of multi-dimensional matrix equality and developed algebraic operations for addition, subtraction, scalar multiplication, and multiplication of two multi-dimensional matrices. The (2, 1)-multiplication provides manipulating multi-dimensional matrices in a way that allows for deeper insights and applications in various mathematical contexts. Let $A = (a_{n,k,l})_{n,k,l\geq 0}$ and $B = (b_{n,k,l})_{n,k,l\geq 0}$ be infinite three-dimensional matrices. The (2, 1)-multiplication of A and B is given by

$$AB = (c_{n,k,l})_{n,k,l\geq 0}, \tag{5}$$

where $c_{n,k,l} = \sum_{x\geq 0} a_{n,x,l} b_{x,k,l}$, and this sum is presumed that the sum contains a limited count of non-zero terms.

Cheon and Jin [17] defined the three-dimensional Riordan array such that $f \in F_1$ and $h, g \in F_0$ and gave the matrix $(m_{n,k,l})_{n,k,l\geq 0} = (g, f, h)$ with the terms $m_{n,k,l} = [z^n]gf^k h^l$. The set of three-dimensional Riordan arrays constitutes a group with respect to the binary operation

$$(g, f, h)(G, F, H) = (gG(f), F(f), hH(f)),$$

where (g, f, h) and (G, F, H) are the three-dimensional Riordan array. The identity element of this group is $(1, t, 1)$, and this group is commonly referred to as the three-dimensional Riordan group.

A three-dimensional array can be perceived as a typical extension of a two-dimensional array, which is usual matrix. Let $U = (u_{n,k,l})_{n,k,l}$ be a three-dimensional arrays of the type $i_1 \times i_2 \times i_3$. The two-dimensional array for fixing l, $0 \le l \le i_3$ is called lth layer matrix of U. Using Riordan arrays and algebraic nature, the authors explained some progress in the theory of generating functions including some special numbers. For $A = (a_{n,k,l}) = (g, f, h)$, the authors gave the matrix $L_l(A)$, which is lth layer matrix of A. Also, this matrix can be obtained by the usual Riordan array given by (gh^l, f). For three-dimensional Riordan arrays $A = (a_{n,k,l}) = (g_1, f_1, h_1)$ and $B = (b_{n,k,l}) = (g_2, f_2, h_2)$, the lth layer matrix of AB is determined by the multiplication of $L_l(A)$ and $L_l(B)$ since

$$L_l(AB) = L_l(A)L_l(B). \tag{6}$$

Garsia [18] considered the formal power series $F(t)$ in the form for

$$F(t) = \frac{t}{R(t)}, \quad R(t) = R_0 + R_1 t + R_2 t^2 + \dots,$$

where $R(t) \in F_0$. Let the right inverse of $F(t)$ be $f(t) = \sum_{n=1}^{\infty} f_n t^n$. The author obtained

$$\sum_{n\ge1} f_n F(tq)\dots F(tq^{n-1}) = R(t).$$

For formal power series $\theta(t) = \sum_{n=0}^{\infty} \theta_n t^n$, the authors gave the operations and the reciprocal operations, respectively, by q and $1/q$ as follows:

$$\hat{\theta}(t) = \sum_{n=0}^{\infty} \theta_n q^{-\binom{n}{2}} t^n, \qquad \check{\theta}(t) = \sum_{n=0}^{\infty} \theta_n q^{\binom{n}{2}} t^n,$$

$$\theta^*(t) = \theta(t)\theta(tq)\theta(tq^2)\dots, \quad \text{and} \quad {}^*\theta(t) = \theta(t)\theta(t/q)\theta(t/q^2)\dots.$$

If $f(t) = t/r(t) = 1 + r_1 t + r_2 t^2 + \dots$, then for formal power series $\Phi(t) = \sum_{n=0}^{\infty} \Phi_n t^n$,

$$\Phi(\underline{f}) = \Phi \mathbin{\check{\circ}} f = \sum_{n=0}^{\infty} \Phi_n f(t) f(q/t) \dots f(t/q^{n-1}) = \frac{(\Phi(t)R^*(t))^\wedge}{\hat{R}^*(t)},$$

$$\Phi(\overline{F}) = \Phi \mathbin{\hat{\circ}} f = \sum_{n=0}^{\infty} \Phi_n F(t)F(qt)\dots F(q^{n-1}t) = \frac{(\Phi(t)^* r(t))^\vee}{{}^*\check{r}(t)}.$$

