Research Article

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Invariant means and lacunary sequence spaces of order (α, β)

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Abstract: In this article, we use the notion of lacunary statistical convergence of order (α, β) to introduce new sequence spaces by lacunary sequence, invariant means defined by Musielak-Orlicz function $\mathcal{M} = (\varkappa_k)$. We also examine some topological properties and prove inclusion relations between newly constructed sequence spaces.

Keywords: difference sequence space, fractional difference operator, Orlicz function, Musielak-Orlicz function, lacunary sequence, invariant mean

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1 Introduction and preliminaries

Consider a mapping σ from the set of positive integers into itself. A continuous linear functional ψ on l_∞, is known to be an invariant mean or σ-mean if and only if

1. $\psi(\zeta) \geq 0$ when the sequence $\zeta = (\zeta_k)$ has $\zeta_k \geq 0$ for all k,
2. $\psi(e) = 1$, where $e = (1, 1, 1, \ldots)$, and
3. $\psi(\zeta_{\sigma(k)}) = \psi(\zeta)$ for all $\zeta \in l_\infty$.

If $\zeta = (\zeta_n)$, we can write $T\zeta = T\zeta_n = (\zeta_{\sigma(n)})$. It can be seen in [1] that

$$V_\sigma = \left\{\zeta \in l_\infty : \lim_k t_{kn}(\zeta) = l, \quad \text{uniformly in } n, \; l = \sigma - \lim \zeta\right\},$$

where

$$t_{kn}(\zeta) = \frac{\zeta_n + \zeta_{\sigma^1 n} + \cdots + \zeta_{\sigma^k n}}{k + 1}.$$

In this case, σ is the translation mapping $n \to n + 1$, σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences (see [2]).

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The first edition of [3] introduced the concept of statistical convergence. Later, this concept was separately proposed by Fast [4] and Steinhaus [5], and some of its fundamental characteristics were investigated by Schoenberg [6], Salat [7], and Fridy [8]. This idea was expanded upon by Çolak [9] using the concept of α-density (for $\alpha = 1$, α-density lowered to natural density) and was given the name statistical convergence of order α. In [10,11], the lacunary statistical convergence of order α was studied. For more development of the topic, we refer to [12–15]. Şengül [16] recently offered an intriguing generalization of this idea by considering (α, β) on behalf of α and characterized statistical convergence of order (α, β) as follows.

The sequence $\zeta = (\zeta_k)$ is known to be statistically convergent of order (α, β), briefly S_α^β-convergence to L if for each $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \le n : |\zeta_k - L| \ge \varepsilon\}|^\beta = 0,$$

where $|\{k \le n : k \in E\}|^\beta$ represents the βth power of number of elements of E not exceeding n, and we write it as $S_\alpha^\beta - \lim \zeta_k = L$.

By a lacunary sequence $\theta = (\theta_r)$ where $\theta_0 = 0$, we mean an increasing sequence of positive integers with $\theta_r - \theta_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be represented by $J_r = (\theta_{r-1}, \theta_r]$ and $t_r = \theta_r - \theta_{r-1}$. The space of lacunary strongly convergent sequences was developed by Freedman et al. [17] as

$$N_\theta = \left\{\zeta \in w : \lim_{r\to\infty} \frac{1}{\phi_r} \sum_{k\in J_r} |\zeta_k - l| = 0, \quad \text{for some } l\right\}.$$

Definition. Consider a lacunary sequence $\theta = (\theta_r)$. The sequence $\zeta = (\zeta_k)$ is $S_\alpha^\beta(\theta)$-statistically convergent (or lacunary statistically convergent of order (α, β)) (see [18]) if there is a real number L such that

$$\lim_{r\to\infty} \frac{1}{\phi_r^\alpha} |\{k \in J_r : |\xi_k - L| \ge \varepsilon\}|^\beta = 0,$$

where $J_r = (\theta_{r-1}, \theta_r]$ and ϕ_r^α represents the αth power $(\phi_r)^\alpha$ of ϕ_r, i.e., $\phi^\alpha = (\phi_r^\alpha) = (\phi_1^\alpha, \phi_2^\alpha, \ldots, \phi_r^\alpha, \ldots)$ and $|\{k \le n : k \subset E\}|^\beta$ represents the βth power of number of elements of E not exceeding n. In this case, the convergence is indicated by $S_\alpha^\beta(\theta) - \lim \xi_k = L$. $S_\alpha^\beta(\theta)$ which will denote the set of all $S_\alpha^\beta(\theta)$-statistically convergent sequences. If $\alpha = \beta = 1$ and $\theta = (2^r)$, then we will write S instead of $S_\alpha^\beta(\theta)$.

