Research Article

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The essential norm of bounded diagonal infinite matrices acting on Banach sequence spaces

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Abstract: We calculate the essential norm of bounded diagonal infinite matrices acting on Köthe sequence spaces. As a consequence of our result, we obtain a recent criteria for the compactness of multiplication operator acting on Köthe sequence spaces.

Keywords: diagonal operators, Banach sequence spaces, compact operators, essential norm

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1 Introduction

The formal study of infinite matrices began with the work of Poincaré in 1884 [1], who gave the rigorous bases for the use of infinite matrices for the calculus of the determinants. Poincaré's results provided Hilbert [2] with the necessary tools to study the eigenvalues of certain integral operators, which can be viewed as infinite matrices operating on specific spaces. This theory enables the study of systems of infinite differential equations such as we can see in the work of Cooke [3]. A walk about the history of infinite matrices can be seen in the excellent article of Bernkopf [4], while the modern approach of this topics and its applications can be found in the article of Shivakumar and Sivakumar [5] and the book of Shivakumar et al. [6].

A complex infinite matrix is denoted by $A = (a_{ij})$, and it is a function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. As usual, the product of an infinite matrix A by the sequence $\mathbf{x} = \{x(n)\}$, denoted as $\mathbf{y} = A \times \mathbf{x}$, is the sequence $\mathbf{y} = \{y(n)\}$ defined by

$$y(n) = \sum_{j=1}^{\infty} a_{n,j} x(j),$$

whenever the aforementioned series is convergent for all $n \in \mathbb{N}$. Thus, if $A \times \mathbf{x}$ is calculable for all $\mathbf{x}$ in a vector space $\mathbb{V}$, then the set

$$\mathbb{W} = \{A \times \mathbf{x} : \mathbf{x} \in \mathbb{V}\} \tag{1}$$

is also a vector space, and we could define the linear operator $T : \mathbb{V} \to \mathbb{W}$ by $T_A(\mathbf{x}) = A \times \mathbf{x}$. So far, we can ask for the following:

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Given a vector space $\mathbb{V}$, what is the vector space $\mathbb{W}$ defined in (1)?.

This last question is related to the well known problem of the multipliers between vector spaces. If the vector spaces $\mathbb{V}$ and $\mathbb{W}$ are Banach spaces, then we can define the norm of the infinite matrix A as the norm of the operator T_A; that is,

$$\|A\|_{\mathbb{V}\to\mathbb{W}} := \sup\left\{\frac{\|A \times \mathbf{x}\|_{\mathbb{W}}}{\|\mathbf{x}\|_{\mathbb{V}}} : \|\mathbf{x}\|_{\mathbb{V}} \neq 0\right\}, \tag{2}$$

where $\|\mathbf{x}\|_{\mathbb{V}}$ denotes the norm of the sequence $\mathbf{x} \in \mathbb{V}$. In this context, we are interested in finding conditions on the infinite matrix A to the supremum in the right side of (2) will be finite; that is, when is $T_A : \mathbb{V} \to \mathbb{W}$ a continuous operator? and, in this case, when is $T_A : \mathbb{V} \to \mathbb{W}$ a compact operator? and, how much is the essential norm of T_A?

Recently, interesting works have appeared that answer the questions in the previous paragraph for certain classes of sequence spaces. Infinite matrices that maps a weighted $l^1(\mathbf{r})$ space into another of the same kind $l^1(\mathbf{s})$ were characterized by Williams and Ye [7], they showed that if A is an infinite matrix, $\mathbf{r}$ and $\mathbf{s}$ are weight sequences (i.e., $r(n) > 0$ for all $n \in \mathbb{N}$), then T_A maps $l^1(\mathbf{r})$ into $l^1(\mathbf{s})$ if and only if

$$\sup_{j\in\mathbb{N}} \sum_{i=1}^{\infty} \frac{s(i)}{r(j)} |a_{i,j}| < \infty,$$

and the most general case when T_A maps $l^p(\mathbf{r})$ into $l^q(\mathbf{s})$ is still an open problem. The infinite matrices that map a l^p space into a Cesàro space ces_p were studied by Foroutannia and Roopaei in [8]. It is of interest to characterize the infinite matrices that define bounded linear operators between certain spaces of sequences and to study other topological properties such as compactness and closedness of the range.

