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Research Article

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Global in time well-posedness of a three-dimensional periodic regularized Boussinesq system

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Abstract: Global in time weak solution to a regularized periodic three-dimensional Boussinesq system is proved to exist in energy spaces. This solution depends continuously on the initial data. In particular, it is unique. The main novelty is the global in time aspect of this solution. The proofs use the coupling between the temperature and the velocity of the fluid, energy methods, and compactness argument.

Keywords: three-dimensional Boussinesq system, global in time weak solution, existence, regularization, uniqueness

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1 Introduction

In this article, we study the periodic regularized Boussinesq system (Bq_α):

$$\partial_t u - \nu\Delta u + (u\cdot\nabla)u - \nabla\cdot(\mathcal{H}^{-1}(\alpha^2\nabla u\cdot\nabla u^T)) = -\nabla p + \theta e_3, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3, \tag{1.1}$$

$$\partial_t\theta - \kappa\Delta\theta + (u\cdot\nabla)\theta = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3, \tag{1.2}$$

$$\operatorname{div}\, u = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3, \tag{1.3}$$

$$(u,\theta)|_{t=0} = (u^0,\theta^0), \quad x\in\mathbb{T}^3, \tag{1.4}$$

where the unknown velocity, the unknown pressure, and the unknown temperature are, respectively, the three-dimensional vector u, the scalars p and θ. The parameters $\nu, \kappa > 0$ denote, respectively, the viscosity and the thermal conductivity of the fluid, $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ is the three-dimensional torus, u^0 is a given free velocity, and θ^0 is a given mean free initial temperature. The vector $e_3 = (0,0,1)^T$. The filtering operator $\mathcal{H}^{-1}$ is the inverse of the Helmholtz operator $\mathcal{H}u = u - \alpha^2\Delta u$.

Starting with a mean free initial temperature, such as sinusoidal initial heating sources, is frequent in natural phenomena and compulsory in many real-word applications (see [1] and multitude references therein for example in the case of industrial applications, or [2] for applications in medicine and health sciences). As mentioned in [3], “the Boussinesq equations model geophysical fluids such as atmospheric fronts and oceanographic turbulence as well as the Rayleigh-Benard convection (see [4,5] for details).” If $\theta = 0$, system (1.1)–(1.4) reduces to the so-called “filtered Clark model for Navier-Stokes equations” considered in [6], where interested readers can find sufficient material about derivation, motivations, and relevance of this regularizing model.

Here, we investigate weak solution to (Bq_α) in homogeneous Sobolev spaces. We obtain *a global in time weak solution that depends continuously on the initial data and in particular it is unique*. To establish this global

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in time well-posedness, we use the coupling between the temperature and the velocity in order to close the energy estimate by a right-hand side, which is a constant that does not depend on time. At the best of our knowledge, this is new. In fact, in existing works in Sobolev spaces (which are energy space and then physically meaningful), the authors obtain always a right-hand side that depends on time while estimating the buoyancy force (see, for example, [3,7] in the 3D case, and [8,9] even in the 2D case, which is supposed to be "well-understood"). Such time dependence makes the solution to be "a large time solution" and not "a global in time solution" because it is defined on $[0, T]$, $T > 0$ and the corresponding energy estimate blows up when the existence time becomes ∞. However, we mention that global in time solutions were established in Besov spaces (which are not energy spaces) for a three-dimensional regularized Boussinesq system with damping in [10], in Besov spaces for a multidimensional damped Boussinesq system in [11] as the initial data are small, and in homogeneous Besov spaces for two-dimensional damped Boussinesq system in [12] under a minimal smallness assumption. Our method applies without any smallness assumption. Contrary to the three works cited earlier, our techniques stay working for the nonregularized Boussinesq system ($\alpha = 0$) while looking for a global in time solution. However, regularization ($\alpha \neq 0$) is compulsory for the continuous dependence and uniqueness . In fact, it is well known that uniqueness for Leray-type solutions to 3D Navier-Stokes and similar equations, such as Boussinesq system, is still an open problem. Mainly, we have the following theorem.