Kızmaz [19], who investigated the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, developed the idea of difference sequence spaces. Et and Çolak [20] introduced the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$, and $c_0(\Delta^n)$ to further expand the idea. The fractional difference operator defined by Baliarsingh [21] was as follows:

Assume that $\xi = (\xi_k) \in w$ and γ be a real number, then the fractional difference operator $\Delta^{(\gamma)}$ is defined by

$$\Delta^{(\gamma)}\xi_k = \sum_{i=0}^{k} \frac{(-\gamma)_i}{i!} \xi_{k-i},$$

where $(-\gamma)_i$ denotes the Pochhammer symbol represented as

$$(-\gamma)_i = \begin{cases} 1, & \text{if } \gamma = 0 \text{ or } i = 0, \\ \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+i-1), & \text{otherwise}. \end{cases}$$

Numerous authors have used the ideas of n-normed spaces [22], difference sequences [23], the Orlicz function [24], and the Musielak-Orlicz function [25] to study sequence spaces along with their applications in approximation theory (see [26–32]). Mohiuddine et al. [33] and Sharma et al. [34] studied some sequence spaces of order (α, β). In this article, we construct some new difference sequence spaces of lacunary statistical convergence defined by Musielak-Orlicz function as follows. We study topological properties of these new sequence spaces and obtain some inclusion results.

An Orlicz function $\aleph$ is a function, $\aleph : [0, \infty) \to [0, \infty)$, which is continuous, nondecreasing, and convex with $\aleph(0) = 0$, $\aleph(x) > 0$ for $x > 0$, and $\aleph(x) \to \infty$ as $x \to \infty$. A sequence $\mathcal{M} = (\aleph_k)$ of Orlicz function is called a Musielak-Orlicz function [35].

Consider $\mathcal{M} = (\aleph_k)$ as a Musielak-Orlicz function, $0 < \alpha \le \beta \le 1$ and $u = (u_k)$ a bounded sequence of positive real numbers. In this article, we define the following sequence spaces:

$$w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0 = \left\{\zeta \in S(n-X) : \lim_{r\to\infty} \frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta = 0 \text{ uniformly in } n, \text{for some } \rho > 0\right\},$$

$$w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma = \left\{\zeta \in S(n-X) : \lim_{r\to\infty} \frac{1}{\phi_r^\alpha}\left[\sum_{k\in I_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k - l)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta = 0 \text{ uniformly in } n, \text{for some } l \text{ and } \rho > 0,\right\}$$

and

$$w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^\infty = \left\{\zeta \in S(n-X) : \sup_{r,n} \frac{1}{\phi_r^\alpha}\left[\sum_{k\in I_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta < \infty \text{ for some } \rho > 0\right\}.$$

If we take $\mathcal{M}(\zeta) = \zeta$, then the previous spaces become $w_\alpha^\beta[u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0$, $w_\alpha^\beta[u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma$, and $w_\alpha^\beta[u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^\infty$.

By taking $u = (u_k) = 1$, then the previous spaces become $w_\alpha^\beta[\mathcal{M}, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0$, $w_\alpha^\beta[\mathcal{M}, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma$ and $w_\alpha^\beta[\mathcal{M}, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^\infty$.

Throughout the article, the following inequality will be used. If $0 \le p_k \le \sup p_k = K$, $D = \max(1, 2^{K-1})$, then

$$|a_k + b_k|^{p_k} \le D\{|a_k|^{p_k} + |b_k|^{p_k}\}, \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

2 Main results

Theorem 2.1. *Let $\mathcal{M} = (\aleph_k)$ be a Musielak-Orlicz function and $u = (u_k)$ be a bounded sequence of positive real numbers. Then, the spaces $w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0$, $w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma$, and $w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^\infty$ are the linear spaces over $\mathbb{C}$ (where $\mathbb{C}$ is the field of complex number).*

Proof. The proof is trivial, so we omit the details. □

Theorem 2.2. *Let $\mathcal{M} = (\aleph_k)$ be a Musielak-Orlicz function and $u = (u_k)$ be a bounded sequence of positive real numbers. Then, $w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0$ is a paranormed space with paranorm defined by*

$$g(\zeta) = \inf\left\{\rho^{\frac{u_r}{K}} : \left(\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right)^{\frac{1}{K}} \le 1, r, n = 1, 2, \ldots\right\},$$

where $K = \max(1, \sup_k u_k < \infty)$.