Tuğlu et al. [19,20] gave q-analog of the nth power of function $f(t)$ for $r \ge 1$ as the following relations:

$$f^{\overline{[r]}}(t) = f(t) *_q f^{\overline{[r-1]}}(t), \quad f^{\overline{[0]}}(t) = 1,$$

and

$$f^{\underline{[r]}}(t) = f(t) *_{1/q} f^{\underline{[r-1]}}(t), \quad f^{\underline{[0]}}(t) = 1,$$

with the operations $*_q$ and $*_{1/q}$ being defined as $f(t) *_q g(t) = f(t)g(qt)$, $f(t) *_{1/q} g(t) = f(t)g(t/q)$, respectively. The operations $*_q$ and $*_{1/q}$ satisfy

$$g(t) *_q f(t) *_q h(t) = g(t) *_q (f(t) *_q h(t))$$

and

$$g(t) *_{1/q} f(t) *_{1/q} h(t) = g(t) *_{1/q} (f(t) *_{1/q} h(t)).$$

Let $g(t) = \sum_{j=0}^{\infty} g_j(q) t^j$ and $f(t) = \sum_{j=0}^{\infty} f_j(q) t^j$, where $f_0(q) = 0, f_1(q) \ne 0$ and $g_0(q) \ne 0$. The q-Riordan array $(g, f)_q$ is $(r_{n,k})_{n,k\ge0}$ with terms

$$r_{n,k} = [t^n] g(t) *_q f^{\overline{[k]}}(t). \tag{7}$$

For $f(t) = t/(t;\, q)$ and $g(t) = 1/(t;\, q)$, they showed that (right) q-compositional inverse of $f(t)$, shown as $\overline{f}(t)$ i.e., $(\overline{f} \mathbin{\hat{\circ}} f)(t) = t$, is

$$\overline{f}(t) = \frac{t}{1 + t/q},$$

and

$$(g \circ f)(t) = \frac{1}{1 + t/q}.$$

Also, the inverse of $\left(\frac{1}{(t;\ q)}, \frac{t}{(t;\ q)}\right)$ is

$$(p_{n,k}) = \left(\frac{1}{(-t/q;\ q)}, \frac{t}{(-t/q;\ q)}\right)_{1/q},$$

where

$$p_{n,k} = \begin{cases} 0, & \text{if } n < k, \\ (-1)^{n-k} q^{\binom{k+1}{2} - n(k+1)} \begin{bmatrix} n \\ k \end{bmatrix}_q, & \text{if } n \geq k. \end{cases} \tag{8}$$

Also, they gave cases of $(g, f)_{q_1}(u, v)_{q_2}$, where $q_1, q_2 \in \{q, 1/q\}$.

In this article, first, three-dimensional q-Riordan arrays are defined using generating functions. These arrays can be used to find new identities. Second, general structure of three-dimensional q-Riordan arrays is given. To illustrate, three-dimensional q-Pascal-like matrix, which is obtained by a three-dimensional q-Riordan array, is examined and inverse of the array is given with the help of definition of three-dimensional q-Riordan array and Heine's binomial formula. Moreover, generalizations of these arrays are found. Third, the product of any two three-dimensional q-Riordan arrays is given for four cases. Finally, using three-dimensional q-Pascal matrix and its inverse, some combinatorial identities involving q-binomial coefficients are obtained for any sequence $(a_n)_{n\geq 1}$. As an application of these identities, taking harmonic sequence instead of $(a_n)_{n\geq 1}$, some interesting sums are found.

2 Structure of three-dimensional q-Riordan arrays

In this section, we provide three-dimensional q-Riordan arrays and explore their applications. The q-Riordan arrays are introduced as a powerful mathematical tool, allowing us to investigate various phenomena in different fields. By establishing their definition based on formal power series, we lay the foundation for utilizing three-dimensional q-Riordan arrays in diverse applications and exploring their potential implications in mathematics and related disciplines.

Definition 2.1. Let $g(t) = \sum_{j=0}^{\infty} g_j t^j$, $f(t) = \sum_{j=0}^{\infty} f_j t^j$, $h(t) = \sum_{j=0}^{\infty} h_j t^j$, with $g_0 \neq 0 \neq h_0$, $f_0 = 0, f_1 \neq 0$. The matrix $(r_{n,k,l})_{n,k,l\geq 0}$ is given by

$$r_{n,k,l} = [t^n] g(t) *_q h^{\overline{[l]}}(t) *_q f^{\overline{[k]}}(t). \tag{9}$$

Then, this matrix is called three-dimensional q-Riordan array or three-dimensional q-Riordan matrix, and we write $(r_{n,k,l})_{n,k,l\geq 0} = (g, f, h)_q$. Similarly, with help of the $*_{1/q}$, we have $(s_{n,k,l})_{n,k,l\geq 0} = (g, f, h)_{1/q}$, which is given by

$$s_{n,k,l} = [t^n] g(t) *_{1/q} h^{\underline{[l]}}(t) *_{1/q} f^{\underline{[k]}}(t). \tag{10}$$

