

Research Article

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Small perturbations of critical nonlocal equations with variable exponents

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Abstract: In this article, we are concerned with the following critical nonlocal equation with variable exponents:

$$\begin{cases} (-\Delta)_{p(x,y)}^s u = \lambda f(x, u) + |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $N \geq 2$, $p \in C(\Omega \times \Omega)$ is symmetric, $f: C(\Omega \times \mathbb{R}) \rightarrow \mathbb{R}$ is a continuous function, and λ is a real positive parameter. We also assume that $\{x \in \mathbb{R}^N : q(x) = p_s^*(x)\} \neq \emptyset$, and $p_s^*(x) = N\tilde{p}(x)/(N - s\tilde{p}(x))$ is the critical Sobolev exponent for variable exponents. We prove the existence of non-trivial solutions in the case of low perturbations (λ small enough) by using the mountain pass theorem, the concentration-compactness principles for fractional Sobolev spaces with variable exponents, and the Moser iteration method. The features of this article are the following: (1) the function f does not satisfy the usual Ambrosetti-Rabinowitz condition and (2) this article contains the presence of critical terms, which can be viewed as a partial extension of the previous results concerning the existence of solutions to this problem in the case of $s = 1$ and subcritical case.

Keywords: fractional p -Laplace operator, $p(\cdot)$ -Laplacian, critical nonlinearity, Moser iteration, concentration-compactness principles, variational methods

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1 Introduction

In this article, we deal with the existence of solution to the following critical nonlocal equations with variable exponents:

$$\begin{cases} (-\Delta)_{p(x,y)}^s u = \lambda f(x, u) + |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $N \geq 2$, $p \in C(\Omega \times \Omega)$ is symmetric, almost everywhere, $p(x, y) = p(y, x)$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, $q \in C(\Omega)$ satisfies

$$\tilde{p}(x) = p(x, x) < q(x) \leq p_s^*(x) := \frac{N\tilde{p}(x)}{N - s\tilde{p}(x)}$$

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for all $x \in \mathbb{R}^N$ and λ is a real parameter. The functional $f(x, t) \in C(\Omega \times \mathbb{R})$ satisfies the following growth assumptions:

$$\begin{aligned} (f_1) \quad & \lim_{t \rightarrow 0} \frac{f(x, t)}{t^{\frac{p(x)-2}{p(x)-2}} t} = 0 \text{ uniformly in } x; \\ (f_2) \quad & \lim_{t \rightarrow \infty} \frac{f(x, t)}{t^{\frac{p(x)-2}{p(x)-2}} t} = +\infty \text{ uniformly in } x. \end{aligned}$$

Our interest in Problem (1.1) is mainly based on theoretical and practical reasons. On the one hand, many scholars pay attention to the study of differential equations and variational issues involving $p(x)$ -growth conditions in recent years. The development of numerous significant models in electrorheological and thermorheological fluids, image processing, and other fields inspired a systematic study of partial differential equations with variable exponents (see [1–4]). As we all know, Lebesgue spaces with variable exponents were initially studied in this article [5] by Orlicz. Then, we can obtain more details on Lebesgue Sobolev spaces related to the $p(\cdot)$ -Laplacian with variable exponents in [6,7]. The Lebesgue-Sobolev spaces related to the $p(\cdot)$ -Laplacian are called variable-exponent Lebesgue-Sobolev spaces and were studied in [8,9]. We also refer the reader to the book [10] for basic knowledge in this area. The literature on the study of such operators is large but we only list a few of them and the recently published articles for interested readers (see e.g., [11–18]). In these articles, the authors used different methods to establish the existence of solutions and some other properties under the subcritical growth condition. Very recently, Cao et al. in [19] established the existence of nontrivial solutions for $p(x)$ -Laplacian equations

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = |u|^{q(x)-2} u + \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $p \in C(\bar{\Omega})$ with $1 < p(x) < N$, $q \in C(\bar{\Omega})$ with $p^+ := \max_{x \in \bar{\Omega}} p(x) < q^- = \min_{x \in \bar{\Omega}} q(x) \leq q^+ := \max_{x \in \bar{\Omega}} q(x) < p^*(x)$, $p^*(x) = \frac{Np(x)}{N-p(x)}$ is the critical exponent and $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function without any growth and Ambrosetti-Rabinowitz conditions. However, the study of critical problem is a very meaningful subject, and the critical problem was initially studied in the seminal paper of Brézis and Nirenberg in [20], which treated for Laplace equations. Since then, there have been extensions of [20] in many directions. Elliptic equations involving critical growth are delicate due to the lack of compactness arising in connection with the variational approach. For such problems, the concentration-compactness principles introduced by Lions in [21,22] and its variants at infinity in [23,24] have played a decisive role in showing that a minimizing sequence or a Palais-Smale sequence is precompact. By using these concentration-compactness principles or extending them to the Sobolev spaces with variable exponents, many authors have been successful to deal with critical problems involving $p(\cdot)$ -Laplacian (see e.g., [24–29] and references therein).

On the other hand, nonlocal operators can be seen as the infinitesimal generators of Lévy stable diffusion processes in [30]. Moreover, they allow us to develop a generalization of quantum mechanics and also to describe the motion of a chain or an array of particles that are connected by elastic springs as well as unusual diffusion processes in turbulent fluid motions and material transports in fractured media (for more details, see, for example, [31,32] and references therein). The study of elliptic equations with nonlocal operators is one of the most fascinating areas of nonlinear analysis. These issues have received a lot of attention in both pure mathematics study and practical applications (see, for example, the recent monograph [33]). The study of fractional Sobolev space with variable exponents is natural extension of variable-exponent Lebesgue Sobolev spaces. As far as we know, the fractional Sobolev spaces with variable exponent and the fractional $p(x)$ -Laplacian were introduced first by Kaufmann et al. in [34]. Here, the authors obtained the embedding result of these spaces to variable-exponent Lebesgue spaces. In addition, they also discussed the existence result of a fractional $p(x)$ -Laplacian problem. Very recently, Ho and Kim obtained fundamental embedding results for the new fractional Sobolev spaces with variable exponents that are a generalization of the well-known fractional Sobolev spaces in [35]. Using this, they demonstrated a priori bounds and multiplicity of solutions of some nonlinear elliptic problems involving the fractional $p(\cdot)$ -Laplacian. We refer to [36–38] for fractional Sobolev spaces with variable exponents and the corresponding nonlocal equations with variable exponents.

Inspired by the works in the aforementioned references, our main purpose in this article is to study the existence of solution for Problem (1.1). Moreover, we do not assume that f satisfy the well-known Ambrosetti-Rabinowitz-type condition, and $q(x)$ can reach the critical exponent $p_s^*(x)$. Therefore, it can be viewed as a partial extension of the results of Cao et al. in [19] concerning the existence of solutions to equation (1.2) in the case of $s = 1$ and subcritical case.

We are ready to state the main result of this article.

Theorem 1.1. *Assume that the assumptions (f_1) and (f_2) hold. Then, there exists $\lambda_1 > 0$ such that Problem (1.1) admits at least one nontrivial solution for $\lambda \in (0, \lambda_1)$.*

The main difficulty in treating Problem (1.1) is the possible lack of compactness due to the presence of critical term and the growth condition of function f . To overcome the difficulties that arise from these features, we use the cutoff function approach and the concentration-compactness principles for fractional Sobolev spaces with variable exponents to prove that auxiliary problem has at least one nontrivial solution. Finally, we obtain nontrivial solutions for original problems with the aid of the Moser iteration method.