Proof. Clearly, $g(\zeta) \ge 0$ for $\zeta = (\zeta_k) \in w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0$. Now, $\aleph_k(0) = 0$, and we have $g(0) = 0$. Conversely, we suppose that $g(\zeta) = 0$, then

$$\inf\left\{\rho^{\frac{u_r}{K}} : \left(\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right)^{\frac{1}{K}} \le 1, r, n = 1, 2, \ldots\right\} = 0.$$

This implies that for a given $\varepsilon > 0$, there exists some $\rho_\varepsilon (0 < \rho_\varepsilon < \varepsilon)$ such that

$$\left(\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho_\varepsilon}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right)^{\frac{1}{K}} \leq 1.$$

Thus,

$$\left(\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\varepsilon}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right)^{\frac{1}{K}}$$
$$\leq \left(\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho_\varepsilon}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right)^{\frac{1}{K}} \leq 1.$$

For each $k \in \mathbb{N}$, we suppose that $\zeta_k \neq 0$. This means that $t_{kn}(\Delta^{(\gamma)}\zeta_k) \neq 0$. Let $\varepsilon \to 0 \Rightarrow \left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\varepsilon}, z_1, \ldots, z_{n-1}\right\| \to \infty$. It follows that

$$\left(\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\varepsilon}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right)^{\frac{1}{K}} \to \infty,$$

which is a contradiction. Therefore, $t_{kn}(\Delta^{(\gamma)}\zeta_k) = 0$ for each k, and thus, $\Delta^{(\gamma)}\zeta_k = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\left(\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho_1}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right)^{\frac{1}{K}} \leq 1$$

and

$$\left(\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\vartheta_k)}{\rho_2}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right)^{\frac{1}{K}} \leq 1,$$

for each r. Suppose that $\rho = \rho_1 + \rho_2$. Therefore, by Minkowski's inequality, we have

$$\left(\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}(\zeta_k + \vartheta_k))}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right)^{\frac{1}{K}}$$
$$\leq \left(\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k) + t_{kn}(\Delta^{(\gamma)}\vartheta_k)}{\rho_1 + \rho_2}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right)^{\frac{1}{K}}$$
$$\leq \left(\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\frac{\rho_1}{\rho_1 + \rho_2}\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho_1}, z_1, \ldots, z_{n-1}\right\|\right)\right.\right.\right.$$
$$\left.\left.\left. + \frac{\rho_2}{\rho_1 + \rho_2}\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\vartheta_k)}{\rho_2}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right)^{\frac{1}{K}}$$
$$\leq \left(\frac{\rho_1}{\rho_1 + \rho_2}\right)\left(\frac{1}{\phi_r^\alpha}\left[\sum_{k\in I_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho_1}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right)^{\frac{1}{K}}$$
$$+ \left(\frac{\rho_2}{\rho_1 + \rho_2}\right)\left(\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\vartheta_k)}{\rho_2}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right)^{\frac{1}{K}} \leq 1.$$

Since ρ's are non-negative, so we have

$$g(\zeta+\vartheta)=\inf\left\{\rho^{\frac{u_r}{H}}:\left(\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)+t_{kn}(\Delta^{(\gamma)}\vartheta_k)}{\rho},z_1,\dots,z_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\right)^{\frac{1}{K}}\leq 1,\ r,n=1,2,\dots\right\}$$

$$\leq\inf\left\{\rho_1^{\frac{u_r}{H}}:\left(\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho_1},z_1,\dots,z_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\right)^{\frac{1}{K}}\leq 1,r,n=1,2,\dots\right\}$$

$$+\inf\left\{\rho_2^{\frac{u_r}{H}}:\left(\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\vartheta_k)}{\rho_2},z_1,\dots,z_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\right)^{\frac{1}{K}}\leq 1,r,n=1,2,\dots\right\}.$$

Thus,

$$g(\zeta+\vartheta)\leq g(\zeta)+g(\vartheta).$$

Finally, we have to show that the scalar multiplication is continuous. Suppose that ν be any complex number. Therefore, by definition,

$$g(\nu\zeta)=\inf\left\{\rho^{\frac{u_r}{H}}:\left(\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\lambda\zeta_k)}{\rho},z_1,\dots,z_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\right)^{\frac{1}{K}}\leq 1,r,n=1,2,\dots\right\}.$$