We start by examining the layer matrices of three-dimensional q-Riordan arrays $R = (r_{n,k,l})_{n,k,l\geq 0} = (g, f, h)_q$ and $S = (s_{n,k,l})_{n,k,l\geq 0} = (g, f, h)_{1/q}$. By (9) and (10), it is seen that the lth layer matrices of R and S, denoted as $L_l(R)$ and $L_l(S)$, respectively, can be given an usual q-Riordan arrays such that

$$(g *_q h^{\overline{[l]}}, f)_q = (r_{n,k,l})_{n,k,l\geq 0} \quad \text{and} \quad (g *_{1/q} h^{\underline{[l]}}, f)_{1/q} = (s_{n,k,l})_{n,k,l\geq 0}.$$

The arrays $(r_{n,k,l})_{n,k,l\geq 0}$ and $(s_{n,k,l})_{n,k,l\geq 0}$ can be expressed using the layer matrices presented as

$$R = [L_0(R), L_1(R), \ldots] \quad \text{and} \quad S = [L_0(S), L_1(S), \ldots].$$

It is clear that (n,k,l)-entry of R and S is the (n,k)-entry of the lth layer matrix of R and S, respectively. Then, by (9) and (10), we have matrices $(r_{n,k,l})_{n,k,l\geq 0}$ and $(s_{n,k,l})_{n,k,l\geq 0}$ as

$$\left[\begin{array}{c}
\left[\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \\ g(t) & g(t)*_q f(t) & g(t)*_q f^{\overline{[2]}}(t) & \cdots \\ \downarrow & \downarrow & \downarrow & \end{array}\right] \longleftarrow L_0(R) \\
\left[\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \\ g(t)*_q h(t) & g(t)*_q h(t)*_q f(t) & g(t)*_q h(t)*_q f^{\overline{[2]}}(t) & \cdots \\ \downarrow & \downarrow & \downarrow & \end{array}\right] \longleftarrow L_1(R) \\
\left[\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \\ g(t)*_q h^{\overline{[2]}}(t) & g(t)*_q h^{\overline{[2]}}(t)*_q f(t) & g(t)*_q h^{\overline{[2]}}(t)*_q f^{\overline{[2]}}(t) & \cdots \\ \downarrow & \downarrow & \downarrow & \end{array}\right] \longleftarrow L_2(R) \\
\vdots
\end{array}\right]$$

and

$$\left[\begin{array}{c}
\left[\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \\ g(t) & g(t)*_{1/q} f(t) & g(t)*_{1/q} f^{\underline{[2]}}(t) & \cdots \\ \downarrow & \downarrow & \downarrow & \end{array}\right] \longleftarrow L_0(S) \\
\left[\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \\ g(t)*_{1/q} h(t) & g(t)*_{1/q} h(t)*_{1/q} f(t) & g(t)*_{1/q} h(t)*_{1/q} f^{\underline{[2]}}(t) & \cdots \\ \downarrow & \downarrow & \downarrow & \end{array}\right] \longleftarrow L_1(S) \\
\left[\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \\ g(t)*_{1/q} h^{\underline{[2]}}(t) & g(t)*_{1/q} h^{\underline{[2]}}(t)*_{1/q} f(t) & g(t)*_{1/q} h^{\underline{[2]}}(t)*_{1/q} f^{\underline{[2]}}(t) & \cdots \\ \downarrow & \downarrow & \downarrow & \end{array}\right] \longleftarrow L_2(S) \\
\vdots
\end{array}\right]$$

respectively. Also, these matrices are called the three-dimensional q-Riordan matrices of $(g,f,h)_q$ and $(g,f,h)_{1/q}$, respectively.

Lemma 1. *Let* $(r_{n,k,l})_{n,k,l\geq 0} = \left(\frac{1}{1-t}, \frac{t}{1-t}, \frac{1}{1-t}\right)_q$ *and* $(s_{n,k,l})_{n,k,l\geq 0} = \left(\frac{1}{1+t/q}, \frac{t}{1+t/q}, \frac{1}{1+t/q}\right)_{1/q}$ *be three-dimensional q-Riordan arrays. We have*

$$r_{n,k,l} = q^{kl+\binom{k+1}{2}}\begin{bmatrix} n+l \\ k+l \end{bmatrix}_q, \tag{11}$$

and

$$s_{n,k,l} = (-1)^{n-k} q^{\binom{k+1}{2}-n(k+l+1)}\begin{bmatrix} n+l \\ k+l \end{bmatrix}_q. \tag{12}$$