The rest of this article is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable-exponent Lebesgue space and fractional Sobolev space with variable-exponent space. In Section 3, we prove the Palais-Smale condition at some special energy levels by using the concentration-compactness principles for fractional Sobolev spaces with variable exponents. In Section 4, we prove the uniform bound of weak solution to auxiliary problem. Section 5 is devoted to proving the existence of nontrivial solutions of Problem (1.1) by the Moser iteration method.

2 Preliminaries

In this section, we briefly review the definitions and list some basic properties of the Lebesgue spaces. Furthermore, we recall and establish some qualitative properties of the new fractional Sobolev spaces with variable exponents.

2.1 Some basic function spaces

We first recall some properties involving Lebesgue-Sobolev spaces with variable exponents in [10,39]. Set Ω be a bounded domain of $\mathbb{R}^N$, and

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : h(x) > 1 \text{ for all } x \in \bar{\Omega}\}.$$

For any $h \in C_+(\bar{\Omega})$, we define

$$h^- = \min_{x \in \bar{\Omega}} h(x), \quad h^+ = \max_{x \in \bar{\Omega}} h(x).$$

For any $p \in C_+(\bar{\Omega})$, we define the variable-exponent Lebesgue space as follows:

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \mid u \text{ is measurable and } \int_{\Omega} |u(x)|^{p(x)} dx < +\infty \right\},$$

which endowed with the norm:

$$\|u\|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Proposition 2.1. [40]

(1) Denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$,

$$\int_{\Omega} |uv| dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{p(x)} \|v\|_{p'(x)}, \quad u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega)$$

holds.

(2) Define mapping $\rho : L^{p(x)}(\Omega) \rightarrow \mathbb{R}$ by $\rho(u) = \int_{\Omega} |u|^{p(x)} dx$, then the following relations hold:

$$|u|_{p(x)} < 1 (=1, >1) \Leftrightarrow \rho(u) < 1 (=1, >1),$$

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+},$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}.$$

Proposition 2.2. [41] Assume that $h \in L_+^{\infty}(\Omega)$, $p \in C_+(\bar{\Omega})$. If $|u|^{h(x)} \in L^{p(x)}(\Omega)$, then we have

$$\min\{|u|_{h(x)p(x)}^{h^-}, |u|_{h(x)p(x)}^{h^+}\} \leq \|u\|_{h(x)p(x)}^{h(x)} \leq \max\{|u|_{h(x)p(x)}^{h^-}, |u|_{h(x)p(x)}^{h^+}\}.$$

Let $s \in (0, 1)$ and $p \in (1, \infty)$ be constants. Define the fractional Sobolev space $W^{s,p}(\Omega)$ as follows:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy < \infty \right\},$$

which endowed with norm:

$$\|u\|_{s,p,\Omega} := \left(\int_{\Omega} |u(x)|^p dx + \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p}.$$

2.2 Fractional Sobolev space with variable exponent

Meanwhile, we recall and find some qualitative properties of the new fractional Sobolev spaces with variable exponents in [34,35]. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. In the following, for brevity, we use $p(x)$ instead of $p(x, x)$. With this notation, $p \in C_+(\bar{\Omega})$. Define

$$W^{s,p(\cdot),\cdot}(\Omega) := \left\{ u \in L^{p(\cdot)}(\Omega) : \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy < +\infty \right\},$$

which endowed with the norm:

$$\|u\|_{s,p,\Omega} := \inf \left\{ \lambda > 0 : M_{\Omega} \left(\frac{u}{\lambda} \right) < 1 \right\},$$

where

$$M_{\Omega}(u) = \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy.$$

Then, $W^{s,p(\cdot),\cdot}(\Omega)$ is a separable reflexive Banach space.

On $W^{s,p(\cdot),\cdot}(\Omega)$, we also make use of the following norm:

$$|u|_{s,p,\Omega} := \|u\|_{L^{p(\cdot)}(\Omega)} + [u]_{s,p,\Omega},$$

where

$$[u]_{s,p,\Omega} := \inf \left\{ \lambda > 0 : \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{N+sp(x,y)}} dx dy < 1 \right\}.$$

Note that $\|\cdot\|_{s,p_1\Omega}$ and $|\cdot|_{s,p,\Omega}$ are equivalent norms on $W^{s,p(\cdot,\cdot)}(\Omega)$ with the relation

$$\frac{1}{2} \|u\|_{s,p,\Omega} \leq |u|_{s,p,\Omega} \leq 2 \|u\|_{s,p,\Omega}, \quad \forall u \in W^{s,p(\cdot,\cdot)}(\Omega).$$

In the follows, when Ω is understood, we use $\|\cdot\|_{s,p}$, $|\cdot|_{s,p}$, and $[\cdot]_{s,p}$ instead of $\|\cdot\|_{s,p,\Omega}$, $|\cdot|_{s,p,\Omega}$, and $[\cdot]_{s,p,\Omega}$, respectively.

Proposition 2.3. [42] *On $W^{s,p(\cdot,\cdot)}(\Omega)$, it holds that*

- (i) *for $u \in W^{s,p(\cdot,\cdot)}(\Omega) \setminus \{0\}$, $\lambda = \|u\|_{s,p}$ if and only if $M_\Omega\left(\frac{u}{\lambda}\right) = 1$;*
- (ii) *$M_\Omega(u) > 1 (=1; <1)$ if and only if $\|u\|_{s,p} > 1 (=1; <1)$, respectively;*
- (iii) *if $\|u\|_{s,p} \geq 1$, then $\|u\|_{s,p}^- \leq M_\Omega(u) \leq \|u\|_{s,p}^+$;*
- (iv) *if $\|u\|_{s,p} < 1$, then $\|u\|_{s,p}^+ \leq M_\Omega(u) \leq \|u\|_{s,p}^-$.*

Proposition 2.4. *For any uniformly continuous function $r \in C_+(\bar{\Omega})$ satisfying $p(x) \leq r(x) \leq p_s^*(x)$ and the embedding*

$$W^{s,p(\cdot,\cdot)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$$

holds. In particular,

$$S_q := \inf_{u \in W^{s,p(\cdot,\cdot)}(\Omega) \setminus \{0\}} \frac{\|u\|}{\|u\|_{L^{q(\cdot)}(\Omega)}}. \quad (2.1)$$

3 An auxiliary problem

In this section, we mainly prove that the energy functional associated with Problem (1.1) satisfies the $(PS)_c$ at some special energy levels. Due to assumption (f_2) , we need to truncate the function f ; thus, there is a constant $Q > 0$ that is large sufficiently so that $f(x, Q) > 0$. For all $x \in \Omega$, let

$$f_Q(x, t) = \begin{cases} f(x, t) & \text{if } 0 < t \leq Q, \\ \frac{f(x, Q)}{Q^{q(x)-1}} t^{q(x)-1} & \text{if } t > Q, \\ 0 & \text{if } t \leq 0, \end{cases} \quad (3.1)$$

where $p^+ < q^- \leq q(x) \leq q^+$. Then, because of the continuity of f , we claim that the cutoff function $f_Q : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Moreover, by assumption (f_1) and (3.1), we know that