Then,

$$g(\nu\zeta_k)=\inf\left\{(|\nu|t)^{\frac{u_r}{H}}:\left(\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{t},z_1,\dots,z_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\right)^{\frac{1}{K}}\leq 1,r,n=1,2,\dots\right\},$$

where $t=\frac{\rho}{|\nu|}$. Since $|\nu|^{u_r}\leq\max(1,|\nu|^{\sup u_r})$, we have

$$g(\nu\zeta)\leq\max(1,|\nu|^{\sup p_r})\inf\left\{t^{\frac{u_r}{H}}:\left(\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{t},z_1,\dots,z_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\right)^{\frac{1}{K}}\leq 1,r,n=1,2,\dots\right\}.$$

From the aforementioned inequality, it follows that scalar multiplication is continuous. □

Theorem 2.3. *Let $\mathcal{M}=(\aleph_k)$ be a Musielak-Orlicz function. Then,*

$$w_{\alpha}^{\beta}[\mathcal{M},u,\Delta^{(\gamma)},\theta,\|\cdot,\dots,\cdot\|]_{\sigma}^{0}\subset w_{\alpha}^{\beta}[\mathcal{M},u,\Delta^{(\gamma)},\theta,\|\cdot,\dots,\cdot\|]_{\sigma}\subset w_{\alpha}^{\beta}[\mathcal{M},u,\Delta^{(\gamma)},\theta,\|\cdot,\dots,\cdot\|]_{\sigma}^{\infty}.$$

Proof. Now, $w_{\alpha}^{\beta}[\mathcal{M},u,\Delta^{(\gamma)},\theta,\|\cdot,\dots,\cdot\|]_{\sigma}^{0}\subset w_{\alpha}^{\beta}[\mathcal{M},u,\Delta^{(\gamma)},\theta,\|\cdot,\dots,\cdot\|]_{\sigma}$ is obvious. We only need to show that $w_{\alpha}^{\beta}[\mathcal{M},u,\Delta^{(\gamma)},\theta,\|\cdot,\dots,\cdot\|]_{\sigma}^{0}\subset w_{\alpha}^{\beta}[\mathcal{M},u,\Delta^{(\gamma)},\theta,\|\cdot,\dots,\cdot\|]_{\sigma}$. For this, let $w_{\alpha}^{\beta}[\mathcal{M},u,\Delta^{(\gamma)},\theta,\|\cdot,\dots,\cdot\|]_{\sigma}$. Then, there exists some positive number ρ_1 such that

$$\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_{k-l})}{\rho_1},z_1,\dots,z_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\to 0,\quad \text{as } r\to\infty \text{ uniformly in } n.$$

Define $\rho=2\rho_1$. Since $\mathcal{M}=(\aleph_k)$ is non-decreasing, convex, and so using Inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \\
&\le \frac{D}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k - l)}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \\
&\quad + \frac{D}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{l}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \\
&\le \frac{D}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k - l)}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \\
&\quad + D\max\left\{1, \left[\aleph_k\left(\left\|\frac{l}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)\right]^K\right\}.
\end{aligned}$$

Thus, $\zeta \in w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\dots,\cdot\|]_\sigma^\infty$. □

Theorem 2.4. *If* $\sup_k[\aleph_k(q)]^{u_k} < \infty$ *for all* $q > 0$, *then*

$$w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\dots,\cdot\|]_\sigma \subset w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\dots,\cdot\|]_\sigma^\infty.$$

Proof. Let $\zeta \in w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\dots,\cdot\|]_\sigma$. Using Inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \\
&\le \frac{D}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k - l)}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta + \frac{D}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{l}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta.
\end{aligned}$$

We can take $\sup_k[\aleph_k(q)]^{u_k} = Q$ as $\sup_k[\aleph_k(q)]^{u_k} < \infty$ is given. Hence, we have $\zeta \in w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\dots,\cdot\|]_\sigma^\infty$. □

Theorem 2.5. *Consider that* $\mathcal{M} = (\aleph_k)$ *satisfies* Δ_2*-condition for all* k, *then*

$$w_\alpha^\beta[u, \Delta^{(\gamma)}, \theta, \|\cdot,\dots,\cdot\|]_\sigma \subset w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\dots,\cdot\|]_\sigma.$$

Proof. Let $\zeta \in w_\alpha^\beta[u, \Delta^{(\gamma)}, \theta, \|\cdot,\dots,\cdot\|]_\sigma$. Then, we have

$$Q_r = \frac{1}{\phi_r^\alpha}\sum_{k\in J_r}\|t_{kn}(\Delta^{(\gamma)}\zeta_k - l), z_1, \dots, z_{n-1}\|^{u_k} \to \infty, \quad \text{as } r \to \infty.$$