Proof. By (3) and (9), we have

$$\begin{aligned} r_{n,k,l} &= [t^n]\frac{1}{1-t} *_q \left(\frac{1}{1-t}\right)^{\overline{[l]}} *_q \left(\frac{t}{1-t}\right)^{\overline{[k]}} \\ &= [t^n]\frac{1}{1-t}\frac{1}{1-qt}\cdots\frac{1}{1-q^l t}\frac{q^{l+1}t}{1-q^{l+1}t}\cdots\frac{q^{k+l}t}{1-q^{k+l}t} \\ &= [t^{n-k}]q^{kl+\binom{k+1}{2}}\sum_{j=0}^{\infty}\begin{bmatrix} j+k+l \\ k+l \end{bmatrix}_q t^j = q^{kl+\binom{k+1}{2}}\begin{bmatrix} n+l \\ k+l \end{bmatrix}_q, \end{aligned} \tag{13}$$

as claimed. Similarly, for $(s_{n,k,l})_{n,k,l\geq 0} = \left(\frac{1}{1+t/q}, \frac{t}{1+t/q}, \frac{1}{1+t/q}\right)_{1/q}$, by (2) and (10), we have proof of (12). □

To illustrate, we choose the three-dimensional q-Riordan array that belongs to (11). We have the following three-dimensional q-Riordan matrix:

$$\left(\frac{1}{1-t},\frac{t}{1-t},\frac{1}{1-t}\right)_q=\begin{bmatrix}\begin{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}_q & & & & \\ \begin{bmatrix}1\\0\end{bmatrix}_q & q\begin{bmatrix}1\\1\end{bmatrix}_q & & & \\ \begin{bmatrix}2\\0\end{bmatrix}_q & q\begin{bmatrix}2\\1\end{bmatrix}_q & q^3\begin{bmatrix}2\\2\end{bmatrix}_q & & \\ \begin{bmatrix}3\\0\end{bmatrix}_q & q\begin{bmatrix}3\\1\end{bmatrix}_q & q^3\begin{bmatrix}3\\2\end{bmatrix}_q & q^6\begin{bmatrix}3\\3\end{bmatrix}_q & \\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}\\ \begin{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}_q & & & & \\ \begin{bmatrix}2\\1\end{bmatrix}_q & q^2\begin{bmatrix}2\\2\end{bmatrix}_q & & & \\ \begin{bmatrix}3\\1\end{bmatrix}_q & q^2\begin{bmatrix}3\\2\end{bmatrix}_q & q^5\begin{bmatrix}3\\3\end{bmatrix}_q & & \\ \begin{bmatrix}4\\1\end{bmatrix}_q & q^2\begin{bmatrix}4\\2\end{bmatrix}_q & q^5\begin{bmatrix}4\\3\end{bmatrix}_q & q^9\begin{bmatrix}4\\4\end{bmatrix}_q & \\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}\\ \begin{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}_q & & & & \\ \begin{bmatrix}3\\2\end{bmatrix}_q & q^3\begin{bmatrix}3\\3\end{bmatrix}_q & & & \\ \begin{bmatrix}4\\2\end{bmatrix}_q & q^3\begin{bmatrix}4\\3\end{bmatrix}_q & q^7\begin{bmatrix}4\\4\end{bmatrix}_q & & \\ \begin{bmatrix}5\\2\end{bmatrix}_q & q^3\begin{bmatrix}5\\3\end{bmatrix}_q & q^7\begin{bmatrix}5\\4\end{bmatrix}_q & q^{12}\begin{bmatrix}5\\5\end{bmatrix}_q & \\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}\\ \vdots\end{bmatrix}.$$

This matrix is called the three-dimensional q-Pascal-like matrix.

Let $(g,f,h)_{q_1}$ and $(u,v,w)_{q_2}$ be the three-dimensional q-Riordan arrays, where $q_1, q_2 \in \{q, 1/q\}$. Now, we will give cases of $(g,f,h)_{q_1}(u,v,w)_{q_2}$.

Theorem 2. *Let $(g,f,h)_q$ and $(u,v,w)_{1/q}$ be the three-dimensional q-Riordan arrays. Then,*

$$(g(t),f(t),h(t))_q(u(t),v(t),w(t))_{1/q}=(g(t),(A\,\hat{\circ}\,f)(q^{2l+2}t),h(q^{l+1}t))_{1/q},\tag{14}$$

where $A(t)=(u *_{1/q} w^{[l]} *_{1/q} v^{[k]})(t)$.