- (e₁) $\lim_{t \rightarrow 0} \frac{f_Q(x, t)}{t^{p(x)-1}} = 0$ uniformly in x ;
- (e₂) $\lim_{t \rightarrow \infty} \frac{F_Q(x, t)}{t^{p^+}} = +\infty$ uniformly in x , where $F_Q(x, t) = \int_0^t f_Q(x, s) ds$;
- (e₃) $|f_Q(x, t)| \leq C_Q^1 |t|^{p(x)-1} + C_Q^2 |t|^{q(x)-1}$ for all $(x, t) \in \Omega \times \mathbb{R}$, where

$$C_Q^1 = \max_{x \in \bar{\Omega}, t \in [0, Q]} \frac{f(x, t)}{t^{p(x)-1}} \quad \text{and} \quad C_Q^2 = \frac{1}{Q^{q^- - 1}} \max_{x \in \bar{\Omega}} f(x, Q);$$

- (e₄) for any $\theta \in (p^+, q^-)$, there exists $\tau = \tau(Q) > 0$ such that

$$\frac{1}{\theta} t f_Q(x, t) - F_Q(x, t) \geq -\tau t^{p(x)} \quad \text{for all } (x, t) \in \Omega \times [0, +\infty).$$

Thus, we consider the following revised version of Problem (1.1):

$$\begin{cases} (-\Delta)_{p(x,y)}^s u = \lambda f_Q(x, u) + |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (3.2)$$

The aim of this section is to prove the existence of nontrivial weak solution for Problem (3.2), that is, $u \in W^{s,p(\cdot,\cdot)}(\Omega)$ is a solution of Problem (3.2) if

$$\langle \mathcal{L}(u), v \rangle = \lambda \int_{\Omega} f_Q(x, u) v dx + \int_{\Omega} |u|^{q(x)-2} u v dx, \quad (3.3)$$

where

$$\langle \mathcal{L}(u), v \rangle := \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{N+sp(x,y)}} dx dy \quad (3.4)$$

for all $v \in W^{s,p(\cdot,\cdot)}(\Omega)$. Problem (3.2) has a variational nature, and the energy functional $J_{\lambda}^Q : W^{s,p(\cdot,\cdot)}(\Omega) \rightarrow \mathbb{R}$ associated with it is defined as follows:

$$J_{\lambda}^Q(u) = T_{p(\cdot,\cdot)}(u) - \lambda \int_{\Omega} F_Q(x, u) dx - \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx, \quad (3.5)$$

where

$$T_{p(\cdot,\cdot)}(u) := \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y) |x - y|^{N+sp(x,y)}} dx dy$$

and $F_Q(x, t) = \int_0^t f_Q(x, s) ds$. It is easy to check that $J_{\lambda}^Q \in C^1(W^{s,p(\cdot,\cdot)}(\Omega), \mathbb{R})$. Therefore, all solutions of Problem (3.2) correspond to critical points of the functional J_{λ}^Q in the weak sense.

Now, we prove that functional J_{λ}^Q has the mountain pass geometry.

Lemma 3.1. *Let $\lambda > 0$ and $Q > 0$. Then, the energy functional J_{λ}^Q satisfies*
(D₁) there exist $\rho_0 > 0$ and $\sigma_0 > 0$ such that $J_{\lambda}^Q(u) \geq \sigma_0$ for any $u \in W^{s,p(\cdot,\cdot)}$ with $\|u\| = \rho_0$;
(D₂) there exists $\varphi_0 \in W^{s,p(\cdot,\cdot)}(\Omega)$, $\varphi_0 > 0$ such that $J_{\lambda}^Q(t\varphi_0) \rightarrow -\infty$ as $t \rightarrow +\infty$.

Proof. First, from (e₁) and (e₃), we know that for any $\varepsilon > 0$, there is a $C_{\varepsilon} > 0$ such that

$$|f_Q(x, t)| \leq \varepsilon |t|^{p(x)-1} + C_{\varepsilon} |t|^{q(x)-1} \quad (3.6)$$

and

$$|F_Q(x, t)| \leq \varepsilon |t|^{p(x)} + C_{\varepsilon} |t|^{q(x)}. \quad (3.7)$$

Hence, for $u \in W^{s,p(\cdot,\cdot)}(\Omega)$ with $\|u\| < 1$, by Propositions 2.1, 2.2, and 2.3, we have

$$\begin{aligned} J_{\lambda}^Q(u) &= T_{p(\cdot,\cdot)}(u) - \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx - \lambda \int_{\Omega} F_Q(x, u) dx \\ &\geq \frac{1}{p^+} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy - \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx - \lambda \varepsilon \int_{\Omega} |u|^{p(x)} dx - \lambda C_{\varepsilon} \int_{\Omega} |u|^{q(x)} dx \\ &\geq \left(\frac{1}{p^+} - \lambda S_{q^+} \varepsilon \right) \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy - \left(\frac{1}{q^-} + \lambda C_{\varepsilon} \right) |u|^{q^-} \\ &\geq \left(\frac{1}{p^+} - \lambda S_{q^+} \varepsilon \right) \|u\|^p - \left(\frac{1}{q^-} + \lambda C_{\varepsilon} \right) S_q \|u\|^{q^-}. \end{aligned} \quad (3.8)$$

Let $\varepsilon = \frac{1}{2\lambda S_q p^+}$, from (3.8), one has

$$J_\lambda^Q(u) \geq \frac{1}{2p^+} \|u\|^{p^+} - \left(\frac{1}{q^-} + \lambda C_\varepsilon \right) S_q \|u\|^{q^-}, \quad (3.9)$$

when $\|u\| < 1$. Then, we can find $\rho_0 > 0$ and $\sigma_0 > 0$ such that $J_\lambda^Q(u) \geq \sigma_0$ for $\|u\| = \rho_0$ since $q^- > p^+$. Hence, (D_1) in Lemma 3.1 holds true.

Now, we verify condition (D_2) of Lemma 3.1. From condition (e_2) , we know that for any $k > 0$, there exists a constant $\delta_k > 1$ large enough such that

$$F_Q(x, t) \geq kt^{p^+} \quad \text{for all } t > \delta_k \quad \text{and } x \in \Omega.$$

It follows from (e_3) that

$$|F_Q(x, t)| \leq \frac{C_Q^1}{p^+} \delta_k^{p^+} + \frac{C_Q^2}{q^-} \delta_k^{q^-} \quad \text{for all } 0 \leq t \leq \delta_k \quad \text{and } x \in \Omega.$$

So, we obtain that there exists a constant $C_k > 0$ such that

$$F_Q(x, t) \geq k |t|^{p^+} - C_k \quad \text{for all } t > 0 \quad \text{and } x \in \Omega. \quad (3.10)$$