Theorem 1.1. *Let $\theta^0 \in L^2(\mathbb{T}^3)$ mean free and let $u^0 \in \dot{H}^1(\mathbb{T}^3)$ be a divergence-free vector field. There exists a unique weak solution (u, θ) of system (Bq_α), such that u belongs to $C(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^2(\mathbb{T}^3))$ and θ belongs to $C(\mathbb{R}_+, L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3))$. Moreover, we have*

$$\begin{aligned} &\|\theta(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + \alpha^2\|\nabla u(t)\|_{L^2}^2 + 2\kappa\int_0^t \|\nabla\theta(\tau)\|_{L^2(\mathbb{T}^3)}^2 \mathrm{d}\tau + \nu\int_0^t (\|\nabla u(\tau)\|_{L^2}^2 + \alpha^2\|\Delta u(\tau)\|_{L^2}^2)\mathrm{d}\tau \\ &\leq C(\alpha, \nu, \kappa, u^0, \theta^0), \end{aligned} \tag{1.5}$$

where

$$C(\alpha, \nu, \kappa, u^0, \theta^0) = \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + \left(1 + \frac{2 + \alpha^4}{\nu\kappa}\right)\|\theta^0\|_{L^2}^2.$$

Furthermore, this solution is continuously dependent on the initial data, $\forall t \in \mathbb{R}$. In particular, it is unique.

Global in time solutions of partial differential equations (PDEs) are known to be of considerable interest from an applicable point of view. They open the way to study all tasks requiring $t \to \infty$ such as the long time behavior, the existence of the attractors (actually under consideration for system (1.1)–(1.4)), the asymptotic stability, among others. For physicists and engineers, global in time solutions are closely related to durable (in time) operating of machines, systems, and networks. That is why in application, we usually try to start with suitable initial data in order to avoid blowup in finite time and ensure global in time functioning. In numerical analysis of nonlinear systems, although the numerical discretization has a local in time aspect, the existence of a unique global in time solution gives the possibility to extend such discretization by translation in time. Clearly, this opens the way to consider tasks related to large but finite time $T < \infty$, while dealing with numerical solutions. In this framework, let us say that extra conditions on initial and boundary conditions are used in the literature to obtain suitable solutions; for example, in [13,14], a range of initial and boundary conditions allow us to obtain a global in time weak solution and to establish decay estimates. Also, in [15,16], the authors used the mean free condition to study the long time behavior and the exponential stability of solutions to the periodic three-dimensional Navier-Stokes equations in critical Sobolev spaces.

As for continuous dependence on initial data (and in particular uniqueness) of PDEs' solution, from the applied mathematical point of view, researchers usually seek for a nearby solution to arise from a nearby initial data. Moreover, we aim not to completely miss solutions either, as our numerical simulation runs on, during time. Particularly, for nonlinear PDEs, where the superposition principle breaks down. For physicists and engineers, one of many features of uniqueness of solution is that when starting with an initial state, the system should evolve – according to the considered PDE – to lead to an only one future state at a later time, especially whenever solution has an infinite lifespan. Furthermore, uniqueness theorems are powerful tools to deduce equalities and identities as the unique solutions can be expressed differently, when computed analytically with different methods in regular geometries, for example.

2 Proof of the main result

2.1 Existence of the global in time weak solution

To prove existence results, it is classical to approximate the regularized Boussinesq system (Bq_α) using any approximating scheme. In the following, we give formal estimates for this approximation. For simplicity of notation, we omit the subscript of the approximating sequences. First, we recall that tensor $(\nabla u \cdot \nabla u^T)$ is given by

$$(\nabla u \cdot \nabla u^T)_{ij} = \nabla u_i \cdot \nabla u_j = \sum_{k=1}^{3} (\partial_{x_k} u_i)(\partial_{x_k} u_j).$$

Using Einstein's summation convention, we have

$$(\partial_{x_k} u_i)(\partial_{x_k} u_j) = \sum_{k=1}^{3} (\partial_{x_k} u_i)(\partial_{x_k} u_j).$$

Applying the Leray projector $\mathbb{P}(u) = u - \nabla(\Delta^{-1}(\nabla \cdot u))$, system (1.1)–(1.4) becomes

$$\partial_t u - \nu\Delta u + \mathbb{P}((u \cdot \nabla)u) + \alpha^2 \mathcal{H}^{-1}\mathbb{P}(((\partial_{x_j} u)\cdot\nabla)\partial_{x_j} u) = \theta e_3, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{T}^3, \tag{2.1}$$

$$\partial_t \theta - \kappa\Delta\theta + (u \cdot \nabla)\theta = 0, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{T}^3, \tag{2.2}$$

$$(u, \theta)|_{t=0} = (u^0, \theta^0), \quad x \in \mathbb{T}^3. \tag{2.3}$$

Taking the inner product in $L^2(\mathbb{T}^3)$ of (2.2) with θ, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^2}^2 + \kappa\|\nabla\theta\|_{L^2}^2 = 0. \tag{2.4}$$

