Research Article

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On Fejér-type inequalities for generalized trigonometrically and hyperbolic *k*-convex functions

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Abstract: For $\mu \in C^1(I)$, $\mu > 0$, and $\lambda \in C(I)$, where I is an open interval of $\mathbb{R}$, we consider the set of functions $f \in C^2(I)$ satisfying the second-order differential inequality $\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\frac{\mathrm{d}f}{\mathrm{d}t}\right) + \lambda f \geq 0$ in I. The considered set includes several classes of generalized convex functions from the literature. In particular, if $\mu \equiv 1$ and $\lambda = k^2, k > 0$, we obtain the class of trigonometrically k-convex functions, while if $\mu \equiv 1$ and $\lambda = -k^2, k > 0$, we obtain the class of hyperbolic k-convex functions. In this article, we establish a Fejér-type inequality for the introduced set of functions without any symmetry condition imposed on the weight function and discuss some special cases of weight functions. Moreover, we provide characterizations of the classes of trigonometrically and hyperbolic k-convex functions.

Keywords: generalized convexity, differential inequalities, trigonometrically k-convex functions, hyperbolic k-convex functions, Fejér inequality, characterization

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1 Introduction

Let $f : I \to \mathbb{R}$ be a convex function, where $I \subset \mathbb{R}$ is an open interval. A result of Hadamard [1] and Hermite [2] states that for all $a, b \in I$ with $a < b$, we have

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

and

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t. \tag{1.2}$$

Moreover, each of the inequalities (1.1) and (1.2) provides a characterization of convex functions (see [3,4]). Fejér [5] established weighted versions of the above inequalities. His result can be stated as follows: Let $f : [a, b] \to \mathbb{R}$ be a convex function and p a nonnegative integrable function that is symmetric w.r.t. $\frac{a+b}{2}$. Then,

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$$\int_a^b f(t)p(t)\mathrm{d}t \le \frac{f(a)+f(b)}{2}\int_a^b p(t)\mathrm{d}t \tag{1.3}$$

and

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)\mathrm{d}t \le \int_a^b f(t)p(t)\mathrm{d}t.$$

The above inequalities have been generalized in various directions. A collection of several results related to Hermite-Hadamard-and Fejér-type inequalities can be found in [6,7]. We also refer to [8–13] (see also references therein).

Let $k > 0$ and $f : I \to \mathbb{R}$. We say that f is trigonometrically k-convex function (see e.g. [14]), if for all $a, b \in I$ with $0 < b - a < \frac{\pi}{k}$, we have

$$f(t) \le \frac{\sin[k(b-t)]}{\sin[k(b-a)]}f(a) + \frac{\sin[k(t-a)]}{\sin[k(b-a)]}f(b), \quad a \le t \le b.$$

In [15], among other results, the first author established the following Fejér-type inequalities involving trigonometrically k-convex functions.

Theorem 1.1. *Let $k > 0$ and $f : I \to \mathbb{R}$ be a trigonometrically k-convex function. Let $a, b \in I$ with $0 < b - a < \frac{\pi}{k}$. Assume that p is an integrable function on $[a, b]$, $p \ge 0$ and p is symmetric w.r.t. $\frac{a+b}{2}$. Then, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b \cos\left[k\left(t-\frac{a+b}{2}\right)\right]p(t)\mathrm{d}t &\le \int_a^b f(t)p(t)\mathrm{d}t, \\ \int_a^b f(t)p(t)\mathrm{d}t &\le \frac{f(a)+f(b)}{2}\sec\left[\frac{k(b-a)}{2}\right]\int_a^b \cos\left[k\left(t-\frac{a+b}{2}\right)\right]p(t)\mathrm{d}t. \end{aligned} \tag{1.4}$$

For other integral inequalities related to trigonometrically k-convex functions, see, e.g. [16–20] and references therein.

Let $k > 0$ and $f : I \to \mathbb{R}$. We say that f is hyperbolic k-convex function (see, e.g., [21]), if for all $a, b \in I$ with $a < b$, we have

$$f(t) \le \frac{\sinh[k(b-t)]}{\sinh[k(b-a)]}f(a) + \frac{\sinh[k(t-a)]}{\sinh[k(b-a)]}f(b), \quad a \le t \le b.$$

In [22], among other results, the first author established the following Fejér-type inequalities involving hyperbolic k-convex functions.