Suppose $\varepsilon > 0$ and choose δ as $0 < \delta < 1$ such that $\aleph_k(q) < \varepsilon$ for $0 \le q \le \delta$ for all k, we have

$$\begin{aligned}
&\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k - l)}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \\
&= \frac{1}{\phi_r^\alpha}\left[\sum^{1}_{k\in J_r, \|t_{kn}(\Delta^{(\gamma)}\zeta - l), z\|\le\delta}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k - l)}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \\
&\quad + \frac{1}{\phi_r^\alpha}\left[\sum^{2}_{k\in J_r, \|t_{kn}(\Delta^{(\gamma)}\zeta - l), z\|>\delta}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k - l)}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta.
\end{aligned}$$

For summation first in the aforementioned equation, we have $\sum^{1} \le \varepsilon^{K}$ and the second summation for all k, we write

$$\|t_{kn}(\Delta^{(\gamma)}\zeta_{k-l}), z_1, \ldots, z_{n-1}\| \le 1 + \left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_{k-l})}{\delta}, z_1, \ldots, z_{n-1}\right\|.$$

Now, this implies that

$$\begin{aligned}&[\aleph_k(\|t_{kn}(\Delta^{(\gamma)}\zeta_k - l), z_1, \ldots, z_{n-1}\|)]\\ &< \left[\aleph_k\left(1 + \left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k - l)}{\delta}, z_1, \ldots, z_{n-1}\right\|\right)\right]\\ &\le \frac{1}{2}(\aleph_k(2)) + \frac{1}{2}\left[\aleph_k\left((2)\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k - l)}{\delta}, z_1, \ldots, z_{n-1}\right\|\right)\right].\end{aligned}$$

Since $\aleph_k$ satisfies Δ_2-condition, we can have

$$\begin{aligned}[\aleph_k(\|t_{kn}(\Delta^{(\gamma)}\zeta_k - l), z_1, \ldots, z_{n-1}\|)] &\le \frac{1}{2}L\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k - l)}{\delta}, z_1, \ldots, z_{n-1}\right\|\aleph_k(2)\\ &\quad + \frac{1}{2}L\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k - l)}{\delta}, z_1, \ldots, z_{n-1}\right\|\aleph_k(2)\\ &= L\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k - l)}{\delta}, z_1, \ldots, z_{n-1}\right\|\aleph_k(2).\end{aligned}$$

So we write

$$\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k - l)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta} \le \varepsilon^{K} + [\max(1, L\aleph_k(2))\delta]^{K} Q_r.$$

Now, $r \to \infty$ implies that $\zeta \in w_{\alpha}^{\beta}[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot, \ldots, \cdot\|]_{\sigma}$. Hence, this completes the proof. □

Theorem 2.6. *The following statements are equivalent*:

i. $w_{\alpha}^{\beta}[u, \Delta^{(\gamma)}, \theta, \|\cdot, \ldots, \cdot\|]_{\sigma}^{\infty} \subset w_{\alpha}^{\beta}[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot, \ldots, \cdot\|]_{\sigma}^{0}$,
ii. $w_{\alpha}^{\beta}[u, \Delta^{(\gamma)}, \theta, \|\cdot, \ldots, \cdot\|]_{\sigma}^{0} \subset w_{\alpha}^{\beta}[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot, \ldots, \cdot\|]_{\sigma}^{0}$,
iii. $\sup_r \frac{1}{\phi_r^{\alpha}} \sum_{k\in J_r} [\aleph_k(q)]^{u_k} < \infty$, *for all* $q > 0$.

Proof. (i)⇒ (ii) We need to show that $w_{\alpha}^{\beta}[u, \Delta^{(\gamma)}, \theta, \|\cdot, \ldots, \cdot\|]_{\sigma}^{0} \subset w_{\alpha}^{\beta}[u, \Delta^{(\gamma)}, \theta, \|\cdot, \ldots, \cdot\|]_{\sigma}^{\infty}$. Let $\zeta \in w_{\alpha}^{\beta}[u, \Delta^{(\gamma)}, \theta, \|\cdot, \ldots, \cdot\|]_{\sigma}^{0}$. Then, there exists $r \ge r_0$, for $\varepsilon > 0$, such that

$$\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)^{u_k}\right]^{\beta} < \varepsilon.$$

Hence, there exists $K > 0$ such that

$$\sup_{r,n}\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)^{u_k}\right]^{\beta} < K,$$

for all n and r. So we obtain $\zeta \in w_{\alpha}^{\beta}[u, \Delta^{(\gamma)}, \theta, \|\cdot, \ldots, \cdot\|]_{\sigma}^{\infty}$.