Proof. Let $(g,f,h)_q=(a_{n,k,l})_{n,k,l\geq 0}=A$, $(u,v,w)_{1/q}=(b_{n,k,l})_{n,k,l\geq 0}=B$. By (5), we obtain

$$(g,f,h)_q(u,v,w)_{1/q}=C=(c_{n,k,l})_{n,k,l\geq 0},$$

where $c_{n,k,l}=\sum_{x\geq 0}a_{n,x,l}b_{x,k,l}$. For fixed l, we show being (n,k)-entry of lth layer matrix of $L_l(AB)$ by $c_{n,k}$. From (9), the lth layer matrix $L_l(A)$ of $(g,f,h)_q$ is

$$\begin{bmatrix} g *_q h^{\boxed{l}}|_{t^0} & & & \\ g *_q h^{\boxed{l}}|_{t^1} & g *_q h^{\boxed{l}} *_q f|_{t^1} & & \\ g *_q h^{\boxed{l}}|_{t^2} & g *_q h^{\boxed{l}} *_q f|_{t^2} & g *_q h^{\boxed{l}} *_q f^{\boxed{2}}|_{t^2} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and the lth layer matrix $L_l(B)$ of $(u, v, w)_{1/q}$ is

$$\begin{bmatrix} u *_{1/q} w^{\underline{l}}|_{t^0} & & & \\ u *_{1/q} w^{\underline{l}}|_{t^1} & u *_{1/q} w^{\underline{l}} *_{1/q} v|_{t^1} & & \\ u *_{1/q} w^{\underline{l}}|_{t^2} & u *_{1/q} w^{\underline{l}} *_{1/q} v|_{t^2} & u *_{1/q} w^{\underline{l}} *_{1/q} v^{\underline{2}}|_{t^2} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $|_{t^n}$ is coefficient of term of t^n. By (6), we write

$$c_{n,k} = \begin{cases} \sum_{j=0}^{\infty} (g *_q h^{\boxed{l}} *_q f^{\boxed{j}}|_{t^n})(u *_{1/q} w^{\underline{l}} *_{1/q} v^{\underline{k}}|_{t^j}), & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases}$$

For matrix $(c_{n,k})_{n,k}$, let C_k denote its kth column. The generating function of C_k is

$$\begin{aligned} c_{k,k}t^k + c_{k+1,k}t^{k+1} + \dots &= \left(\sum_{j=0}^{\infty} (g *_q h^{\boxed{l}} *_q f^{\boxed{j}}|_{t^k})(u *_{1/q} w^{\underline{l}} *_{1/q} v^{\underline{k}}|_{t^j})\right)t^k \\ &\quad + \left(\sum_{j=0}^{\infty} (g *_q h^{\boxed{l}} *_q f^{\boxed{j}}|_{t^{k+1}})(u *_{1/q} w^{\underline{l}} *_{1/q} v^{\underline{k}}|_{t^j})\right)t^{k+1} + \dots \\ &= ((g *_q h^{\boxed{l}} *_q f^{\boxed{0}}|_{t^k})(u *_{1/q} w^{\underline{l}} *_{1/q} v^{\underline{k}}|_{t^0}) + (g *_q h^{\boxed{l}} *_q f^{\boxed{1}}|_{t^k})(u *_{1/q} w^{\underline{l}} *_{1/q} v^{\underline{k}}|_{t^1}) + \dots)t^k \\ &\quad + ((g *_q h^{\boxed{l}} *_q f^{\boxed{0}}|_{t^{k+1}})(u *_{1/q} w^{\underline{l}} *_{1/q} v^{\underline{k}}|_{t^0}) + \dots)t^{k+1} + \dots \\ &= (u *_{1/q} w^{\underline{l}} *_{1/q} v^{\underline{k}}|_{t^0})(g *_q h^{\boxed{l}} *_q f^{\boxed{0}}) + (u *_{1/q} w^{\underline{l}} *_{1/q} v^{\underline{k}}|_{t^1})(g *_q h^{\boxed{l}} *_q f^{\boxed{1}}) + \dots \\ &= g *_q h^{\boxed{l}} *_q ((u *_{1/q} w^{\underline{l}} *_{1/q} v^{\underline{k}}|_{t^0})f^{\boxed{0}} + (u *_{1/q} w^{\underline{l}} *_{1/q} v^{\underline{k}}|_{t^1})f^{\boxed{1}} + \dots) \\ &= g *_q h^{\boxed{l}} *_q ((u *_{1/q} w^{\underline{l}} *_{1/q} v^{\underline{k}}) \hat{\circ} f) \\ &= g *_q h^{\boxed{l}} *_q A \hat{\circ} f. \end{aligned}$$

From here, by combinatorial operations, we can write

$$\begin{aligned} g(t) *_q h^{\boxed{l}}(t) *_q (A \hat{\circ} f)(t) &= g(t)h(qt)\dots h(q^l t)(A \hat{\circ} f)(q^{l+1}t) \\ &= g(t)h(q^l t)\dots h(qt)(A \hat{\circ} f)(q^{l+1}t) \\ &= g(t) *_{1/q} h^{\underline{l}}(q^{l+1}t) *_{1/q} (A \hat{\circ} f)(q^{2l+2}t). \end{aligned}$$