Taking the $\dot{H}^{-1}(\mathbb{T}^3)$ duality action on (2.1) with $v = \mathcal{H}u$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2 + \alpha^2\|\nabla u\|_{L^2}^2) + \nu(\|\nabla u\|_{L^2}^2 + \alpha^2\|\Delta u\|_{L^2}^2) + \langle \mathbb{P}((u \cdot \nabla)u), v\rangle_{\dot{H}^{-1}} + \alpha^2\langle \mathbb{P}((\partial_{x_j} \cdot \nabla)\partial_{x_j}), u\rangle_{\dot{H}^{-1}} = \langle \theta e_3, v\rangle_{\dot{H}^{-1}}.$$

Using the identity $\langle \mathbb{P}((u \cdot \nabla)u), v\rangle_{\dot{H}^{-1}} = -\langle \mathbb{P}((u \cdot \nabla)v), u\rangle_{\dot{H}^{-1}}$ and integrating by part, it holds that

$$\langle \mathbb{P}((u \cdot \nabla)u), v\rangle_{\dot{H}^{-1}} + \alpha^2\langle \mathbb{P}((\partial_{x_j} \cdot \nabla)\partial_{x_j}), u\rangle_{\dot{H}^{-1}} = 0.$$

It follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2 + \alpha^2\|\nabla u\|_{L^2}^2) + \nu(\|\nabla u\|_{L^2}^2 + \alpha^2\|\Delta u\|_{L^2}^2) = \langle \theta e_3, v\rangle_{\dot{H}^{-1}}.$$

Integrating (2.2) with respect to x, we deduce that the first Fourier coefficient of θ is conserved during time, i.e., $C_0(\theta(t)) = C_0(\theta^0) = 0$, $\forall t > 0$, as θ^0 is mean free. So,

$$\langle \theta e_3, v\rangle_{\dot{H}^{-1}} = \sum_{k \neq (0,0,0)} \hat{\theta}_n(k)\ \hat{v}_n^3(k). \tag{2.5}$$

Now, as $\theta(t)$ is mean free, $\forall t > 0$, using Cauchy-Schwarz inequality and Young's product inequality, we obtain

$$\langle \theta e_3, v\rangle_{\dot{H}^{-1}} \leq 4/\nu\|\theta\|_{L^2}^2 + 4\alpha^4/\nu\|\nabla\theta\|_{L^2}^2 + \nu/2\|\nabla u\|_{L^2}^2.$$

Thus,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||u||_{L^2}^2 + \alpha^2||\nabla u||_{L^2}^2) + \frac{\nu}{2}(||\nabla u||_{L^2}^2 + \nu\alpha^2||\Delta u||_{L^2}^2) \le \frac{4}{\nu}(||\theta||_{L^2}^2 + \alpha^4||\nabla\theta||_{L^2}^2). \tag{2.6}$$

Summing up (2.4) and (2.6) and integrating with respect to time, we obtain

$$\begin{aligned}&||\theta(t)||_{L^2}^2 + ||u(t)||_{L^2}^2 + \alpha^2||\nabla u(t)||_{L^2}^2 + 2\kappa\int_0^t||\nabla\theta(\tau)||_{L^2}^2\mathrm{d}\tau + \nu\int_0^t(||\nabla u(\tau)||_{L^2}^2 + 2\alpha^2||\Delta u(\tau)||_{L^2}^2)\mathrm{d}\tau\\ &\le \; ||\theta^0||_{L^2}^2 + ||u^0||_{L^2}^2 + \alpha^2||\nabla u^0||_{L^2}^2 + \frac{4}{\nu}\int_0^t||\theta(\tau)||_{L^2}^2\mathrm{d}\tau + \frac{4\alpha^4}{\nu}\int_0^t||\nabla\theta(\tau)||_{L^2}^2\mathrm{d}\tau.\end{aligned}$$

Using, respectively, Poincaré inequality and the integral with respect to time of (2.4), we obtain

$$\int_0^t||\theta||_{L^2}^2\mathrm{d}\tau \le \int_0^t||\nabla\theta||_{L^2}^2\mathrm{d}\tau \le \frac{1}{2\kappa}||\theta^0||_{L^2}^2.$$