Theorem 1.2. *Let $k > 0$ and $f : I \to \mathbb{R}$ be a hyperbolic k-convex function. Let $a, b \in I$ with $a < b$. Assume that p is an integrable function on $[a, b]$, $p \ge 0$ and p is symmetric w.r.t. $\frac{a+b}{2}$. Then, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b \cosh\left[k\left(t-\frac{a+b}{2}\right)\right]p(t)\mathrm{d}t &\le \int_a^b f(t)p(t)\mathrm{d}t, \\ \int_a^b f(t)p(t)\mathrm{d}t &\le \frac{f(a)+f(b)}{2}\operatorname{sech}\left[\frac{k(b-a)}{2}\right]\int_a^b \cosh\left[k\left(t-\frac{a+b}{2}\right)\right]p(t)\mathrm{d}t. \end{aligned} \tag{1.5}$$

For other integral inequalities related to hyperbolic k-convex functions, see, e.g., [23–25] and references therein.

On the contrary, it was shown in [14] that $f \in C^2(I)$ is trigonometrically k-convex function ($k > 0$) if and only if f satisfies the differential inequality

$$f'' + k^2 f \geq 0 \quad \text{in } I. \tag{1.6}$$

In [21], it was proven that $f \in C^2(I)$ is hyperbolic k-convex function ($k > 0$) if and only if f satisfies the differential inequality

$$f'' - k^2 f \geq 0 \quad \text{in } I. \tag{1.7}$$

Motivated by the above properties, we introduce in this article a more general class of functions including the sets of trigonometrically and hyperbolic k-convex functions. Namely, for $\mu \in C^1(I)$, $\mu > 0$, and $\lambda \in C(I)$, we consider the class of functions $f \in C^2(I)$ satisfying the second-order differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\frac{\mathrm{d}f}{\mathrm{d}t}\right) + \lambda f \geq 0 \quad \text{in } I. \tag{1.8}$$

Observe that if $\mu \equiv 1$ and $\lambda \equiv 0$, then (1.8) reduces to $f'' \geq 0$ in I, which means that f is convex. If $\mu \equiv 1$ and $\lambda = k^2$, $k > 0$, then (1.8) reduces to (1.6), which means that f is trigonometrically k-convex function. If $\mu \equiv 1$ and $\lambda = -k^2$, $k > 0$, then (1.8) reduces to (1.7), which means that f is hyperbolic k-convex function. In this article, we first establish a Fejér-type inequality for the considered class of functions without assuming any symmetry condition on the weight function. We next discuss some special cases of weight functions. Next, we provide characterizations of the classes of trigonometrically and hyperbolic k-convex functions.

The organization of the article is as follows: In Section 2, we introduce the class of generalized convex functions in the above sense, under certain (natural) assumptions. In Section 3, a Fejér-type inequality involving a general weight function p is established, and some special cases of p are discussed. Finally, characterizations of the classes of trigonometrically and hyperbolic k-convex functions are provided in Section 4.

Throughout this article I denotes an open interval of $\mathbb{R}$.

2 Generalized convex functions

Let $\mu \in C^1(I)$, $\mu > 0$, and $\lambda \in C(I)$. We consider the class of functions

$$X_{\mu,\lambda}(I) = \left\{f \in C^2(I) : \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\frac{\mathrm{d}f}{\mathrm{d}t}\right) + \lambda f \geq 0\right\}.$$

The above set recovers several classes of generalized convex functions.

- If $\mu \equiv 1$ and $\lambda \equiv 0$, then

$$X_{1,0}(I) = \{f \in C^2(I) : f \text{ is convex}\}.$$

- If $\mu \equiv 1$ and $\lambda \equiv k^2$, $k > 0$, then (see [14])

$$\begin{aligned} X_{1,k^2}(I) &= \{f \in C^2(I) : f'' + k^2 f \geq 0\} \\ &= \{f \in C^2(I) : f \text{ is trigonometrically } k\text{-convex function}\}. \end{aligned}$$

- If $\mu \equiv 1$ and $\lambda \equiv -k^2$, $k > 0$, then (see [21])

$$\begin{aligned} X_{1,-k^2}(I) &= \{f \in C^2(I) : f'' - k^2 f \geq 0\} \\ &= \{f \in C^2(I) : f \text{ is hyperbolic } k\text{-convex function}\}. \end{aligned}$$

Throughout this article, it is assumed that

(H1) There exist $a, b \in I$ with $a < b$ such that the homogeneous differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu(t)\frac{\mathrm{d}y}{\mathrm{d}t}(t)\right) + \lambda(t)y(t) = 0, \quad a < t < b \tag{2.1}$$

admits two linearly independent solutions y_1 and y_2 satisfying

$$y_1(a) = y_2(b) = 0.$$

(H2) In the interval $[a, b]$, any nonzero solution to (2.1) may have at most one zero.