Fix $\varphi_0 \in W^{s,p(\cdot,\cdot)}(\Omega)$ with $\varphi_0 > 0$ on Ω and $t > 1$. From (3.10), we have

$$\begin{aligned} J_\lambda^Q(t\varphi_0) &= T_{p(\cdot,\cdot)}(t\varphi_0) - \int_\Omega \frac{1}{q(x)} |t\varphi_0|^{q(x)} dx - \lambda \int_\Omega F_Q(x, t\varphi_0) dx \\ &\leq T_{p(\cdot,\cdot)}(t\varphi_0) - \lambda t^{p^+} k \int_\Omega \varphi_0^{p^+} dx + \lambda C_k |\Omega| \\ &\leq |t|^{p^+} \left[T_{p(\cdot,\cdot)}(\varphi_0) - \lambda k \int_\Omega \varphi_0^{p^+} dx + \lambda C_k |\Omega| \right]. \end{aligned} \quad (3.11)$$

Let k large enough such that

$$T_{p(\cdot,\cdot)}(\varphi_0) - \lambda k \int_\Omega \varphi_0^{p^+} dx + \lambda C_k |\Omega| < 0,$$

then

$$\lim_{t \rightarrow +\infty} J_\lambda^Q(t\varphi_0) = -\infty.$$

we can also obtain the conclusion (D_2) in Lemma 3.1. This completes the proof of Lemma 3.1. $\square$

It follows from Lemma 3.1 that there exists a $(PS)_{c_\lambda^Q}$ sequence $\{u_n\} \subset W^{s,p(\cdot,\cdot)}(\Omega)$ such that

$$J_\lambda^Q(u_n) \rightarrow c_\lambda^Q \quad \text{and} \quad J_\lambda^{Q'}(u_n) \rightarrow 0 \quad (3.12)$$

at the minimax level

$$c_\lambda^Q = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J_\lambda^Q(\gamma(t)) > 0,$$

where $\Gamma = \{\gamma \in C^1([0,1], W^{s,p(\cdot,\cdot)}(\Omega)) : \gamma(0) = 0, J_\lambda^Q(\gamma(1)) < 0\}$.

Furthermore, we can obtain the following compactness result.

Lemma 3.2. Assume that the assumptions (f_1) and (f_2) hold. Then, there exists $\lambda_0 > 0$ such that J_λ^Q satisfies $(PS)_{c_\lambda^Q}$ -condition for any $\lambda \leq \lambda_0$, where

$$c_\lambda^Q < \left(\frac{1}{\theta} - \frac{1}{q^-} \right) \min\{(S_q^{\bar{p}})^{\xi^+}, (S_q^{\bar{p}})^{\xi^-}\} \quad (3.13)$$

and $\xi(x) = \frac{q(x)}{q(x) - \bar{p}}$ and S_q is defined as in (2.1).

Proof. Let $\{u_n\} \subset W^{s,p(\cdot,\cdot)}(\Omega)$ be a Palais-Smale sequence and $|J_\lambda^Q(u_n)| \leq c_\lambda^Q$ for some $c_\lambda^Q > 0$ and all $n \geq 1$ and $(J_\lambda^Q)'(u_n) \rightarrow 0$.

First, we claim that $\{u_n\}$ is bounded in $W^{s,p(\cdot,\cdot)}(\Omega)$.

In fact, it is easy to obtain that we have done if $\|u_n\| \leq 1$. If $\|u_n\| \geq 1$, then from (e_4) , Propositions 2.2 and 2.3, we have

$$\begin{aligned} c_\lambda^Q + o_n(1)\|u_n\| &\geq J_\lambda^Q(u_n) - \frac{1}{\theta} \langle (J_\lambda^Q)'(u_n), u_n \rangle \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy + \lambda \int_{\Omega} \left(\frac{1}{\theta} f_Q(x, u_n) u_n - F_Q(x, u_n) \right) dx \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{q^-} \right) \int_{\Omega} |u_n|^{q(x)} dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_n\|^{p^-} - \lambda \tau \int_{\Omega} |u_n|^{p(x)} dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta} - \lambda \tau S_q \right) \|u_n\|^{p^-}, \end{aligned} \quad (3.14)$$

where $o_n(1)$ denotes a quantity that goes to zero as $n \rightarrow +\infty$. Now, choose $\lambda_0 > 0$ so that

$$\frac{1}{p^+} - \frac{1}{\theta} - \lambda_0 \tau S > 0.$$

According to (3.14), we obtain

$$\begin{aligned} c_\lambda^Q + o_n(1)\|u_n\| &\geq J_\lambda^Q(u_n) - \frac{1}{\theta} \langle (J_\lambda^Q)'(u_n), u_n \rangle \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta} - \lambda_0 \tau S_q \right) \|u_n\|^{p^-}, \lambda \leq \lambda_0. \end{aligned} \quad (3.15)$$

This fact implies that $\{u_n\}_n$ is bounded in $W^{s,p(\cdot,\cdot)}(\Omega)$.

Next, it follows from the boundedness of sequences $\{u_n\}$ that $u_n \rightarrow u$ a.e. in $\mathbb{R}^3$ and

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } W^{s,p(\cdot,\cdot)}(\Omega), \\ \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(s,y)}} dy &\overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega), \\ |u_n|^{q(x)} &\overset{*}{\rightharpoonup} \nu \text{ in } \mathcal{M}(\Omega), \end{aligned}$$

where μ and ν are the bounded nonnegative measures on Ω . Then, as proof in Ho and Kim [35], up to a subsequence, there exists a (at most countable) set of distinct points $\{x_i\}_{i \in I} \subset \Omega$ and a family of positive numbers $\{v_i\}_{i \in I}$ such that

$$\begin{aligned} U_n(x) &\overset{*}{\rightharpoonup} \mu \geq U(x) + \sum_{i \in I} \delta_{x_i} \mu_i \text{ in the sense of measures in } \mathcal{M}(\Omega), \\ |u_n|^{q(x)} &\overset{*}{\rightharpoonup} \nu = |u|^{q(x)} + \sum_{i \in I} \delta_{x_i} v_i \text{ in the sense of measures in } \mathcal{M}(\Omega) \end{aligned}$$

and

$$S_q v_i^{\frac{1}{p_s^-}} \leq \mu_i^{\frac{1}{p^-}} \quad \text{for all } i \in I, \quad (3.16)$$

where

$$U_n(x) = \int_{\Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dy \quad \text{and} \quad U(x) = \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dy \quad \text{for } x \in \Omega.$$

In the following, we shall prove that $I = \emptyset$.