Thus, we reach our target in closing the estimate by a constant that is independent of time:

$$\begin{aligned}&||\theta(t)||_{L^2}^2 + ||u(t)||_{L^2}^2 + \alpha^2||\nabla u(t)||_{L^2}^2 + 2\kappa\int_0^t||\nabla\theta(\tau)||_{L^2}^2\mathrm{d}\tau + \nu\int_0^t(||\nabla u(\tau)||_{L^2}^2 + \alpha^2||\Delta u(\tau)||_{L^2}^2)\mathrm{d}\tau\\ &\le \; C(\alpha,\nu,\kappa,u^0,\theta^0),\end{aligned} \tag{2.7}$$

where

$$C(\alpha,\nu,\kappa,u^0,\theta^0) = ||u^0||_{L^2}^2 + \alpha^2||\nabla u^0||_{L^2}^2 + \left(1 + \frac{2+\alpha^4}{\nu\kappa}\right)||\theta^0||_{L^2}^2.$$

A standard compactness argument finishes the proof. Such compactness method is a universal method used in nonlinear PDEs to prove existence results. It is based on Aubin-Lion lemma [17,18]. Among other references dealing with fluid mechanic equations, one can see [19] for the application of compactness method in the case of Navier-Stokes equations, [20–23] for its application in the case of Burger equation, [24–27] for its application in the case of magneto-hydrodynamical systems, [28–32] in the case of Boussinesq system, and [33] in the case of wave equation. Also in the case of regularized stochastic Navier-Stokes equations [34], where the compactness method is used conjointly with probabilistic tools. Continuity in time can be proved in a classical manner following details in [35] for 3D Boussinesq system, for example.

2.2 Continuous dependence of solutions with respect to initial data and uniqueness

Consider two solutions (u_1, θ_1) and (u_2, θ_2) of (Bq_α) corresponding to (u_1^0, θ_1^0) and (u_2^0, θ_2^0). Let $\delta u = u_1 - u_2$ and $\delta\theta = \theta_1 - \theta_2$. Then,

$$\begin{aligned}&\partial_t\delta u - \nu\Delta\delta u + \mathbb{P}((\delta u\cdot\nabla)u_1) + \mathbb{P}((\delta u_2\cdot\nabla)\delta u) + \alpha^2\mathcal{H}^{-1}\mathbb{P}(((\partial_{x_j}\delta u)\cdot\nabla)\partial_{x_j}u_1) + \alpha^2\mathcal{H}^{-1}\mathbb{P}(((\partial_{x_j}\delta u_2)\cdot\nabla)\partial_{x_j}\delta u)\\ &\quad = \delta\theta e_3\end{aligned} \tag{2.8}$$

$$\partial_t\delta\theta - \Delta\delta\theta + (\delta u\cdot\nabla)\theta_1 + (u_2\cdot\nabla)\delta\theta = 0 \tag{2.9}$$

$$\mathrm{div}\;\delta u = 0 \tag{2.10}$$

$$(\delta u, \delta\theta)_{t=0} = (u_1^0 - u_2^0, \theta_1^0 - \theta_2^0). \tag{2.11}$$

Using (2.7), we deduce that $\frac{\mathrm{d}}{\mathrm{d}t}\delta\theta$ and $\frac{\mathrm{d}}{\mathrm{d}t}\delta u$ belong to $L^2(\mathbb{R}_+, H^{-1})$. Also, $\delta\theta$ and δu belong to $L^2(\mathbb{R}_+, H^1)$. The duality actions on (2.8) with $\mathcal{H}\delta u$, and on (2.9) with $\delta\theta$ give

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\delta\theta, \delta\theta \right\rangle_{\dot{H}^{-1}} + \kappa\|\nabla\delta\theta\|_{L^2}^2 + \langle \delta u \cdot \nabla\theta_1, \delta\theta\rangle_{\dot{H}^{-1}} = 0$$

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\delta u, \mathcal{H}\delta u \right\rangle_{\dot{H}^{-1}} + \nu(\|\nabla\delta u\|_{L^2}^2 + \alpha^2\|\Delta\delta u\|_{L^2}^2) + \langle \mathbb{P}((\delta u \cdot\nabla)u_1) + \mathbb{P}((u_2 \cdot\nabla)\delta u, \mathcal{H}\delta u)\rangle_{\dot{H}^{-1}}$$
$$+ \alpha^2\langle \mathcal{H}^{-1}\mathbb{P}(((\partial_{x_j}\delta u)\cdot\nabla)\partial_{x_j}u_1) + \mathcal{H}^{-1}\mathbb{P}(((\partial_{x_j}u_2)\cdot\nabla)\partial_{x_j}\delta u), \mathcal{H}\delta u\rangle_{\dot{H}^{-1}} = \langle\delta\theta, \mathcal{H}\delta u\rangle_{\dot{H}^{-1}},$$