In this article, we consider the case when A is an infinite diagonal matrix acting on the general setting of Köthe sequence spaces, which include $l^p(\mathbf{r})$ spaces, Orlicz sequence spaces, Lorentz sequence spaces, and Cesàro sequence spaces among others. Given a sequence $\mathbf{u} = \{u(n)\}$, the infinite matrix diagonal with entries $\mathbf{u}$, denoted by $A = \mathrm{diag}(\mathbf{u})$, is the matrix $A = (a_{i,j})$ such that $a_{i,j} = u(i)$ for $i = j$ and $a_{i,j} = 0$ for $i \neq j$. In this case, the operator $T_A = T_{\mathrm{diag}(\mathbf{u})}$ is known as the multiplication operator with symbol $\mathbf{u}$, and it is denoted by M_u and satisfies that

$$M_u\mathbf{x} = \mathrm{diag}(\mathbf{u}) \times \mathbf{x} = \mathbf{y} = \{u(n)\cdot x(n)\}$$

for all $\mathbf{x} \in \mathbb{V}$. This operator is well known and continues to be the subject of extensive study by a large number of researchers. In the context of Banach spaces of sequences, we can mention that if $\mathbb{V}$ is a Köthe-type sequence space (see the next section for the definition and properties of Köthe sequence spaces), then M_u maps $\mathbb{V}$ into itself if and only if $\mathbf{u} \in l^\infty$ (see, for instances, [9]), and more recently, Ramos-Fernández and Salas-Brown [10] have proven that, in this case, $M_u : \mathbb{V} \to \mathbb{V}$ is a compact operator if and only if $\mathbf{u} \in c_0$, that is, $u_n \to 0$ as $n \to \infty$. Similar results have been obtained by various authors in different Banach sequence spaces, such as Lorentz sequence spaces [11], Orlicz-Lorentz sequence spaces [12,13], Cesàro sequence spaces [14], Cesàro-Orlicz sequence spaces [15], among others.

In the context of compactness, one of the objectives is to estimate the essential norm of an operator or the non-compactness measure. Given a Banach space $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$, the essential norm of a continuous linear operator $T : \mathbb{V} \to \mathbb{V}$, denoted by $\|T\|_e$, is defined as the distance to the class $\mathcal{K}(\mathbb{V})$ of all compact operators on $\mathbb{V}$, that is,

$$\|T\|_e = \inf\{\|T - K\| : K \in \mathcal{K}(\mathbb{V})\},$$

where $\|T\|$ denotes the operator norm of T. Since $\mathcal{K}(\mathbb{V})$ is a closed subspace of $\mathcal{B}(X)$, the class of all bounded linear operator (with the topology induced by the operator norm), we can see that an operator $T \in \mathcal{B}(\mathbb{V})$ is compact if and only if $\|T\|_e = 0$. Recently, interesting works have appeared about the essential norm of multiplication operators acting on Banach sequences spaces. In the case of Lorentz sequence spaces $l_{(p,q)}$, Castillo et al. in [16] proved that the essential norm of the operator $M_u : l_{(p,q)} \to l_{(p,q)}$ is given by

$$\|M_u\|_e = \limsup_{n\to\infty} |u(n)|,$$

which implies the result of Arora et al. in [11]. Ramos-Fernández et al. [17] showed that the aforementioned result is also valid for Cesàro sequence spaces, and more recently, Ramos-Fernández and Salas-Brown [18] have proven that the same is still valid for Orlicz sequence spaces. The main objective of this article is to prove that the result obtained by Castillo et al. in [16] is valid for the more general setting of Köthe-type spaces, which generalize all the results mentioned in this paragraph, and the proof of this aim is given in Section 3. The article is completed in Section 2 where we gather some properties of Köthe sequence spaces.