On the contrary, we suppose that $I \neq \emptyset$. Let $i \in I$ and we can use a smooth cut-off function $\psi_{\varepsilon,i}$ centered at z_i such that

$$0 \leq \psi_{\varepsilon,i}(x) \leq 1, \quad \psi_{\varepsilon,i}(x) = 1 \quad \text{in } B_{\varepsilon/2}(z_i), \quad \psi_{\varepsilon,i}(x) = 0 \quad \text{in } B_{\varepsilon}(z_i)^c, \quad |\nabla \psi_{\varepsilon,i}(x)| \leq 4/\varepsilon,$$

for any $\varepsilon > 0$ small. It is easy to see that $\{u_n \psi_{\varepsilon,i}\}$ is a bounded sequence in $W^{s,p(\cdot,\cdot)}(\Omega)$. Obviously, $\langle (J_{\lambda}^Q)'(u_n), u_n \psi_{\varepsilon,i} \rangle \rightarrow 0$, i.e.,

$$\langle \mathcal{L}(u_n), u_n \psi_{\varepsilon,i} \rangle = \lambda \int_{\Omega} f_Q(x, u_n) u_n \psi_{\varepsilon,i} dx + \int_{\Omega} |u_n|^{q(x)} u_n \psi_{\varepsilon,i} dx + o_n(1),$$

where

$$\langle \mathcal{L}(u_n), u_n \psi_{\varepsilon,i} \rangle := \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2} (u_n(x) - u_n(y)) (u_n(x) \psi_{\varepsilon,i}(x) - u_n(y) \psi_{\varepsilon,i}(y))}{|x - y|^{N+sp(x,y)}} dx dy.$$

Note that

$$\langle \mathcal{L}(u_n), u_n \psi_{\varepsilon,i} \rangle = \int_{\mathbb{R}^N} U_n(x) \psi_{\varepsilon,i} dx - \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2} (u_n(x) - u_n(y)) u_n(y) (\psi_{\varepsilon,i}(y) - \psi_{\varepsilon,i}(x))}{|x - y|^{N+sp(x,y)}} dx dy.$$

Hence,

$$\begin{aligned} & \int_{\mathbb{R}^N} U_n(x) \psi_{\varepsilon,i} dx - \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2} (u_n(x) - u_n(y)) u_n(y) (\psi_{\varepsilon,i}(y) - \psi_{\varepsilon,i}(x))}{|x - y|^{N+sp(x,y)}} dx dy \\ &= \lambda \int_{\Omega} f_Q(x, u_n) u_n \psi_{\varepsilon,i} dx + \int_{\Omega} |u_n|^{q(x)} u_n \psi_{\varepsilon,i} dx + o_n(1). \end{aligned} \quad (3.17)$$

Since $\{u_n\}$ is bounded in $W^{s,p(\cdot,\cdot)}(\Omega)$. By the Hölder inequality and the Lebesgue dominant convergence, we have

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{\Omega} f_Q(x, u_n) u_n \psi_{\varepsilon,i} dx = 0. \quad (3.18)$$

Moreover, let $\delta > 0$ be arbitrary and fixed. And $\{u_n\}$ is bounded in $L^{q(\cdot)}(\Omega)$. According to the Young inequality, we yield

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2} (u_n(x) - u_n(y)) u_n(y) (\psi_{\varepsilon,i}(y) - \psi_{\varepsilon,i}(x))}{|x - y|^{N+sp(x,y)}} dx dy \\ & \leq \delta \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy - C \iint_{\mathbb{R}^{2N}} |u_n(y)|^{p(x,y)} \frac{|\psi_{\varepsilon,i}(x) - \psi_{\varepsilon,i}(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \\ & \leq C\delta - C \iint_{\mathbb{R}^{2N}} |u_n(y)|^{p(x,y)} \frac{|\psi_{\varepsilon,i}(x) - \psi_{\varepsilon,i}(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx. \end{aligned} \quad (3.19)$$

In view of Lemma 4.4 in [6], we obtain

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \iint_{\mathbb{R}^{2N}} |u_n(y)|^{p(x,y)} \frac{|\psi_{\varepsilon,i}(x) - \psi_{\varepsilon,i}(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy = 0.$$

Consequently, taking the limit superior in (3.17) as $n \rightarrow \infty$ and then the limit superior as $\varepsilon \rightarrow 0$,

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2} (u_n(x) - u_n(y)) u_n(y) (\psi_{\varepsilon,i}(y) - \psi_{\varepsilon,i}(x))}{|x - y|^{N+sp(x,y)}} dx dy = 0, \quad (3.20)$$

since $\delta > 0$ is arbitrary. So, $\psi_{\varepsilon,i}$ has compact support. Then, letting $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$ in (3.20), we imply from (3.18) and (3.20) that

$$\mu_i \leq \nu_i.$$

Inserting this into (3.16), we yield

$$\nu_i \geq (S_q^{\bar{p}})^{\frac{q(z_i)}{q(z_i)-\bar{p}}} \geq \min\{(S_q^{\bar{p}})^{\xi^+}, (S_q^{\bar{p}})^{\xi^-}\}, \quad (3.21)$$

where $\xi = \frac{q(z_i)}{q(z_i)-\bar{p}}$. By (3.14) and (3.21), we have

$$\begin{aligned} c_\lambda^Q &= \lim_{n \rightarrow \infty} J_\lambda^Q(u_n) = \lim_{n \rightarrow \infty} \left(J_\lambda^Q(u_n) - \frac{1}{\theta} \langle (J_\lambda^Q)'(u_n), u_n \rangle \right) \\ &\geq \left(\frac{1}{\theta} - \frac{1}{q^-} \right) \int_{\Omega} |u_n|^{q(x)} dx \geq \left(\frac{1}{\theta} - \frac{1}{q^-} \right) \int_{\Omega} |u_n|^{q(x)} \psi_{\varepsilon,i} dx \\ &\geq \left(\frac{1}{\theta} - \frac{1}{q^-} \right) \nu_i \geq \left(\frac{1}{\theta} - \frac{1}{q^-} \right) \min\{(S_q^{\bar{p}})^{\xi^+}, (S_q^{\bar{p}})^{\xi^-}\} \end{aligned}$$

for all $\lambda \in (0, \lambda_0)$. In view of (3.13), it is a contradiction. Hence, $I = \emptyset$. Moreover,

$$\limsup_{n \rightarrow \infty} \int_{\Omega} |u_n|^{q(x)} dx = \int_{\Omega} |u|^{q(x)} dx.$$

The Brézis-Lieb Lemma 3.9 of [43] for the variable-exponent Lebesgue spaces yields that

$$\int_{\Omega} |u_n - u|^{q(x)} dx \rightarrow 0.$$

Consequently, we have

$$\int_{\Omega} |u_n|^{q(x)-2} u_n (u_n - u) dx \rightarrow 0. \quad (3.22)$$

By (e₃), we have

$$\begin{aligned} \left| \int_{\Omega} f_Q(x, u_n)(u_n - u) dx \right| &\leq \int_{\Omega} |f_Q(x, u_n)(u_n - u)| dx \\ &\leq C_Q^1 \| |u_n|^{p(x)-1} \|_{\frac{p(x)}{p(x)-1}} \|u_n - u\|_{p(x)} + C_Q^2 \| |u_n|^{q(x)-1} \|_{\frac{q(x)}{q(x)-1}} \|u_n - u\|_{q(x)} \\ &\leq C_Q^1 \max\{|u_n|_{p(x)}^{p^+-1}, |u_n|_{p(x)}^{p^--1}\} \|u_n - u\|_{p(x)} + C_Q^2 \max\{|u_n|_{q(x)}^{q^+-1}, |u_n|_{q(x)}^{q^--1}\} \|u_n - u\|_{q(x)} \\ &\rightarrow 0 \quad \text{as } n \rightarrow +\infty. \end{aligned} \quad (3.23)$$