The definition of the three-dimensional q-Riordan matrices yields that

$$(g(t), f(t), h(t))_q (u(t), v(t), w(t))_{1/q} = (g(t), (A \hat{\circ} f)(q^{2l+2}t), h(q^{l+1}t))_{1/q}.$$

The proof is complete. □

Theorem 3. *Let $(g, f, h)_{1/q}$ and $(u, v, w)_{1/q}$ be the three-dimensional q-Riordan arrays. Then,*

$$(g(t), f(t), h(t))_{1/q}(u(t), v(t), w(t))_{1/q} = (g(t), (A \check{\circ} f)(t), h(t))_{1/q}, \tag{15}$$

where $A(t)$ is as before.

Theorem 4. *Let $(g, f, h)_{1/q}$ and $(u, v, w)_q$ be the three-dimensional q-Riordan arrays. Then,*

$$(g(t), f(t), h(t))_{1/q}(u(t), v(t), w(t))_q = (g(t), (B \check{\circ} f)(t/q^{2(l+1)}), h(t/q^{l+1}))_q, \tag{16}$$

where $B(t) = (u *_q w^{\boxed{l}} *_q v^{\boxed{k}})(t)$.

Theorem 5. *Let $(g, f, h)_q$ and $(u, v, w)_q$ be the three-dimensional q-Riordan arrays. Then,*

$$(g(t), f(t), h(t))_q(u(t), v(t), w(t))_q = (g(t), (B \hat{\circ} f)(t), h(t))_q, \tag{17}$$

where $B(t)$ is as before.

The following theorem presents a systematic approach for computing the inverse of a q-Pascal-like matrix:

Theorem 6. *The inverse matrix of the three-dimensional q-Pascal-like matrix is*

$$\left(\frac{1}{1+t/q}, \frac{t}{1+t/q}, \frac{1}{1+t/q}\right)_{1/q}.$$

Proof. Let $A(t) = (u *_{1/q} w^{\boxed{l}} *_{1/q} v^{\boxed{k}})(t)$. From (2) and (3), we obtain

$$\begin{aligned} A(t) &= \frac{1}{1+t/q} *_{1/q} \left(\frac{1}{1+t/q}\right)^{\boxed{l}} *_{1/q} \left(\frac{t}{1+t/q}\right)^{\boxed{k}} \\ &= \frac{1}{1+t/q}\frac{1}{1+t/q^2}\cdots\frac{1}{1+t/q^{l+1}}\frac{t/q^{l+1}}{1+t/q^{l+2}}\cdots\frac{t/q^{l+k}}{1+t/q^{l+k+1}} \\ &= \frac{1}{q^{kl+\binom{k+1}{2}}}\sum_{j=0}^{\infty}\begin{bmatrix} j+k+l \\ k+l \end{bmatrix}_{1/q}\frac{(-1)^j}{q^j}t^{j+k} \\ &= \frac{1}{q^{kl+\binom{k+1}{2}}}\sum_{j=k}^{\infty}\begin{bmatrix} j+l \\ k+l \end{bmatrix}_{q}\frac{(-1)^{j-k}}{q^{(j-k)(k+l+1)}}t^{j}. \end{aligned}$$

Then,

$$(A \hat{\circ} f)(t) = \frac{1}{q^{kl+\binom{k+1}{2}}}\sum_{j=k}^{\infty}\begin{bmatrix} j+l \\ k+l \end{bmatrix}_{q}\frac{(-1)^{j-k}}{q^{(j-k)(k+l+1)}}(f(t))^{\boxed{j}}.$$

Finally, we take

$$(m_{n,k,l})_{n,k,l\geq 0} = \left(\frac{1}{1-t}, \frac{t}{1-t}, \frac{1}{1-t}\right)_q \left(\frac{1}{1+t/q}, \frac{t}{1+t/q}, \frac{1}{1+t/q}\right)_{1/q}.$$