2 Some remarks about Köthe sequence spaces

In this section, we use the notation $\mathbf{a} = \{a(k)\}$ for a sequence of complex number, while $(\mathbf{x}_n)$ will denote a sequence of sequences. With these notations, a Banach space of sequence $(X, \|\cdot\|)$ is a Köthe sequence space if it satisfies:

(i) If $\mathbf{a}, \mathbf{b}$ are sequences with $|a(k)| \le |b(k)|$ for all $k \in \mathbb{N}$ and $\mathbf{b} \in X$, then $\mathbf{a} \in X$ and $\|\mathbf{a}\|_X \le \|\mathbf{b}\|_X$.

(ii) If $A = \{n_1, n_2, \ldots, n_m\}$ is a finite subset of $\mathbb{N}$, then the sequence $\mathbf{1}_A$ defined by

$$\mathbf{1}_A(k) = \begin{cases} 1, & k \in A, \\ 0, & k \notin A, \end{cases}$$

belongs to X.

Observe that condition (ii) is equivalent to say that X contains the canonical basis $(\mathbf{e}_n)$, where, as is usual, $\mathbf{e}_n = \mathbf{1}_{\{n\}}$ is the sequence with 1 at the nth position and $e_n(k) = 0$ for all $k \neq n$. It is important to remark that two sequences $\mathbf{a}$ and $\mathbf{b}$ are equal if and only if $a(k) = b(k)$ for all $k \in \mathbb{N}$. The null sequence is $\mathbf{0} = (0, 0, \ldots)$. The spaces l^p with $1 \le p \le \infty$, c_0, the space of all sequences converging to zero, and the Lorentz sequence spaces $l_{(p,q)}$ with $1 < p \le \infty$ and $1 \le q \le \infty$ are important examples of Köthe sequence spaces. The study of Köthe sequence spaces began with the works of Köthe and Toeplitz in [19] and later by Köthe in [20,21]. Their theory has been generalized by Dieudonné [22], Cooper [23], and Lorentz and Wertheim [24].

The Köthe dual X' of the Köthe sequence space X is a Banach space, which can be identified with the space of all functionals possessing a series representation, that is, $\mathbf{c} \in X'$ if and only if

$$\|\mathbf{c}\|_{X'} \coloneqq \sup_{\|\mathbf{a}\|_X \le 1} \sum_{k=1}^{\infty} |c(k)a(k)| < +\infty.$$

The space $(X', \|\cdot\|_{X'})$ is also a Köthe sequence space, and if $\mathbf{c} \in X'$, then the relation $h_c : X \to \mathbb{C}$ defined by

$$h_c(\mathbf{a}) = \sum_{k=1}^{\infty} c(k)a(k)$$

is a bounded functional on X and $\|h_c\| = \|\mathbf{c}\|_{X'}$. It is known that if X has order continuous norm (i.e., $\|\mathbf{x}_n\|_X \to 0$ for all sequences $(\mathbf{x}_n) \subset X$ such that $\mathbf{x}_n \downarrow 0$), then the dual space X^* can be naturally identified with X' [25]. In particular, for each $n \in \mathbb{N}$, the canonical sequence $\mathbf{e}_n$ belongs to X', and we have the following useful property. We include its proof for benefit of the reader.

Proposition 1. *For each $n \in \mathbb{N}$, we have*

$$\|\mathbf{e}_n\|_X \|\mathbf{e}_n\|_{X'} = 1.$$

Proof. Suppose that $\mathbf{a} = \{a(n)\} \in X$ satisfies $\|a\|_X \le 1$, then

$$|a(k)e_n(k)| \le |a(k)|$$

for all $k \in \mathbb{N}$, and hence, the sequence $\mathbf{a} \cdot \mathbf{e}_n = \{a(k)e_n(k)\}$ belongs to X and also