From (3.22) and (3.23), we know that $\lim_{n \rightarrow \infty} \langle \mathcal{L}(u), u_n - u \rangle = 0$. Clearly, $\langle (J_\lambda^Q)'(u_n), u_n - u \rangle \rightarrow 0$ as $n \rightarrow \infty$. Hence,

$$\begin{aligned} o(1) &= \langle (J_\lambda^Q)'(u_n) - (J_\lambda^Q)'(u), u_n - u \rangle \\ &= \langle \mathcal{L}(u_n), u_n - u \rangle - \langle \mathcal{L}(u), u_n - u \rangle + \langle \mathcal{L}(u), u_n - u \rangle - \lambda \int_{\mathbb{R}^N} (f(x, u_n)u_n - f(x, u)u)(u_n - u) dx \\ &\quad - \int_{\mathbb{R}^N} (|u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u)(u_n - u) dx \\ &= [\langle \mathcal{L}(u_n), u_n - u \rangle - \langle \mathcal{L}(u), u_n - u \rangle] + o(1). \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} [\langle \mathcal{L}(u_n), u_n - u \rangle - \langle \mathcal{L}(u), u_n - u \rangle] = 0.$$

Consequently, we obtain

$$\lim_{n \rightarrow \infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy = \lim_{n \rightarrow \infty} (\langle L(u_n), u_n - u \rangle - \langle L(u), u_n - u \rangle) = 0.$$

This fact implies that $\{u_n\}$ strongly converges to u in $W^{s,p(\cdot,\cdot)}(\Omega)$. Hence, the proof is complete. $\square$

4 L^∞ estimate

In this section, we will prove the uniform bound of weak solution to Problem (1.1) by using the Moser iteration method. To this end, we need some auxiliary lemmas.

Lemma 4.1. *Let $\beta(x) > 0$ and $p(x) \geq 1$. Then, we obtain*

$$\left(\frac{1}{\beta(x)} \right)^{1/p(x)} \left(\frac{p(x) + \beta(x) - 1}{p(x)} \right) \geq 1. \quad (4.1)$$

Proof. If $p(x) = 1$, we can obtain the result easily. Hence, we can assume $p(x) > 1$. So, it is obvious that the function $t \mapsto t^{p(x)}$ is convex. Therefore,

$$\beta(x) - 1 \geq p(x)(\beta(x)^{1/p(x)} - 1).$$

Moreover,

$$p(x) + \beta(x) - 1 \geq p(x)\beta(x)^{1/p(x)}.$$

So we can deduce

$$\left(\frac{1}{\beta(x)} \right)^{1/p(x)} \left(\frac{p(x) + \beta(x) - 1}{p(x)} \right) \geq 1.$$

Hence, the proof is complete. $\square$

Next, we define the monotone increasing function $J_{p(x)}(t) = |t|^{p(x)-2}t$ for every $1 < p(x) < \infty$ to obtain some important conclusions.

Lemma 4.2. *Let the function $f: \mathbb{R} \rightarrow \mathbb{R}$ is C^1 and convex. Then, we obtain*

$$J_{p(x)}(a - b)[AJ_{p(x),\tau}(f'(a)) - BJ_{p(x),\tau}(f'(b))] \geq (\tau(a - b)^2 + (f(a) - f(b))^2)^{(p(x)-2)/2} \times (f(a) - f(b))(A - B) \quad (4.2)$$

for all $\tau \geq 0$, $a, b \in \mathbb{R}$ and for any $A, B \geq 0$, where $J_{p(x),\tau}(t) = (\tau + |t|^2)^{(p-2)/2}t$, $t \in \mathbb{R}$.

Proof. If $a = b$, we obtain the inequality. Then, we assume $a \neq b$. And we can deduce that f is a C^1 convex function. Then, one can obtain

$$f(a) - f(b) \leq f'(a)(a - b) \text{ and } f(a) - f(b) \geq f'(b)(a - b).$$

According to (4.2), we can deduce that

$$\begin{aligned} & J_{p(x)}(a - b)[AJ_{p(x),\tau}(f'(a)) - BJ_{p(x),\tau}(f'(b))] \\ &= (\tau(a - b)^2 + (f'(a)(a - b))^2)^{(p(x)-2)/2} f'(a)(a - b)A - (\tau(a - b)^2 + (f'(b)(a - b))^2)^{(p(x)-2)/2} f'(b)(a - b)B. \end{aligned} \quad (4.3)$$

Therefore, we deduce inequality (4.2). $\square$

Remark 4.1. When $\tau = 0$, for all $a, b \in \mathbb{R}$ and $A, B \geq 0$, we can rewrite (4.2) as the following expression:

$$J_{p(x)}(a - b)[AJ_{p(x)}(f'(a)) - BJ_{p(x)}(f'(b))] \geq (f(a) - f(b))^{p(x)-2}(f(a) - f(b))(A - B). \quad (4.4)$$

Lemma 4.3. Suppose $1 < p(x) < \infty$. Define

$$G(t) = \int_0^t g'(\tau)^{1/p(x)} d\tau, \quad t \in \mathbb{R},$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ to be an increasing function. Then, one can deduce that

$$J_{p(x)}(a - b)(g(a) - g(b)) \geq |G(a) - G(b)|^{p(x)}.$$

Proof. Suppose $a > b$ (the result is the same as $a < b$). In view of the Jensen inequality, we obtain

$$\begin{aligned} J_{p(x)}(a - b)(g(a) - g(b)) &= (a - b)^{p(x)-1} \int_b^a g'(\tau) d\tau = (a - b)^{p(x)-1} \int_b^a G'(\tau)^{p(x)} d\tau \\ &\geq \left(\int_b^a G'(\tau) d\tau \right)^{p(x)} = |G(a) - G(b)|^{p(x)}. \end{aligned} \quad (4.5)$$

Thus, the proof is completed. $\square$

Lemma 4.4. Suppose that (f_1) and (f_2) are fulfilled. Let u be a solution to Problem (3.2), then $u \in L^\infty(\Omega)$. Moreover, there exists $T > 0$ such that $|u|_{L^\infty} \leq T$.

Proof. First, for all $\varepsilon > 0$, we define the function $g_\varepsilon(t) = (\varepsilon^2 + t^2)^{\frac{1}{2}}$. And it is easy to prove that the function g_ε is convex and Lipschitz. Then, we set $v = \phi |g'_\varepsilon(u)|^{p(x,y)-2} g'_\varepsilon(u)$ as the test function in (3.3) for all $0 < \phi \in C_c^\infty(\Omega)$. Moreover, let us set $a = u(x)$, $b = u(y)$, $A = \phi(x)$, and $B = \phi(y)$ in (4.4). Thus, for all $0 < \phi \in C_c^\infty(\Omega)$, we obtain

$$\begin{aligned} &\iint_{\Omega \times \Omega} \frac{|g_\varepsilon(u(x)) - g_\varepsilon(u(y))|^{p(x,y)-2} (g_\varepsilon(u(x)) - g_\varepsilon(u(y)))(\phi(x) - \phi(y))}{|x - y|^{N+sp(x,y)}} dx dy \\ &\leq \int_{\Omega} (|\lambda f_Q(x, u) + |u|^{q(x)-1}|) |g'_\varepsilon(u)|^{p(x)-1} \phi dx. \end{aligned} \quad (4.6)$$

In view of $g_\varepsilon(t) \rightarrow |t|$ as $\varepsilon \rightarrow 0$ and $|g'_\varepsilon(t)| \leq 1$, combining with Fatou's lemma, and according to the limit $\varepsilon \rightarrow 0$ in (4.6), we deduce that

$$\iint_{\Omega \times \Omega} \frac{||u(x)| - |u(y)||^{p(x,y)-2} (|u(x)| - |u(y)|)(\phi(x) - \phi(y))}{|x - y|^{N+sp(x,y)}} dx dy \leq \int_{\Omega} (|\lambda f_Q(x, u) + |u|^{q(x)-1}|) \phi dx \quad (4.7)$$

for all $\phi \in C_c^\infty(\Omega)$.