(ii) ⇒ (iii) Suppose that (iii) does not imply. Then, for some $q > 0$,

$$\sup_{r}\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}[\aleph_k(q)]^{u_k}\right] = \infty,$$

and now, we can take a subinterval $J_{r(m)}$ of J_r such that

$$\frac{1}{\phi^{\alpha}_{r(m)}}\left[\sum_{k\in J_{r(m)}}\left[\aleph_k\left(\frac{1}{m}\right)\right]^{u_k}\right]^{\beta} > m, \quad m = 1, 2, \ldots. \tag{2.1}$$

Let us define $\zeta = (\zeta_k)$ as follows: $\zeta_k = \frac{1}{m}$ if $k \in J_{r(m)}$ and $\zeta_k = 0$ if $k \notin J_{r(m)}$. Then, $\zeta \in w^{\beta}_{\alpha}[u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]^0_{\sigma}$ but by equation (2.1), $\zeta \notin w^{\beta}_{\alpha}[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]^0_{\sigma}$, which is a contradiction. Thus, our supposition is wrong. Hence, (iii) must hold.

(iii) ⇒ (i) Suppose that (i) does not holds; thus, for $\zeta \in w^{\beta}_{\alpha}[u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]^{\infty}_{\sigma}$, we have

$$\sup_{r,n}\frac{1}{\phi^{\alpha}_{r}}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right] = \infty. \tag{2.2}$$

Let $q = \left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|$ for each k and fixed n, so that equation (2.2) becomes

$$\sup_{r}\frac{1}{\phi^{\alpha}_{r}}\left[\sum_{k\in J_r}[\aleph_k(q)]^{u_k}\right]^{\beta} = \infty,$$

which is a contradiction. Thus, our supposition is wrong. Hence, (i) must hold. □

Theorem 2.7. *The following statements are equivalent*:

i. $w^{\beta}_{\alpha}[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]^0_{\sigma} \subset w^{\beta}_{\alpha}[u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]^0_{\sigma}$,
ii. $w^{\beta}_{\alpha}[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]^0_{\sigma} \subset w^{\beta}_{\alpha}[u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]^{\infty}_{\sigma}$,
iii. $\inf_r \sum_{k\in J_r}[\aleph_k(q)]^{u_k} > 0$, *for all* $q > 0$.

Proof. (i) ⇒ (ii) : The proof is straightforward so we omit the details.

(ii) ⇒ (iii) Let us assume that (iii) does not hold. Then,

$$\inf_{r}\frac{1}{\phi^{\alpha}_{r}}\left[\sum_{k\in J_r}[\aleph_k(q)]^{u_k}\right]^{\beta} = 0, \quad \text{for some } q > 0,$$

and now, we can take a subinterval $J_{r(m)}$ of J_r such that

$$\frac{1}{\phi^{\alpha}_{r}}\left[\sum_{k\in J_{r(m)}}[\aleph_k(m)]^{u_k}\right]^{\beta} < \frac{1}{m}, \quad m = 1, 2, \ldots. \tag{2.3}$$

Now, we take $\zeta_k = m$ if $k \in J_{r(m)}$ and $\zeta_k = 0$ if $k \notin J_{r(m)}$. Therefore, by equation (2.3), $\zeta \in w^{\beta}_{\alpha}[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]^0_{\sigma}$ but $\zeta \notin w^{\beta}_{\alpha}[u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]^{\infty}_{\sigma}$, which is a contradiction. Thus, our supposition is wrong. Therefore, (iii) holds.