$$\|\mathbf{a}\cdot\mathbf{e}_n\|_X \le \|\mathbf{a}\|_X \le 1.$$

Thus, $|a(n)|\|\mathbf{e}_n\|_X = \|\mathbf{a}\cdot\mathbf{e}_n\|_X \le 1$, and so

$$\sum_{k=1}^{\infty}|a(k)||e_n(k)| = |a(n)| \le \frac{1}{\|e_n\|_X}$$

for all $a \in X$ such that $\|\mathbf{a}\|_X \le 1$, which in turn means that

$$\|\mathbf{e}_n\|_{X'} = \sup_{\|\mathbf{a}\|_X\le 1}\sum_{k=1}^{\infty}|a(k)||e_n(k)| \le \frac{1}{\|\mathbf{e}_n\|_X};$$

that is, $\|\mathbf{e}_n\|_X\|\mathbf{e}_n\|_{X'} \le 1$. Furthermore, for $n \in \mathbb{N}$ fixed, the sequence

$$\mathbf{c} = \frac{\mathbf{e}_n}{\|\mathbf{e}_n\|_X}$$

belongs to X and satisfies that $\|\mathbf{c}\|_X = 1$; thus,

$$\|\mathbf{e}_n\|_{X'} \ge \sum_{k=1}^{\infty}|c(k)||e_n(k)| = \frac{1}{\|\mathbf{e}_n\|_X},$$

and we obtain $\|\mathbf{e}_n\|_X\|\mathbf{e}_n\|_{X'} = 1$. □

3 Main result

The aim of this section is to obtain an expression for the essential norm of the multiplication operator acting on Köthe sequence spaces, which implies the results obtained by Castillo et al. [16], Ramos-Fernández et al. [17], and Ramos-Fernández and Salas-Brown [18] among others. The essential norm of the bounded operator is given by

$$\|T\|_e = \inf\{\|T-K\| : K \in \mathcal{K}(X)\},$$

where $\mathcal{K}(X)$ denotes the class of all compact operators defined on X and $\|T\|$ is the norm of the operator $T : X \to X$. In particular, the operator $T : X \to X$ is compact if and only if $\|T\|_e = 0$; and hence, we are going to obtain a result that implies, in many cases, the following theorem due to Ramos-Fernández and Salas-Brown [10]:

Theorem 1. [10] *Suppose that $\mathbf{u} \in l^\infty$ and let $(X, \|\cdot\|_X)$ be a Köthe sequence space, the multiplication operator $M_u : X \to X$ is compact if and only if*

$$\lim_{n\to\infty}|u(n)| = 0.$$

Also, it is convenient to recall that a sequence $(\mathbf{x}_n) \subset X$ is a Schauder basis for the Banach space $(X, \|\cdot\|_X)$ if for every $\mathbf{x} \in X$ we can find a complex sequence $\mathbf{a} = \{a(k)\}$ such that

$$\mathbf{x} = \sum_{k=1}^{\infty}a(k)\mathbf{x}_k;$$

this finally means that

$$\lim_{H\to\infty}\left\|\mathbf{x} - \sum_{k=1}^{H}a(k)\mathbf{x}_k\right\|_X = 0.$$

With these notations, we have the following result:

Theorem 2. (Main) *Suppose that* $\mathbf{u} \in l^\infty$ *and let* $(X, \|\cdot\|_X)$ *be a Köthe sequence space that has order continuous norm and such that the canonical sequence* $(\mathbf{e}_n)$ *is a Schauder basis for* X'. *Then for the multiplication operator* $M_u : X \to X$, *we have*

$$\|M_u\|_e = \limsup_{n\to\infty} |u(n)|. \tag{3}$$

The proof of Theorem 2 is divided into four steps:

Step 1. Establish $\limsup_{n\to\infty}|u(n)|$ is an upper bound for $\|M_u\|_e$.