Using $tg(t) = th(t) = f(t) = t/(1-t)$ and $tu(t) = tw(t) = v(t) = t/(1+t/q)$ in (14) and by (4), we have

$$\begin{aligned} m_{n,k,l} &= [t^n]\frac{1}{1-t} *_{1/q} \left(\frac{1}{1-q^{l+1}t}\right)^{\boxed{l}} *_{1/q} \left(\frac{1}{q^{kl+\binom{k+1}{2}}}\sum_{j=k}^{\infty}\begin{bmatrix} j+l \\ k+l \end{bmatrix}_{q}\frac{(-1)^{j-k}}{q^{(j-k)(k+l+1)}}\left(\frac{q^{2l+2}t}{1-q^{2l+2}t}\right)^{\boxed{j}}\right) \\ &= [t^n]\frac{1}{1-t}\frac{1}{1-q^l t}\cdots\frac{1}{1-qt}\frac{1}{q^{kl+\binom{k+1}{2}}}\sum_{j=k}^{\infty}\begin{bmatrix} j+l \\ k+l \end{bmatrix}_{q}\frac{(-1)^{j-k}}{q^{(j-k)(k+l+1)}}\frac{q^{l+1}t}{1-q^{l+1}t}\frac{q^{l+2}t}{1-q^{l+2}t}\cdots\frac{q^{l+j}t}{1-q^{l+j}t} \\ &= [t^n]\frac{1}{q^{kl+\binom{k+1}{2}}}\sum_{j=k}^{\infty}\begin{bmatrix} j+l \\ k+l \end{bmatrix}_{q}\frac{(-1)^{j-k}}{q^{(j-k)(k+l+1)}}\frac{1}{1-t}\frac{1}{1-qt}\cdots\frac{q^{l+j}t}{1-q^{l+j}t} \\ &= [t^n]\frac{1}{q^{kl+\binom{k+1}{2}}}\sum_{j=k}^{\infty}\begin{bmatrix} j+l \\ k+l \end{bmatrix}_{q}\frac{(-1)^{j-k}}{q^{(j-k)(k+l+1)}}\frac{q^{jl+\binom{j+1}{2}}t^j}{(t;\,q)_{j+l}} = t^k, \end{aligned}$$

which means kth column is $[0, 0, \cdots, 0, 1, 0, \cdots]^T$. So, proof is complete. □

Now, we give a generalization of the three-dimensional q-Riordan arrays in Lemma 1 without proof.

Lemma 7. *Let* $(r_{n,k,l})_{n,k,l\geq 0} = \left(\frac{1}{(1-t)^{m+1}}, \frac{t}{1-t}, \frac{1}{1-t}\right)_q$ *and* $(s_{n,k,l})_{n,k,l\geq 0} = \left(\frac{1}{(1+t/q)^{m+1}}, \frac{t}{1+t/q}, \frac{1}{1+t/q}\right)_{1/q}$ *be the three-dimensional q-Riordan arrays, where m is non-negative integer. We have*

$$r_{n,k,l} = q^{k(m+l)+\binom{k+1}{2}}\begin{bmatrix} n+m+l \\ k+m+l \end{bmatrix}_q \tag{18}$$

and

$$s_{n,k,l} = (-1)^{n-k}q^{\binom{k+1}{2}-n(k+m+l+1)}\begin{bmatrix} n+m+l \\ k+m+l \end{bmatrix}_q. \tag{19}$$

Theorem 8. *The inverse of* $\left(\frac{1}{(1-t)^{m+1}}, \frac{t}{1-t}, \frac{1}{1-t}\right)_q$ *is*

$$\left(\frac{1}{(1+t/q)^{m+1}}, \frac{t}{1+t/q}, \frac{1}{1+t/q}\right)_{1/q}.$$

Proof. Using similar way in Theorem 6, the result can be found. □

Theorem 9. *Let* $n \geq 1$, $l \geq 0$ *be integers and* $a(t) = \sum_{j=1}^{\infty} a_j t^j$. *Then, the following identities hold:*

$$\sum_{k=1}^{n} a_k q^{kl+\binom{k+1}{2}}\begin{bmatrix} n+l \\ k+l \end{bmatrix}_q = \sum_{j=1}^{n}\sum_{k=1}^{j} a_k q^{j(l+1)+\binom{k}{2}}\begin{bmatrix} n-j+l \\ l \end{bmatrix}_q\begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_q, \tag{20}$$

and

$$\sum_{k=1}^{n}(-1)^k q^{\binom{k+1}{2}-n(k+l+1)} a_k\begin{bmatrix} n+l \\ k+l \end{bmatrix}_q = \sum_{j=1}^{n}\sum_{k=1}^{j}(-1)^k q^{\binom{k-1}{2}-n(l+1)-kj+1} a_k\begin{bmatrix} n-j+l \\ l \end{bmatrix}_q\begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_q. \tag{21}$$

Proof. First, we will show the proof of (20). For fixed l, let $(d_{n,k})_{n,k\geq 0}$ be the lth layer matrix of $(r_{n,k,l})_{n,k,l\geq 0} = \left(\frac{1}{1-t}, \frac{t}{1-t}, \frac{1}{1-t}\right)_q$ and $C = (c_n)_n = (d_{n,k})_{n,k}(a_k)_k$, where

$$(a_k)_k = [0, a_1, a_2, \ldots]^T.$$

From (13) and definition of matrix multiplication,

$$c_n = \sum_{k=1}^{n} d_{n,k}a_k = \sum_{k=1}^{n} q^{kl+\binom{k+1}{2}}\begin{bmatrix} n+l \\ k+l \end{bmatrix} a_k. \tag{22}$$