Next, for all $k > 0$, let us define that $u_k = \min\{(u - 1)^+, k\} \in W^{s,p(\cdot,\cdot)}(\Omega)$. Therefore, for all $\beta > 0$ and $\delta > 0$, we make $\phi = (u_k + \delta)^\beta - \delta^\beta$ to be the test function in (4.7). Then, we obtain

$$\begin{aligned} &\iint_{\Omega \times \Omega} \frac{||u(x)| - |u(y)||^{p(x,y)-2} (|u(x)| - |u(y)|)((u_k + \delta)^\beta - (u_k + \delta)^\beta)}{|x - y|^{N+sp(x,y)}} dx dy \\ &\leq \int_{\Omega} |\lambda f_Q(x, u) + |u|^{q(x)-1}| ((u_k + \delta)^\beta - \delta^\beta) dx. \end{aligned} \quad (4.8)$$

According to Lemma 4.3 and choosing $h(u) = (u_k + \delta)^\beta$, we can deduce

$$\begin{aligned}
& \iint_{\Omega \times \Omega} \frac{|((u_k(x) + \delta)^{(\beta+p(x,y)-1)/p(x,y)} - (u_k(y) + \delta)^{(\beta+p(x,y)-1)/p(x,y)})|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dx dy \\
& \leq \int_Q \left(\frac{(\beta + p(x,y) - 1)^{p^+}}{\beta p(x,y)^{p(x,y)}} \right) \frac{\|u(x) - u(y)\|^{p(x,y)-2} (|u(x)| - |u(y)|) (u_k(x) + \delta)^\beta - (u_k(y) + \delta)^\beta}{|x-y|^{N+sp(x,y)}} dx dy \\
& \leq \left(\frac{(\beta + p^+ - 1)^{p^+}}{\beta (p^+)^{p^+}} \right) \iint_{\Omega \times \Omega} \frac{\|u(x) - u(y)\|^{p(x,y)-2} (|u(x)| - |u(y)|) (u_k(x) + \delta)^\beta - (u_k(y) + \delta)^\beta}{|x-y|^{N+sp(x,y)}} dx dy \\
& \leq \left(\frac{(\beta + p^+ - 1)^{p^+}}{\beta (p^+)^{p^+}} \right) \int_{\Omega} (|\lambda f_Q(x, u) + |u|^{q(x)-1}|) ((u_k + \delta)^\beta - \delta^\beta) dx \\
& \leq C_1 \left(\frac{(\beta + p^+ - 1)^{p^+}}{\beta (p^+)^{p^+}} \right) \left\| \int_{\{u \geq 1\}} |u|^{q(x)-1} ((u_k + \delta)^\beta - \delta^\beta) dx \right\| \\
& \leq C_2 \left(\frac{(\beta + p^+ - 1)^{p^+}}{\beta (p^+)^{p^+}} \right) \left\| \int_{\Omega} |u|^{q(x)-1} ((u_k + \delta)^\beta - \delta^\beta) dx \right\| \\
& \leq C_3 \left(\frac{(\beta + p^+ - 1)^{p^+}}{\beta (p^+)^{p^+}} \right) \|u\|_{r^*}^{q(x)-1} \|(u_k + \delta)^\beta\|_t,
\end{aligned} \tag{4.9}$$

where $t = r^*/(r^* - q(x) + 1)$, $(r^*) < (p_s^*)^-$. According to the Sobolev embedding theorem from Proposition 2.4, we can obtain

$$\begin{aligned}
& \iint_{\Omega \times \Omega} \frac{|((u_k(x) + \delta)^{(\beta+p(x,y)-1)/p(x,y)} - (u_k(y) + \delta)^{(\beta+p(x,y)-1)/p(x,y)})|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dx dy \\
& \geq C \|(u_k + \delta)^{(\beta+p(x,y)-1)/p(x,y)} - \delta^{(\beta+p(x,y)-1)/p(x,y)}\|_{r^*}^{p(x,y)}.
\end{aligned} \tag{4.10}$$

In view of triangle inequality and $(u_k + \delta)^{\beta+p(x,y)-1} \geq \delta^{p(x,y)-1}(u_k + \delta)^\beta$, we obtain

$$\begin{aligned}
& \left[\int_{\Omega} ((u_k + \delta)^{(\beta+p(x,y)-1)/p(x,y)} - \delta^{(\beta+p(x,y)-1)/p(x,y)})^{r^*} dx \right]^{p(x,y)/r^*} \\
& \geq \left(\frac{\delta}{2} \right)^{p(x,y)-1} \left[\int_{\Omega} (u_k + \delta)^{r^* \beta/p(x,y)} dx \right]^{p(x,y)/r^*} - \delta^{\beta+p(x,y)-1} |\Omega|^{p(x,y)/r^*}.
\end{aligned} \tag{4.11}$$

Hence, combining (4.9) and (4.10) with (4.11), we deduce

$$\|(u_k + \delta)^{\beta/p(x,y)}\|_{r^*}^{p(x,y)} \leq C_1 \left[C \left(\frac{2}{\delta} \right)^{p(x,y)-1} \left(\frac{(\beta + p^+ - 1)^{p^+}}{\beta (p^+)^{p^+}} \right) \|u\|_{r^*}^{q(x)-1} \|(u_k + \delta)^\beta\|_t + \delta^\beta |\Omega|^{p(x,y)/r^*} \right]. \tag{4.12}$$

Therefore, according to Lemma 4.1, we can obtain

$$\begin{aligned}
\delta^\beta |\Omega|^{p(x,y)/r^*} & \leq \left(\frac{(\beta + p^+ - 1)^{p^+}}{\beta (p^+)^{p^+}} \right) |\Omega|^{p(x,y)/r^*-1/t} \|(u_k + \delta)^\beta\|_t \\
& \leq \left(\frac{(\beta + p^+ - 1)^{p^+}}{\beta (p^+)^{p^+}} \right) |\Omega|^{p^+/r^*-1/t} \|(u_k + \delta)^\beta\|_t.
\end{aligned} \tag{4.13}$$

In view of (4.13) and (4.12), we obtain

$$\left\| (u_k + \delta) \frac{\beta}{p^+} \right\|_{r^*}^{p^+} \leq C_1 \left[\frac{1}{\beta} \left(\frac{\beta + p^+ - 1}{p^+} \right)^{p^+} \|(u_k + \delta)^\beta\|_t \left(\frac{C \|u\|_{r^*}^{q(x)-1}}{\delta^{p-1}} + |\Omega|^{p^+/r^*-1/t} \right) \right]. \tag{4.14}$$