For any $N \in \mathbb{N}$, we consider the set $A_N = \{1, 2, \ldots, N\}$ and we define the sequence $\mathbf{u}_N = \mathbf{u} \cdot \mathbf{1}_{A_N} = \{u(k)\mathbf{1}_{A_N}(k)\}$, then $\mathbf{u}_N \in l^\infty$ and the operator $M_{u_N} : X \to X$ is continuous. Furthermore,

$$\lim_{n\to\infty} |u_N(n)| = 0$$

and by Theorem 1, the operator $M_{u_N} : X \to X$ is also compact. Thus, by definition of the essential norm, we can write

$$\|M_u\|_e \le \|M_u - M_{u_N}\| = \|M_{u-u_N}\|.$$

Next, for any sequence $\mathbf{a} = \{a(k)\} \in X$ such that $\|\mathbf{a}\|_X = 1$ and for any $k \in \mathbb{N}$, we have

$$|(u(k) - u_N(k))\cdot a(k)| \le S_N |a(k)|,$$

where

$$S_N = \sup\{|u(k)| : k \ge N\}.$$

Hence, the sequence $(\mathbf{u} - \mathbf{u}_N)\cdot \mathbf{a} = \{(u(k) - u_N(k))\cdot a(k)\}$ belongs to X and

$$\|(\mathbf{u} - \mathbf{u}_N)\cdot \mathbf{a}\|_X \le S_N \|\mathbf{a}\|_X.$$

Therefore,

$$\|M_{u-u_N}(\mathbf{a})\|_X = \|(u - u_N)\cdot \mathbf{a}\|_X \le S_N \|\mathbf{a}\|_X \le S_N,$$

and we have $\|M_u\|_e \le S_N$ for any $N \in \mathbb{N}$, which proves that

$$\|M_u\|_e \le \limsup_{n\to\infty} |u(n)|.$$

This give us the upper bound for $\|M_u\|_e$.

Step 2. Establish that X' is contained into a weighted c_0 space.

The following property is a generalization of Proposition 1 in [17].

Lemma 1. *Let* $(X, \|\cdot\|_X)$ *be a Köthe sequence space such that the canonical sequence* $(\mathbf{e}_n)$ *is a Schauder basis for* X'. *Then for any sequence* $\mathbf{c} = \{c(k)\} \in X'$, *we have*

$$\lim_{n\to\infty} \frac{1}{\|\mathbf{e}_n\|_X} |c(n)| = \lim_{n\to\infty} \|\mathbf{e}_n\|_{X'} |c(n)| = 0.$$

Proof. Let $\mathbf{c} = \{c(k)\}$ be any sequence in X'. Since $(\mathbf{e}_n)$ is a Schauder basis for X', we have

$$\lim_{n\to\infty} \left\| \sum_{m=n}^{\infty} c(m)\mathbf{e}_m \right\|_{X'} = 0.$$

Next, we can observe that for $n \in \mathbb{N}$ fixed, and for all $k \in \mathbb{N}$, the following inequality holds

$$|c(n)\mathbf{e}_n(k)| \le \left| \sum_{m=n}^{\infty} c(m)\mathbf{e}_m(k) \right|,$$

and since X' is a Köthe sequence space, we obtain

$$|c(n)|\|\mathbf{e}_n\|_{X'} = \|c(n)\mathbf{e}_n\|_{X'} \leq \left\| \sum_{m=n}^{\infty} c(m)\mathbf{e}_m \right\|_{X'}.$$

This last fact implies the result. □

Step 3. Establish that the normalized canonical basis converges weakly to zero.

Next we shall go to prove the following result:

Lemma 2. *Let $(X, \|\cdot\|_X)$ be a Köthe sequence space, which has an order continuous norm such that the canonical sequence $(\mathbf{e}_n)$ is a Schauder basis for X'. Then the sequence $(\mathbf{f}_n) \subset X$ defined by*

$$\mathbf{f}_n = \frac{1}{\|\mathbf{e}_n\|_X}\mathbf{e}_n \tag{4}$$

converges weakly to zero. In particular,

$$\lim_{n\to\infty} \|K(\mathbf{f}_n)\|_X = 0 \tag{5}$$

for all compact operator $K : X \to X$.