3 Fejér-type inequalities

In this section, we establish a Fejér-type inequality for the class of functions $X_{\mu,\lambda}(I)$. No symmetry condition is imposed on the weight function, we only assume that the weight function is continuous and nonnegative. Just before, we need the following lemmas that will play a crucial role in our proofs.

3.1 Preliminaries

Lemma 3.1. *Let $\mu \in C^1(I)$, $\mu > 0$, and $\lambda \in C(I)$. Assume that* (H1) *and* (H2) *hold. Let $G = G(t, s) : [a, b] \times [a, b] \to \mathbb{R}$ be Green's function of* (2.1) *under the Dirichlet boundary conditions $y(a) = y(b) = 0$. We have $G \leq 0$.*

For the proof of the above lemma, see, e.g. [26].

Lemma 3.2. *Under the assumptions of Lemma* 3.1, *Green's function G is given by*

$$G(t, s) = \begin{cases} \dfrac{y_1(s)y_2(t)}{\mu(s)W(s)} & \text{if } a \leq s \leq t \leq b, \\ \dfrac{y_1(t)y_2(s)}{\mu(s)W(s)} & \text{if } a \leq t \leq s \leq b, \end{cases}$$

where W is the Wronskian of the solutions to the homogeneous problem (2.1), *i.e.,*

$$W(s) = y_1(s)y_2'(s) - y_1'(s)y_2(s).$$

The proof of the above result can be found in [27].

Lemma 3.3. *Let $p \in C(I)$ be a nonnegative function. Under the assumptions of Lemma* 3.1, *the Dirichlet boundary value problem*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\mu(t)\dfrac{\mathrm{d}g}{\mathrm{d}t}(t)\right) + \lambda(t)g(t) = -p(t), \quad a < t < b, \\ g(a) = g(b) = 0 \end{cases} \tag{3.1}$$

admits a unique solution given by

$$g(t) = -\int_a^b G(t, s)p(s)\mathrm{d}s, \quad a \leq t \leq b.$$

Moreover, the function g satisfies the following properties:

(*i*) $g \geq 0$;

(*ii*) $g'(a) = -y_1'(a)\int_a^b \frac{y_2(s)p(s)}{\mu(s)W(s)}\mathrm{d}s$;

(*iii*) $g'(b) = -y_2'(b)\int_a^b \frac{y_1(s)p(s)}{\mu(s)W(s)}\mathrm{d}s$.

The results mentioned in the above lemma follow immediately from Lemmas 3.1 and 3.2.

3.2 A Fejér-type inequality for the class of functions $X_{\mu,\lambda}(I)$

Our main result in this section is stated below.

Theorem 3.1. *Let $\mu \in C^1(I)$, $\mu > 0$, $\lambda \in C(I)$ and assume that* (H1) *and* (H2) *hold. Let $f \in X_{\mu,\lambda}(I)$ and $p \in C(I)$, $p \geq 0$. Then, it holds that*

$$\int_a^b f(t)p(t)\mathrm{d}t \leq f(a)\left[-\mu(a)y_1'(a)\int_a^b \frac{y_2(s)p(s)}{\mu(s)W(s)}\mathrm{d}s\right] + f(b)\left[\mu(b)y_2'(b)\int_a^b \frac{y_1(s)p(s)}{\mu(s)W(s)}\mathrm{d}s\right]. \tag{3.2}$$

Proof. From (3.1), we have

$$\int_a^b f(t)p(t)\mathrm{d}t = -\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu(t)\frac{\mathrm{d}g}{\mathrm{d}t}(t)\right)f(t)\mathrm{d}t - \int_a^b \lambda(t)g(t)f(t)\mathrm{d}t. \tag{3.3}$$

Integrating by parts, we obtain

$$-\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu(t)\frac{\mathrm{d}g}{\mathrm{d}t}(t)\right)f(t)\mathrm{d}t = -\left[\mu(t)\frac{\mathrm{d}g}{\mathrm{d}t}(t)f(t)\right]_{t=a}^b + \int_a^b \frac{\mathrm{d}g}{\mathrm{d}t}(t)\left[\mu(t)\frac{\mathrm{d}f}{\mathrm{d}t}(t)\right]\mathrm{d}t. \tag{3.4}$$