There exist $\delta > 0$ such that $\delta^{p-1} = C \|u\|_{r^*}^{q(x)-1} (|\Omega|^{p^+/r^*-1/t})^{-1}$. Moreover, when $\beta \geq 1$, we find that $((\beta + p^* - 1)/p^*)^{p^*} \leq \beta^{p^*}$. And choosing suitable t and r^* such that $\eta = \frac{r^*}{tp^*} > 1$ and $\tau = t\beta$, we can change inequality (4.14) into

$$\|(u_k + \delta)\|_{\eta\tau} \leq (C |\Omega|^{p^+/r^*-1/t})^{t/\tau} \left(\frac{\tau}{t}\right)^{t/\tau} \|(u_k + \delta)\|_{\tau}. \quad (4.15)$$

Choose $\tau_0 = t$ and $\tau_{m+1} = \eta\tau_m = \eta^{m+1}t$. After using m iterations, one can obtain

$$\|(u_k + \delta)\|_{\tau_{m+1}} \leq (C |\Omega|^{p^+/r^*-1/t})^{(\sum_{i=0}^m t/\tau_i)} \left(\prod_{i=0}^m \left(\frac{\tau_i}{t}\right)^{t/\tau_i}\right)^{p^*-1} \|(u_k + \delta)\|_t. \quad (4.16)$$

According to $\eta > 1$, it is obvious that

$$\sum_{i=0}^{\infty} \frac{t}{\tau_i} = \sum_{i=0}^{\infty} \frac{1}{\eta^i} = \frac{\eta}{\eta - 1}$$

and

$$\prod_{i=0}^{\infty} \left(\frac{\tau_i}{t}\right)^{t/\tau_i} = \eta^{\eta/(\eta-1)^2}.$$

Hence, in view of (4.16), we can obtain

$$\|u_k\|_{\infty} \leq (C |\Omega|^{p^+/r^*-1/t})^{\eta/(\eta-1)} (C' \eta^{\eta/(\eta-1)^2})^{p^*-1} \|(u_k + \delta)\|_t \quad (4.17)$$

as $k \rightarrow \infty$. Moreover, by using the triangle inequality with $u_k \leq (u - 1)^+$ in (4.17), we can deduce that

$$\|u_k\|_{\infty} \leq C(\eta^{\eta/(\eta-1)^2})^{p^*-1} (|\Omega|^{p^+/r^*-1/t})^{\eta/(\eta-1)} (\|(u - 1)^+\|_t + \delta |\Omega|^{1/t}). \quad (4.18)$$

Then, letting $k \rightarrow \infty$ in (4.18), we obtain

$$\|(u - 1)^+\|_{\infty} \leq C(\eta^{\eta/(\eta-1)^2})^{p^*-1} (|\Omega|^{p^+/r^*-1/t})^{\eta/(\eta-1)} (\|(u - 1)^+\|_t + \delta |\Omega|^{1/t}). \quad (4.19)$$

This means that $u \in L^{\infty}(\Omega)$. Therefore, there is $T > 0$ such that $|u|_{L^{\infty}} \leq T$. $\square$

5 Proof of Theorem 1.1

Now, we are ready to prove Theorem 1.1.

Proof. According to Lemma 3.2, for all $\lambda > 0$ and $Q > 0$, there exists a $u_{\lambda} \in W^{s,p(\cdot,\cdot)}(\Omega)$ such that

$$J_{\lambda}^Q(u_{\lambda}) = C_{\lambda}^Q \quad \text{and} \quad (J_{\lambda}^Q)'(u_{\lambda}) = 0. \quad (5.1)$$

Then, we can choose $\varphi_1 \in W^{s,p(\cdot,\cdot)}(\Omega)$, $\varphi_1 > 0$ and $t_1 > 0$ large enough such that $J_{\lambda}^Q(t_1\varphi_1) < 0$. Furthermore, choosing $\psi_1 = t_1\varphi_1$, we define the function $\gamma : [0, 1] \rightarrow W^{s,p(\cdot,\cdot)}(\Omega)$ as $\gamma(t) = t\psi_1$. According to $\gamma \in \Gamma$ and $f_Q(x, \gamma(t)) = 0$ for every $x \in \Omega$ and $t \in [0, 1]$, we deduce

$$\begin{aligned} C_{\lambda}^Q &\leq \max_{t \in [0,1]} J_{\lambda}^Q(\gamma(t)) = \max_{t \in [0,1]} \left[T_{p(\cdot,\cdot)}(t\psi_1) - \int_{\Omega} \frac{1}{q(x)} |t\psi_1|^{q(x)} dx \right] \\ &\leq \max_{t \in [0,1]} \left[t^{p^+} T_{p(\cdot,\cdot)}(\psi_1) - \frac{t^{q^+}}{q^+} \int_{\Omega} |\psi_1|^{q(x)} dx \right] \\ &\leq \Lambda_{p,q}, \end{aligned} \quad (5.2)$$

where $\Lambda_{p,q}$ is a positive constant, and it is independent of λ and Q , and

$$\Lambda_{p,q} = \max_{t \in [0, +\infty)} \left[t^{p^+} T_{p(\cdot, \cdot)}(\psi_1) - \frac{t^{q^+}}{q^+} \int_{\Omega} |\psi_1|^{q(x)} dx \right].$$

Furthermore, in view of (5.1), Propositions 2.1 and 2.2, we can obtain that

$$\begin{aligned} C_{\lambda}^Q &= J_{\lambda}^Q(u_{\lambda}) - \frac{1}{\theta} \langle (J_{\lambda}^Q)'(u_{\lambda}), u_{\lambda} \rangle \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \iint_{\Omega \times \Omega} \frac{|u_{\lambda}(x) - u_{\lambda}(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy - \lambda \tau \int_{\Omega} |u_{\lambda}|^{p(x)} dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta} - \lambda \tau S_q \right) \iint_{\Omega \times \Omega} \frac{|u_{\lambda}(x) - u_{\lambda}(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy. \end{aligned} \quad (5.3)$$

If we set $\lambda_1 \in \left(0, \frac{\theta - p^+}{p^+ \theta \tau S_q}\right)$, then $\frac{1}{p^+} - \frac{1}{\theta} - \lambda \tau S_q \triangleq \frac{1}{\lambda_1} > 0$. Hence, in view of (5.2) and (5.3), we can deduce that

$$\begin{aligned} \Lambda_1 \Lambda_{p,q} &\geq \iint_{\Omega \times \Omega} \frac{|u_{\lambda}(x) - u_{\lambda}(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \\ &\geq \begin{cases} \|u_{\lambda}\|^{p^-}, & \text{if } \|u_{\lambda}\| > 1 \\ \|u_{\lambda}\|^{p^+}, & \text{if } \|u_{\lambda}\| \leq 1 \end{cases} \quad \text{for all } \lambda \in (0, \lambda_1). \end{aligned} \quad (5.4)$$

So, we obtain

$$\|u_{\lambda}\| \leq (\Lambda_1 \Lambda_{p,q})^{\frac{1}{p^-}} + (\Lambda_1 \Lambda_{p,q})^{\frac{1}{p^+}}.$$

According to Lemma 4.4, we can yield that

$$\|u_{\lambda}\|_{\infty} < T.$$

Let $\lambda_* = \{\lambda_0, \lambda_1\}$. Then, for any $\lambda \in (0, \lambda_1)$, u_{λ} is also a solution to Problem (1.1). $\square$

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