for almost every time t in $\mathbb{R}$. Previously, we used that $\langle \mathbb{P}((u_2 \cdot\nabla)\delta u, \delta u)\rangle_{\dot{H}^{-1}} = 0$. Applying the Lions-Magenes lemma concerning the derivatives of a function with value in Banach spaces (see, e.g., [19], Chap. 3, Lemma 1.2) and summing up, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\delta u\|_{L^2}^2 + \alpha^2\|\nabla\delta u\|_{L^2}^2 + \|\delta\theta\|_{L^2}^2) + \nu(\|\nabla\delta u\|_{L^2}^2 + \alpha^2\|\Delta\delta u\|_{L^2}^2) + \kappa\|\nabla\delta\theta\|_{L^2}^2 = \sum_{l=0}^{5} I_l, \tag{2.12}$$

where

$$\begin{aligned}
I_0 &= \langle\delta\theta, \mathcal{H}\delta u\rangle_{\dot{H}^{-1}},\\
I_1 &= -\langle\delta u \cdot\nabla\theta_1, \delta\theta\rangle_{\dot{H}^{-1}},\\
I_2 &= -\langle\mathbb{P}((\delta u \cdot\nabla)u_1, \mathcal{H}\delta u)\rangle_{\dot{H}^{-1}},\\
I_3 &= -\langle\mathbb{P}((u_2 \cdot\nabla)\delta u, \mathcal{H}\delta u)\rangle_{\dot{H}^{-1}},\\
I_4 &= -\langle\alpha^2\mathcal{H}^{-1}\mathbb{P}(((\partial_{x_j}\delta u)\cdot\nabla)\partial_{x_j}u_1), \mathcal{H}\delta u\rangle_{\dot{H}^{-1}},\\
I_5 &= -\langle\alpha^2\mathcal{H}^{-1}\mathbb{P}(((\partial_{x_j}u_2)\cdot\nabla)\partial_{x_j}\delta u), \mathcal{H}\delta u\rangle_{\dot{H}^{-1}}.
\end{aligned}$$

To deal with the buoyancy effects, we use, respectively, the Cauchy-Schwarz inequality and Young's inequality for products to obtain

$$|I_0| \le \|\delta\theta\|_{L^2}^2 + \|\delta u\|_{L^2}^2 + \kappa/2\|\nabla\delta u\|_{L^2}^2 + C\|\nabla\delta u\|_{L^2}^2. \tag{2.13}$$

Remark 2.1. In a first step, we will deal with a local in time existence, and there is no need to apply our method in the subsection of existence result. At the end of this subsection, we will make clear how we will deduce the global in time dependence on initial data and uniqueness from the local one.

To investigate I_1, we use the Cauchy Schwartz inequality twice to obtain

$$|I_1| \le \|\delta u\|_{L^3}\|\nabla\theta_1\|_{L^2}\|\delta\theta\|_{L^6}.$$

Let us recall the following Sobolev inequalities: for all $\vartheta \in \dot{H}^1(\mathbb{T}^3)$, it holds that

$$\|\vartheta\|_{L^3(\mathbb{T}^3)} \le c\|\vartheta\|_{L^2(\mathbb{T}^3)}^{1/2}\|\vartheta\|_{\dot{H}^1(\mathbb{T}^3)}^{1/2} \quad\text{and}\quad \|\vartheta\|_{L^6(\mathbb{T}^3)} \le \|\vartheta\|_{\dot{H}^1(\mathbb{T}^3)}.$$

Then, it follows that

$$|I_1| \le c\|\delta u\|_{L^2}^{1/2}\|\delta u\|_{\dot{H}^1}^{1/2}\|\nabla\theta_1\|_{L^2}\|\delta\theta\|_{\dot{H}^1} \le c\|\delta u\|_{L^2}^{1/2}\|\nabla\delta u\|_{L^2}^{1/2}\|\nabla\theta_1\|_{L^2(\mathbb{T}^3)}\|\nabla\delta\theta\|_{L^2}.$$

Using twice Young product inequality, one obtains

$$|I_1| \le c(\|\delta u\|_{L^2}^2 + \|\nabla\delta u\|_{L^2}^2)\|\nabla\theta_1\|_{L^2}^2 + \frac{\kappa}{2}\|\nabla\delta\theta\|_{L^2}^2. \tag{2.14}$$