Integrating again by parts, we obtain

$$\int_a^b \frac{\mathrm{d}g}{\mathrm{d}t}(t)\left[\mu(t)\frac{\mathrm{d}f}{\mathrm{d}t}(t)\right]\mathrm{d}t = \left[g(t)\mu(t)\frac{\mathrm{d}f}{\mathrm{d}t}(t)\right]_{t=a}^b - \int_a^b g(t)\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu(t)\frac{\mathrm{d}f}{\mathrm{d}t}(t)\right]\mathrm{d}t.$$

Since $g(a) = g(b) = 0$, it holds that

$$\int_a^b \frac{\mathrm{d}g}{\mathrm{d}t}(t)\left[\mu(t)\frac{\mathrm{d}f}{\mathrm{d}t}(t)\right]\mathrm{d}t = -\int_a^b g(t)\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu(t)\frac{\mathrm{d}f}{\mathrm{d}t}(t)\right]\mathrm{d}t. \tag{3.5}$$

Then, combining (3.4) with (3.5), we obtain

$$-\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu(t)\frac{\mathrm{d}g}{\mathrm{d}t}(t)\right)f(t)\mathrm{d}t = -\left[\mu(t)\frac{\mathrm{d}g}{\mathrm{d}t}(t)f(t)\right]_{t=a}^b - \int_a^b g(t)\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu(t)\frac{\mathrm{d}f}{\mathrm{d}t}(t)\right]\mathrm{d}t. \tag{3.6}$$

Next, (3.3) and (3.6) yield

$$\int_a^b f(t)p(t)\mathrm{d}t = -\left[\mu(t)\frac{\mathrm{d}g}{\mathrm{d}t}(t)f(t)\right]_{t=a}^b - \int_a^b g(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu(t)\frac{\mathrm{d}f}{\mathrm{d}t}(t)\right] + \lambda(t)f(t)\right)\mathrm{d}t.$$

Since $g \geq 0$ by Lemma 3.3 (i) and $f \in X_{\mu,\lambda}(I)$, we have

$$\int_a^b g(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu(t)\frac{\mathrm{d}f}{\mathrm{d}t}(t)\right] + \lambda(t)f(t)\right)\mathrm{d}t \geq 0.$$

Consequently, we obtain

$$\int_a^b f(t)p(t)\mathrm{d}t \leq \mu(a)g'(a)f(a) - \mu(b)g'(b)f(b).$$

Finally, (3.2) follows from the above inequality and Lemma 3.3 (ii) and (iii). □

Let us now discuss some special cases of Theorem 3.1.

3.3 The case: $\lambda \equiv 0$

In this case,

$$X_{\mu,0}(I) = \left\{ f \in C^2(I) : \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu \frac{\mathrm{d}f}{\mathrm{d}t}\right) \geq 0 \right\},$$

where $\mu \in C^1(I)$, $\mu > 0$. Let us check that assumptions (H1) and (H2) hold in this case. Let $a, b \in I$ with $a < b$ and consider problem (2.1) with $\lambda \equiv 0$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu(t)\frac{\mathrm{d}y}{\mathrm{d}t}(t)\right) = 0, \quad a < t < b. \tag{3.7}$$

Then, the functions

$$y_1(t) = \int_a^t \frac{1}{\mu(s)}\mathrm{d}s, \quad y_2(t) = \int_t^b \frac{1}{\mu(s)}\mathrm{d}s$$

are two linearly independent solutions to (3.7). Moreover, we have

$$y_1(a) = y_2(b) = 0.$$

This shows that (H1) holds for all $a, b \in I$ with $a < b$. On the contrary, if $t_0, t_1 \in [a, b]$ with $t_0 < t_1$, then the unique solution to the boundary-valued problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\mu(t)\dfrac{\mathrm{d}y}{\mathrm{d}t}(t)\right) = 0, \quad t_0 < t < t_1, \\ y(t_1) = y(t_2) = 0 \end{cases}$$

is the zero function. This shows that (H2) holds. Hence, from Theorem 3.1, if $f \in X_{\mu,0}(I)$ and $p \in C(I)$ is nonnegative, then (3.2) holds for all $a, b \in I$ with $a < b$. In our case, we have

$$y_1'(a) = \frac{1}{\mu(a)}, \quad y_2'(b) = -\frac{1}{\mu(b)}$$

and

$$\begin{aligned} W(t) &= y_1(t)y_2'(t) - y_1'(t)y_2(t) \\ &= -\frac{1}{\mu(t)}\int_a^t \frac{1}{\mu(s)}\mathrm{d}s - \frac{1}{\mu(t)}\int_t^b \frac{1}{\mu(s)}\mathrm{d}s \\ &= -\frac{1}{\mu(t)}\int_a^b \frac{1}{\mu(s)}\mathrm{d}s. \end{aligned}$$

Injecting the above terms in (3.2), we obtain the following result.