On the other hand, we can write matrix C as

$$\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \frac{1}{1-t} * \left(\frac{1}{1-t}\right)^{\overline{[l]}} & \frac{1}{1-t} * \left(\frac{1}{1-t}\right)^{\overline{[l]}} * \frac{t}{1-t} & \frac{1}{1-t} * \left(\frac{1}{1-t}\right)^{\overline{[l]}} * \frac{t}{1-t} * \frac{t}{1-t} & \cdots \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}\begin{bmatrix} 0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}. \tag{23}$$

From here, by (3), matrix multiplication and production of power series, we have the nth term of column matrix C, namely, c_n is

$$\begin{aligned} c_n &= [t^n]\left(a_1\frac{1}{1-t}\frac{1}{1-qt}\cdots\frac{1}{1-q^lt}\frac{q^{l+1}t}{1-q^{l+1}t} + a_2\frac{1}{1-t}\frac{1}{1-qt}\cdots\frac{1}{1-q^lt}\frac{q^{l+1}t}{1-q^{l+1}t}\frac{q^{l+2}t}{1-q^{l+2}t} + \cdots\right) \\ &= [t^n]\frac{1}{1-t}\cdots\frac{1}{1-q^lt}\left(a_1\frac{q^{l+1}t}{1-q^{l+1}t} + a_2\frac{q^{l+1}t}{1-q^{l+1}t}\frac{q^{l+2}t}{1-q^{l+2}t} + \cdots\right) \\ &= [t^n]\sum_{n=0}^{\infty}\begin{bmatrix} n+l \\ l \end{bmatrix}_q t^n \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_k q^{n(l+1)+\binom{k}{2}}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q t^n \\ &= \sum_{j=1}^{n}\begin{bmatrix} n-j+l \\ l \end{bmatrix}_q \sum_{k=1}^{j} a_k q^{j(l+1)+\binom{k}{2}}\begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_q. \end{aligned} \tag{24}$$

Since left-hand sides of (22) and (23) are equal, we have (20). Similarly, if we use the three-dimensional q-Riordan array $\left(\frac{1}{1+t/q}, \frac{t}{1+t/q}, \frac{1}{1+t/q}\right)_{1/q}$ and (2), we can obtain (21). So, the proof is complete. □

Corollary 10. *For integers $n \geq 1$ and $l \geq 0$, we have*

$$\sum_{k=1}^{n} q^{kl+\binom{k+1}{2}} \begin{bmatrix} n+l \\ k+l \end{bmatrix}_q H_k = \sum_{j=1}^{n}\sum_{k=1}^{j} q^{j(l+1)+\binom{k}{2}} \begin{bmatrix} n-j+l \\ l \end{bmatrix}_q \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_q H_k$$

and

$$\sum_{k=1}^{n} (-1)^k q^{\binom{k+1}{2}-n(k+l+1)} \begin{bmatrix} n+l \\ k+l \end{bmatrix}_q H_k = \sum_{j=1}^{n}\sum_{k=1}^{j} (-1)^k q^{\binom{k-1}{2}-n(l+1)-kj+1} \begin{bmatrix} n-j+l \\ l \end{bmatrix}_q \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_q H_k.$$

Proof. Taking $\alpha(t) = \frac{-\ln(1-t)}{1-t}$ in Theorem 9 and by (1), one can reach the desired result immediately. □

3 Conclusions and future work

In conclusion, the study of three-dimensional q-Riordan arrays has provided valuable insights into the realm of combinatorial identities and matrix algebra. The defined structures and their properties offer a versatile tool for exploring new relationships and generating interesting sums. The obtained results highlight the significance of q-binomial coefficients and their connection to various mathematical sequences.

Moving forward, there may be several avenues for further research. First, investigating the properties and applications of three-dimensional q-Riordan arrays in other areas of mathematics, such as graph theory or number theory, would be beneficial. Additionally, exploring the connection between three-dimensional q-Riordan and other algebraic structures could provide deeper insights. Furthermore, extending the concept of three-dimensional q-Riordan arrays to higher dimensions or exploring different parameterizations could yield novel results.

Overall, the study of the three-dimensional q-Riordan arrays opens up exciting possibilities for future research and applications in various branches of mathematics. The developed techniques and identities pave the way for exciting avenues of research, opening doors to explore new connections and delve deeper into the properties of the three-dimensional q-Riordan arrays. Future studies can focus on investigating their applications in combinatorics, algebra, or other related fields.

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