(iii) ⇒ (i) It is trivial, so we omit the details. □

Theorem 2.8. *The inclusion* $w^{\beta}_{\alpha}[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]^{\infty}_{\sigma} \subset w^{\beta}_{\alpha}[u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]^0_{\sigma}$ *holds if and only if*

$$\lim_{r\to\infty}\frac{1}{\phi^{\alpha}_{r}}\left[\sum_{k\in I_r}[M_k(t)]^{u_k}\right]^{\beta} = \infty. \tag{2.4}$$

Proof. Let $w^{\beta}_{\alpha}[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]^{\infty}_{\sigma} \subset w^{\beta}_{\alpha}[u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]^0_{\sigma}$. Let us assume that equation (2.4) does not hold. Therefore, there is a subinterval $J_{r(m)}$ of J_r and a number $q_0 > 0$, where $q_0 = \left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots z_{n-1}\right\|$ for all k and n, such that

$$\frac{1}{\phi^{\alpha}_{r(m)}}\left[\sum_{k\in J_{r(m)}}[\aleph_k(q_0)]^{u_k}\right]^{\beta} \le M < \infty, \quad m = 1, 2, \ldots. \tag{2.5}$$

Let us define $\zeta_k = q_0$ if $k \in J_{r(m)}$ and $\zeta_k = 0$ if $k \notin J_{r(m)}$. Then, by equation (2.5), $\zeta \in w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^\infty$. But $\zeta \notin w_\alpha^\beta[u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^0$. Hence, equation (2.5) holds.

Conversely, let us suppose that equation (2.5) holds and $\zeta \in w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^\infty$. Then, we have

$$\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \leq M < \infty. \tag{2.6}$$

Now, suppose that $\zeta \notin w_\alpha^\beta[u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^0$. Then, for $\varepsilon > 0$ and a subinterval J_{r_i} of J_r, there is k_0 such that $||t_{kn}(\Delta^{(\gamma)}\zeta_k), z_1, \ldots, z_{n-1}||^{u_k} > \varepsilon$ for $k \geq k_0$. Thus, we have

$$\left[\aleph_k\left(\frac{\varepsilon}{\rho}\right)\right]^{u_k} \leq \left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k},$$

which is a contradiction to equation (2.5) using equation (2.6). Hence, $w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^\infty \subset w_\alpha^\beta[u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^0$ holds if and only if $\lim_{r\to\infty}\frac{1}{\phi_r^\alpha}[\sum_{k\in I_r}[M_k(t)]^{u_k}]^\beta = \infty$. □

Theorem 2.9. *The following inclusions hold for Musielak-Orlicz functions $\mathcal{M} = (\aleph_k)$ and $\mathcal{M}' = (\aleph'_k)$:*

(i) $w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^\infty \cap w_\alpha^\beta[\mathcal{M}'^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^\infty$
$$\subset w_\alpha^\beta[\mathcal{M} + \mathcal{M}'^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^0;$$

(ii) $w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma \cap w_\alpha^\beta[\mathcal{M}'^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma$
$$\subset w_\alpha^\beta[\mathcal{M} + \mathcal{M}'^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma;$$

(iii) $w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^0 \cap w_\alpha^\beta[\mathcal{M}'^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^0$
$$\subset w_\alpha^\beta[\mathcal{M} + \mathcal{M}'^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^0.$$

Proof. Suppose that $\zeta \in w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^\infty \cap w_\alpha^\beta[\mathcal{M}'^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^\infty$. Then,

$$\sup_{r,n}\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta < \infty \quad \text{uniformly in } n$$

and

$$\sup_{r,n}\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph'_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta < \infty \quad \text{uniformly in } n.$$

Now, using Inequality (1.1), we obtain

$$\begin{aligned}
&\left[(\aleph_k + \aleph'_k)\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k} \\
&\leq D\left[\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right] + D\left[\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right] \\
&\Rightarrow \frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[(\aleph_k + \aleph'_k)\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \\
&\leq \frac{D}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta + \frac{D}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph'_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \quad \text{uniformly in } n.
\end{aligned}$$

This proves $w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^\infty \cap w_\alpha^\beta[\mathcal{M}'^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^\infty$
$$\subset w_\alpha^\beta[\mathcal{M} + \mathcal{M}'^{(\gamma)}, \theta, ||\cdot,\ldots,\cdot||]_\sigma^0.$$

Similarly, we prove

$$w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma \cap w_\alpha^\beta[\mathcal{M}'^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma \subset w_\alpha^\beta[\mathcal{M} + \mathcal{M}'^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma$$

and

$$w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0 \cap w_\alpha^\beta[\mathcal{M}'^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0 \subset w_\alpha^\beta[\mathcal{M} + \mathcal{M}'^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0. \qquad \square$$

Theorem 2.10. *If* $0 < u_k \le v_k$ *and* $\left(\frac{v_k}{u_k}\right)$ *be bounded. Then, for each k, we have*

(*i*) $w_\alpha^\beta[\mathcal{M}, v, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]\infty_\sigma \subset w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^\infty$;
(*ii*) $w_\alpha^\beta[\mathcal{M}, v, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0 \subset w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma$;
(*iii*) $w_\alpha^\beta[\mathcal{M}, v, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0 \subset w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0$.