Corollary 3.1. *Let $\mu \in C^1(I)$, $\mu > 0$, $f \in X_{\mu,0}(I)$, and $p \in C(I)$, $p \geq 0$. Then, for all $a, b \in I$ with $a < b$, it holds that*

$$\int_a^b f(t)p(t)\mathrm{d}t \leq \left(\int_a^b \frac{1}{\mu(t)}\mathrm{d}t\right)^{-1} \int_a^b \left(f(a)\int_t^b \frac{1}{\mu(s)}\mathrm{d}s + f(b)\int_a^t \frac{1}{\mu(s)}\mathrm{d}s\right)p(t)\mathrm{d}t. \tag{3.8}$$

Let us consider the special case of Corollary 3.1 when $\mu \equiv 1$. In this case $f \in X_{1,0}(I)$ means that $f \in C^2(I)$ is convex. Taking $\mu \equiv 1$ in (3.8), we obtain the following Fejér inequality for twice continuously differentiable and convex functions.

Corollary 3.2. *Let $f \in C^2(I)$ be a convex function and $p \in C(I)$, $p \geq 0$. Then, for all $a, b \in I$ with $a < b$, it holds that*

$$\int_a^b f(t)p(t)\mathrm{d}t \leq \frac{1}{b-a}\int_a^b (f(a)(b-t) + f(b)(t-a))p(t)\mathrm{d}t. \tag{3.9}$$

Remark 3.1. Assume in addition that p is symmetric w.r.t. $\frac{a+b}{2}$, that is,

$$p(a+b-t) = p(t), \quad a \leq t \leq b.$$

In this case, one has

$$\int_a^b (b-t)p(t)\mathrm{d}t = \int_a^b (t-a)p(t)\mathrm{d}t,$$

which yields

$$\begin{aligned}\int_a^b (f(a)(b-t) + f(b)(t-a))p(t)\mathrm{d}t &= (f(a)+f(b))\int_a^b (b-t)p(t)\mathrm{d}t \\ &= \frac{f(a)+f(b)}{2}\int_a^b (b-t+t-a)p(t)\mathrm{d}t = (b-a)\frac{f(a)+f(b)}{2}\int_a^b p(t)\mathrm{d}t.\end{aligned}$$

Hence, (3.9) reduces to Fejér inequality (1.3).

3.4 The case: $\mu \equiv 1$ and $\lambda = k^2$, $k > 0$

In this case, $X_{1,k^2}(I)$ is the set of twice continuously differentiable and trigonometrically k-convex functions (see Section 2). Let us check that assumptions (H1) and (H2) hold. For $a, b \in I$ with $0 < b - a < \frac{\pi}{k}$, consider the homogeneous differential equation (2.1) with $\mu \equiv 1$ and $\lambda = k^2$, that is,

$$y''(t) + k^2 y(t) = 0, \quad a < t < b. \tag{3.10}$$

Then, the functions

$$y_1(t) = \sin[k(t-a)], \quad y_2(t) = \sin[k(b-t)]$$

are two linearly independent solutions to (3.10). Moreover, we have

$$y_1(a) = y_2(b) = 0.$$

This shows that (H1) is satisfied. On the contrary, the first eigenvalue of the Dirichlet boundary value problem

$$\begin{cases} y''(t) + \theta y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases}$$

where $\theta > 0$, is $\theta_1 = \frac{\pi^2}{(b-a)^2}$. So, if $0 < b - a < \frac{\pi}{k}$ ($k^2 < \theta_1$), the only solution to

$$\begin{cases} y''(t) + k^2 y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases}$$

is the function zero. This shows that (H2) is satisfied. Consequently, for all $a, b \in I$ with $0 < b - a < \frac{\pi}{k}$, (H1)–(H2) hold, and Theorem 3.1 applies. Moreover, we have

$$y_1'(a) = k, \quad y_2'(b) = -k$$

and

$$W(t) = -k\sin[k(b-a)].$$

Hence, injecting the above terms in (3.2), we obtain the following result.