As for $\theta = 0$, system (1.1)–(1.4) reduces to the one considered in [6], we recall estimates established there for I_j, $2 \le j \le 5$:

$$|I_2| \le C\|\nabla u_1\|_{L^2}^4\|\nabla\delta u\|_{L^2}^2 + c\|\mathcal{H}\delta u\|_{L^2}^{4/3}\|\Delta\delta u\|_{L^2}^{2/3}, \tag{2.15}$$

$$|I_3| \leq C\|\Delta u_2\|_{L^2}^2\|\nabla\delta u\|_{L^2}^2 + c\|\mathcal{H}\delta u\|_{L^2}^2, \tag{2.16}$$

$$|I_4| \leq C\alpha^2\|\nabla u_1\|_{L^2}^4\|\nabla\delta u\|_{L^2}^2 + c\|\Delta\delta u\|_{L^2}^2, \tag{2.17}$$

$$|I_5| \leq C\alpha^2\|\nabla u_2\|_{L^2}^4\|\nabla\delta u\|_{L^2}^2 + c\|\Delta\delta u\|_{L^2}^2. \tag{2.18}$$

In the periodic case, the Fourier multipliers of the operator $\mathcal{H}$ and the Laplace operator Δ are, respectively, $1 + \alpha^2\,|k|^2$ and $|k|^2$, summing up (2.13) to (2.18), estimate (2.12) implies that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\delta u\|_{L^2}^2 + \alpha^2\|\nabla\delta u\|_{L^2}^2 + \|\delta\theta\|_{L^2}^2)\\ &\leq C(\|\Delta u_2\|_{L^2}^2 + \|\nabla u_1\|_{L^2}^4 + \|\nabla u_2\|_{L^2}^4 + \|\nabla\theta_1\|_{L^2}^2 + 1)(\|\delta u\|_{L^2}^2 + \alpha^2\|\nabla\delta u\|_{L^2}^2 + \|\delta\theta\|_{L^2}^2).\end{aligned} \tag{2.19}$$

Previously, for the fluid viscous term, the Laplace part cancels out and the gradient part is dropped from the left-hand side. Also, for the temperature, the viscous term that is only a gradient contribution cancels out. Thanks to the Grönwall inequality, we obtain

$$\begin{aligned}&\|\delta u(t)\|_{L^2}^2 + \alpha^2\|\nabla\delta u(t)\|_{L^2}^2 + \|\delta\theta(t)\|_{L^2}^2\\ &\leq \exp\left(C\int_0^t g(\tau)\mathrm{d}\tau\right)(\|\delta u\|_{L^2}^2(0) + \alpha^2\|\nabla\delta u\|_{L^2}^2(0) + \|\delta\theta\|_{L^2}^2(0)),\end{aligned}$$

where

$$g(\tau) = 1 + \|\Delta u_2(\tau)\|_{L^2}^2 + \|\nabla u_1(\tau)\|_{L^2}^4 + \|\nabla u_2(\tau)\|_{L^2}^4 + \|\nabla\theta_1(\tau)\|_{L^2}^2.$$

By estimate (2.7), the inequality $\|\vartheta\|_{L^4} \leq C\|\vartheta\|_{L^2}^{1/4}\|\vartheta\|_{\dot{H}^1}^{3/4}$, for $\vartheta \in \dot{H}^1$, we deduce that g is integrable on any interval of time $[0, T],\ T > 0$. Thus, the solution depends continuously on the initial data on $[0, T],\ T > 0$, and in particular, it is unique. As the global solution is *time continuous on* $\mathbb{R}_+$, continuous dependence with respect to the initial data and uniqueness persist for all time $t \in \mathbb{R}_+$.

3 Conclusion

We presented a new method to build a global in time solution to a three-dimensional model of the Boussinesq system. This method uses the coupling between the velocity equation and the temperature equation. It is elaborated using classical analysis without any tedious calculation. To do so, we considered a mean free initial temperature. We explained that mean free initial temperature is a common hypothesis in natural phenomena, and sometimes, a compulsory hypothesis in real-word applications related to industry and medicine for example. Our method still applies in the case of nonregularized model. Regularization is used only to deal with continuous dependence on the initial data and uniqueness. Also, we pointed out why global in time solutions, continuous dependence, and uniqueness aspects are so interesting in industry, physics, and even in theoretical mathematics. Moreover, we made the difference between global in time solutions and large time solutions that are confused in many references.

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