Proof. (i) Let $\zeta \in w_\alpha^\beta[\mathcal{M}, v, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^\infty$. Then,

$$\sup_{r,n} \frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{v_k}\right]^\beta < \infty \quad \text{uniformly in } n.$$

Write $\mu_{k,n} = \left[\left[\aleph_k\left(\left\|\frac{t_{kn}(\Delta^{(\gamma)}\zeta_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{v_k}\right]^\beta$ and $\lambda_k = \frac{u_k}{v_k}$. Since $u_k \le v_k$, $0 < \lambda < \lambda_k \le 1$. Define $x_{k,n} = \mu_{k,n}$, $x_{k,n} = 0$ if $\mu_{k,n} \ge 1$ and $y_{k,n} = \mu_{k,n}$, $y_{k,n} = 0$ if $\mu_{k,n} \ge 1$. So $\mu_{k,n} = x_{k,n} + y_{k,n}$ and $\mu_{k,n}^{\lambda_k} = x_{k,n}^{\lambda_k} + y_{k,n}^{\lambda_k}$. Now, it follows that $y_{k,n}^{\lambda_k} \le x_{k,n} \le y_{k,n}$ and $y_{k,n}^{\lambda_k} \le y_{k,n}^{\lambda}$. Therefore,

$$\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\mu_{k,n}^{\lambda_k}\right] = \frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}(x_{k,n}^{\lambda_k} + y_{k,n}^{\lambda_k})\right] \le \frac{1}{\phi_r^\alpha}\sum_{k\in J_r} y_{k,n} + \frac{1}{\phi_r^\alpha}\sum_{k\in J_r} y_{k,n}^\lambda.$$

Since $\lambda < 1$ such that $\frac{1}{\lambda} > 1$, for each n and therefore by Holder's inequality, we obtain

$$\begin{aligned}
\frac{1}{\phi_r^\alpha}\sum_{k\in J_r} y_{k,n}^\lambda &= \sum_{k\in J_r}\left(\frac{1}{\phi_r^\alpha} z_{k,n}\right)^\lambda \left(\frac{1}{\phi_r^\alpha}\right)^{1-\lambda} \\
&\le \left(\sum_{k\in J_r}\left[\left(\frac{1}{\phi_r^\alpha} y_{k,n}\right)^\lambda\right]^{\frac{1}{\lambda}}\right)^\lambda \left(\sum_{k\in J_r}\left[\left(\frac{1}{\phi_r^\alpha}\right)^{1-\lambda}\right]^{\frac{1}{(1-\lambda)}}\right)^{1-\lambda} \\
&= \left(\frac{1}{\phi_r^\alpha}\sum_{k\in J_r} y_{k,n}\right)^\lambda.
\end{aligned}$$

Thus, we have

$$\frac{1}{\phi_r^\alpha}\sum_{k\in J_r}\mu_{k,n}^{\lambda_k} \le \frac{1}{\phi_r^\alpha}\sum_{k\in J_r}\mu_{k,n} + \left[\frac{1}{\phi_r^\alpha}\sum_{k\in J_r} y_{k,n}\right]^\lambda.$$

Thus, $\zeta \in w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^\infty$. Hence,

$$w_\alpha^\beta[\mathcal{M}, v, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]\infty_\sigma \subset w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^\infty.$$

Similarly, we prove

$$w_\alpha^\beta[\mathcal{M}, v, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0 \subset w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma$$

and

$$w_\alpha^\beta[\mathcal{M}, v, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0 \subset w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, \|\cdot,\ldots,\cdot\|]_\sigma^0. \qquad \square$$

3 Conclusion

Baliarsingh [21] defined the fractional difference operator and Cai et al. [36] and Sharma et al. [34] studied some sequence spaces of order (α, β). We have constructed here some new lacunary sequence spaces of fractional difference operator defined by Musielak-Orlicz function as $w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot, \ldots, \cdot||]_\sigma^0$, $w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot, \ldots, \cdot||]_\sigma$, and $w_\alpha^\beta[\mathcal{M}, u, \Delta^{(\gamma)}, \theta, ||\cdot, \ldots, \cdot||]_\sigma^\infty$ and studied some topological properties. We also established some inclusion relation between these sequence spaces. Readers may find these spaces interesting of further scope to study their geometric properties, matrix transformations, and corresponding compact matrix operators [37].

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