Corollary 3.3. *Let $k>0$, $f \in X_{1,k^2}(I)$, and $p \in C(I)$, $p \geq 0$. Then, for all $a, b \in I$ with $0 < b-a < \frac{\pi}{k}$, it holds that*

$$\int_a^b f(t)p(t)\mathrm{d}t \leq \frac{f(a)}{\sin[k(b-a)]}\int_a^b \sin[k(b-s)]p(s)\mathrm{d}s + \frac{f(b)}{\sin[k(b-a)]}\int_a^b \sin[k(s-a)]p(s)\mathrm{d}s. \tag{3.11}$$

Remark 3.2. Assume in addition that p is symmetric w.r.t. $\frac{a+b}{2}$. In this case, we have

$$\begin{aligned}\int_a^b \sin[k(b-s)]p(s)\mathrm{d}s &= \int_a^b \sin[k(t-a)]p(a+b-t)\mathrm{d}t \\ &= \int_a^b \sin[k(t-a)]p(t)\mathrm{d}t,\end{aligned}$$

which yields

$$\begin{aligned}2\int_a^b \sin[k(b-s)]p(s)\mathrm{d}s &= \int_a^b (\sin[k(b-t)] + \sin[k(t-a)])p(t)\mathrm{d}t \\ &= 2\sin\left[\frac{k(b-a)}{2}\right]\int_a^b \cos\left[\frac{k(a+b-2t)}{2}\right]p(t)\mathrm{d}t,\end{aligned}$$

that is,

$$\begin{aligned}\int_a^b \sin[k(b-s)]p(s)\mathrm{d}s &= \int_a^b \sin[k(s-a)]p(s)\mathrm{d}s \\ &= \sin\left[\frac{k(b-a)}{2}\right]\int_a^b \cos\left[\frac{k(a+b-2s)}{2}\right]p(s)\mathrm{d}s.\end{aligned}$$

Hence, (3.11) reduces to (1.4).

3.5 The case: $\mu \equiv 1$ and $\lambda \equiv k^2$, $k > 0$

In this case, $X_{1,-k^2}(I)$ is the set of twice continuously differentiable and hyperbolic k-convex functions (see Section 2). Let us check that assumptions (H1) and (H2) hold. For $a, b \in I$ with $a < b$, consider the homogeneous differential equation (2.1) with $\mu \equiv 1$ and $\lambda = -k^2$, that is,

$$y''(t) - k^2 y(t) = 0, \quad a < t < b. \tag{3.12}$$

Then, the functions

$$y_1(t) = \sinh[k(t-a)], \quad y_2(t) = \sinh[k(b-t)]$$

are two linearly independent solutions to (3.12). Moreover, we have

$$y_1(a) = y_2(b) = 0.$$

This shows that (H1) is satisfied. On the contrary, the only solution to the Dirichlet boundary-value problem

$$\begin{cases} y''(t) - k^2 y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases}$$

is the function zero. This shows that (H2) is satisfied. Consequently, for all $a, b \in I$ with $a < b$, (H1)–(H2) hold, and Theorem 3.1 applies. Moreover, we have

$$y_1'(a) = k, \quad y_2'(b) = -k$$

and

$$W(t) = -k \sinh[k(b-a)].$$

Hence, injecting the above terms in (3.2), we obtain the following result.

Corollary 3.4. *Let $k > 0$, $f \in X_{1,-k^2}(I)$, and $p \in C(I)$, $p \geq 0$. Then, for all $a, b \in I$ with $a < b$, it holds that*

$$\int_a^b f(t)p(t)\mathrm{d}t \leq \frac{f(a)}{\sinh[k(b-a)]}\int_a^b \sinh[k(b-s)]p(s)\mathrm{d}s + \frac{f(b)}{\sinh[k(b-a)]}\int_a^b \sinh[k(s-a)]p(s)\mathrm{d}s. \tag{3.13}$$