Proof. Let h be any bounded functional defined on X. Since X has an order continuous norm, there exists a sequence $\mathbf{c} \in X'$ such that

$$h(\mathbf{a}) = \sum_{k=1}^{\infty} c(k)a(k)$$

for all sequence $\mathbf{a} \in X$. In particular,

$$|h(\mathbf{f}_n)| = \frac{|c(n)|}{\|\mathbf{e}_n\|_X},$$

and the weak convergence follows from Proposition 1.

Furthermore, it is well known that every compact operator $K : X \to X$ is completely continuous (see Chapter 6, Section 3, Proposition 3.3 in [26]), and hence, $\|K(\mathbf{f}_n)\|_X \to 0$ as $n \to \infty$. This concludes the proof of the result. □

Step 4. Establish that $\limsup_{n\to\infty}|u(n)|$ is a lower bound for $\|M_u\|_e$.

Now we can finish the proof of the main result. Since $\|\mathbf{f}_n\|_X = 1$ for all $n \in \mathbb{N}$, then by definition of the operator norm and for any compact operator $K : X \to X$, we can write

$$\begin{aligned}
\|M_u - K\| &\geq \|M_u(\mathbf{f}_n) - K(\mathbf{f}_n)\|_X \\
&\geq \left\| M_u\left(\frac{\mathbf{e}_n}{\|\mathbf{e}_n\|_X}\right) \right\|_X - \|K(\mathbf{f}_n)\|_X \\
&= \frac{1}{\|\mathbf{e}_n\|_X} \|u \cdot \mathbf{e}_n\|_X - \|K(\mathbf{f}_n)\|_X \\
&= |u(n)| - \|K(\mathbf{f}_n)\|_X.
\end{aligned}$$

Therefore, by taking limit when n goes to infinite, we obtain

$$\|M_u - K\| \geq \limsup_{n\to\infty}|u(n)|,$$

where we have used relation (5). The result follows since the compact operator $K : X \to X$ was arbitrary.

By combining Step 1 with this last step, it follows

$$\|M_u\|_e = \limsup_{n\to\infty}|u(n)|,$$

which completes the proof of Theorem 2.

As an immediate consequence of our result, we have (see [10,13–18]).

Corollary 1. *Suppose that* $\mathbf{u} \in l^\infty$ *and let* $(X, \|\cdot\|_X)$ *be a Köthe sequence space, which has an order continuous norm, and such that the canonical sequence* $(\mathbf{e}_n)$ *is a Schauder basis for* X'*. Then the multiplication operator* $M_u : X \to X$ *is compact if and only if*

$$\limsup_{n\to\infty} |u(n)| = 0. \tag{6}$$

Proof. This follows from the fact that the operator $T : X \to X$ is compact if and only if $\|T\|_e = 0$. □

Final comment. There is an important result due to Axler et al. [27] about the essential norm of operators from a Banach space X into c_0. However, there are a lot of Köthe sequence spaces X, which are not subspace of c_0; for instance, if $\omega = \{\omega(n)\}$ is a sequence weight such that $\omega(n) > 0$ for all $n \in \mathbb{N}$, then for $p > 1$, the weighted l^p space, denoted by $l^p(\omega)$, of all complex sequences $\boldsymbol{x} = \{x(n)\}$ such that

$$\|\boldsymbol{x}\|_{l^p(\omega)} = \left(\sum_{n=1}^{\infty} |x(n)|^p \omega(n)\right)^{\frac{1}{p}} < \infty,$$

in general, is not contained in c_0. In these kinds of spaces, our Theorem 2 can be applied. Furthermore, our findings are applicable to various Banach sequence spaces, including c_0, l^p spaces with $1 < p < \infty$, Orlicz and Lorentz spaces, and Cesàro spaces among others, and it has as consequence the compactness results in [10,13–18]. Our result can not be applied to any Banach sequence space; for instance, if we put $X = l^1$, then $X' = X^* = l^\infty$, and the canonical basis $(\mathbf{e}_n)$ is not a Schauder basis for l^∞ because this space is not separable.

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