Remark 3.3. Assume in addition that p is symmetric w.r.t. $\frac{a+b}{2}$. In this case, we have

$$\begin{aligned} \int_a^b \sinh[k(b-s)]w(s)\mathrm{d}s &= \int_a^b \sinh[k(t-a)]w(a+b-t)\mathrm{d}t \\ &= \int_a^b \sinh[k(t-a)]w(t)\mathrm{d}t, \end{aligned}$$

which yields

$$\begin{aligned} 2\int_a^b \sinh[k(b-s)]w(s)\mathrm{d}s &= \int_a^b (\sinh[k(b-t)] + \sinh[k(t-a)])w(t)\mathrm{d}t \\ &= 2\sinh\left[\frac{k(b-a)}{2}\right]\int_a^b \cosh\left[\frac{k(a+b-2t)}{2}\right]w(t)\mathrm{d}t, \end{aligned}$$

that is,

$$\begin{aligned} \int_a^b \sinh[k(b-s)]w(s)\mathrm{d}s &= \int_a^b \sinh[k(s-a)]w(s)\mathrm{d}s \\ &= \sinh\left[\frac{k(b-a)}{2}\right]\int_a^b \cosh\left[\frac{k(a+b-2s)}{2}\right]w(s)\mathrm{d}s. \end{aligned}$$

Hence, (3.13) reduces to (1.5).

4 Characterizations of $X_{1,\pm k^2}(I)$

In this section, we establish characterizations of the class of twice continuously differentiable and trigonometrically k-convex functions and the class of twice continuously differentiable and hyperbolic k-convex functions, namely the sets $X_{1,\pm k^2}(I)$, $k > 0$.

We first consider the set $X_{1,k^2}(I)$.

Theorem 4.1. *Let $f \in C^2(I)$ and $k > 0$. The following statements are equivalent:*

(i) $f \in X_{1,k^2}(I)$;

(ii) *For all $a, b \in I$ with $0 < b - a < \frac{\pi}{k}$, it holds that*

$$\int_a^b f(t)\mathrm{d}t \leq \tan\left[\frac{k(b-a)}{2}\right]\frac{f(a)+f(b)}{k}. \tag{4.1}$$

Proof. (i) $\Rightarrow$ (ii) follows immediately from (3.11) with $p \equiv 1$. Assume now that (ii) holds, and let $t \in I$ be fixed. Using (4.1) with $a = t - \varepsilon$ and $b = t + \varepsilon$, where $0 < \varepsilon < \frac{\pi}{2k}$, we obtain

$$\int_{t-\varepsilon}^{t+\varepsilon} f(s)\mathrm{d}s \leq \frac{1}{k}\tan(k\varepsilon)(f(t-\varepsilon)+f(t+\varepsilon)). \tag{4.2}$$

We now consider the function

$$g_\varepsilon(s) = \frac{1}{k^2}\left(\frac{\cos[k(t-s)]}{\cos(k\varepsilon)} - 1\right), \quad t-\varepsilon \leq s \leq t+\varepsilon.$$

Elementary calculations show that

$$\begin{cases} g_\varepsilon''(s) + k^2 g_\varepsilon(s) = -1, & t-\varepsilon < s < t+\varepsilon, \\ g_\varepsilon(t-\varepsilon) = g_\varepsilon(t+\varepsilon) = 0. \end{cases} \tag{4.3}$$

Moreover, we have $g_\varepsilon \geq 0$ and

$$g_\varepsilon'(t-\varepsilon) = \frac{1}{k}\tan(k\varepsilon), \quad g_\varepsilon'(t+\varepsilon) = -\frac{1}{k}\tan(k\varepsilon). \tag{4.4}$$

Making use of (4.3), (4.4), and integrating by parts, we obtain

$$\begin{aligned} \int_{t-\varepsilon}^{t+\varepsilon} f(s)\mathrm{d}s &= -\int_{t-\varepsilon}^{t+\varepsilon} f(s)g_\varepsilon''(s)\mathrm{d}s - k^2\int_{t-\varepsilon}^{t+\varepsilon} f(s)g_\varepsilon(s)\mathrm{d}s \\ &= -[f(s)g_\varepsilon'(s)]_{s=t-\varepsilon}^{t+\varepsilon} - \int_{t-\varepsilon}^{t+\varepsilon}(f''(s)+k^2 f(s))g_\varepsilon(s)\mathrm{d}s \\ &= \frac{1}{k}\tan(k\varepsilon)(f(t-\varepsilon)+f(t+\varepsilon)) - \int_{t-\varepsilon}^{t+\varepsilon}(f''(s)+k^2 f(s))g_\varepsilon(s)\mathrm{d}s, \end{aligned}$$

which implies by (4.2) that

$$\int_{t-\varepsilon}^{t+\varepsilon}(f''(s)+k^2 f(s))g_\varepsilon(s)\mathrm{d}s \geq 0.$$

Since $g_\varepsilon \geq 0$, we deduce from the above inequality that there exists

$$s_\varepsilon \in [t-\varepsilon, t+\varepsilon]$$

such that

$$f''(s_\varepsilon) + k^2 f(s_\varepsilon) \geq 0.$$

Since $f \in C^2(I)$, passing to the limit in the above inequality as $\varepsilon \to 0^+$, we obtain $f''(t) + k^2 f(t) \geq 0$, which proves that $f \in X_{1,k^2}(I)$, that is, (i) holds. □

We now consider the set $X_{1,-k^2}(I)$.

Theorem 4.2. *Let $f \in C^2(I)$ and $k > 0$. The following statements are equivalent:*
(i) $f \in X_{1,-k^2}(I)$;
(ii) *For all $a, b \in I$ with $a < b$, it holds that*

$$\int_a^b f(t)\mathrm{d}t \leq \tanh\left[\frac{k(b-a)}{2}\right]\frac{f(a)+f(b)}{k}. \tag{4.5}$$

Proof. (i) $\Rightarrow$ (ii) follows immediately from (3.13) with $p \equiv 1$. Assume now that (ii) holds, and let $t \in I$ be fixed. Using (4.5) with $a = t - \varepsilon$ and $b = t + \varepsilon$, where $\varepsilon > 0$ is small enough, we obtain

$$\int_{t-\varepsilon}^{t+\varepsilon} f(s)\mathrm{d}s \leq \frac{1}{k}\tanh(k\varepsilon)(f(t-\varepsilon)+f(t+\varepsilon)). \tag{4.6}$$

Consider now the function h_ε defined by

$$h_\varepsilon(s) = \frac{1}{k^2}\left(\frac{\cosh[k(t-s)]}{\cosh(k\varepsilon)} - 1\right), \quad t-\varepsilon \leq s \leq t+\varepsilon.$$

Elementary calculations show that

$$\begin{cases} h_\varepsilon''(s) - k^2 h_\varepsilon(s) = -1, & t-\varepsilon < s < t+\varepsilon, \\ h_\varepsilon(t-\varepsilon) = h_\varepsilon(t+\varepsilon) = 0. \end{cases} \tag{4.7}$$

Moreover, we have $h_\varepsilon \geq 0$ and

$$h_\varepsilon'(t-\varepsilon) = \frac{1}{k}\tanh(k\varepsilon), \quad h_\varepsilon'(t+\varepsilon) = -\frac{1}{k}\tanh(k\varepsilon). \tag{4.8}$$

Using (4.6)–(4.8) and proceeding as in the proof of Theorem 4.1, we obtain

$$\int_{t-\varepsilon}^{t+\varepsilon} (f''(s) - k^2 f(s))h_\varepsilon(s)\mathrm{d}s \geq 0.$$

Since $h_\varepsilon \geq 0$, we deduce from the above inequality that there exists

$$s_\varepsilon \in [t-\varepsilon, t+\varepsilon]$$

such that

$$f''(s_\varepsilon) - k^2 f(s_\varepsilon) \geq 0.$$

Passing to the limit in the above inequality as $\varepsilon \to 0^+$, we obtain $f''(t) - k^2 f(t) \geq 0$, which proves that $f \in X_{1,-k^2}(I)$, that is, (i) holds. □

5 Conclusion

For $\mu \in C^1(I)$, $\mu > 0$, and $\lambda \in C(I)$, we considered the set of functions $f \in C^2(I)$ satisfying the second-order differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\frac{\mathrm{d}f}{\mathrm{d}t}\right) + \lambda f \geq 0.$$

This set of functions includes different classes of generalized convex functions: convex functions ($\mu \equiv 1$, $\lambda \equiv 0$), trigonometrically k-convex functions ($\mu \equiv 1$, $\lambda \equiv k^2$), and hyperbolic k-convex functions ($\mu \equiv 1$, $\lambda \equiv -k^2$). We first established a Fejér-type inequality for the considered class of functions without any symmetry condition imposed on the weight function (see Theorem 3.1). Next, we discussed some special cases of μ and λ. In particular, when $\mu \equiv 1$ and $\lambda = \pm k^2$, $k > 0$, we provided characterizations of the sets $X_{1,\pm k^2}(I)$ (see Theorems 4.1 and 4.2).

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