Research Article

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Hahn Laplace transform and its applications

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Abstract: Like q-calculus, Hahn calculus (or q, ω-calculus) is constructed by defining a difference derivative operator and an integral operator. The q, ω-analogs of the integral representations of the Laplace transform and related special functions, such as gamma and beta, are proposed in this article. Then, some basic properties similar to classical and q-analogs are investigated. Finally, a few examples are given to solve q, ω-initial value problems via the newly introduced q, ω-Laplace transform.

Keywords: q, ω-gamma function, q, ω-beta function, q, ω-Laplace transform, q, ω-differential equations, Hahn calculus, q-calculus

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1 Introduction

Quantum calculus (q-calculus) is based on the derivative defined by the difference ratio without limit. It has essential applications in mathematics, engineering and mathematical physics such as quantum chromodynamics, quantum theory, the theory of relativity, basic hypergeometric functions, string theory, etc. (see [1–4]). One type of q-calculus is Hahn calculus (q, ω-calculus). Let $q \in (0,1)$, $\omega > 0$, $\omega_0 := \frac{\omega}{1-q}$, and I be an interval of $\mathbb{R}$ containing ω_0. The Hahn difference operator (see [5,6]) of f, which is defined on I, is as follows:

$$D_{q,\omega}f(x) := \begin{cases} \dfrac{f(qx+\omega)-f(x)}{(qx+\omega)-x}, & x \neq \omega_0, \\ f'(\omega_0), & x = \omega_0. \end{cases} \tag{1}$$

After the definition of q, ω-integral in the study by Annaby et al. [7], interest in studies on the Hahn calculus has increased. Many physical and mathematical problems in classical and q-calculus are extended to Hahn calculus. The theory and applications of q, ω-differential equations were examined by Hamza and Ahmed [8,9]. The q, ω-Sturm-Liouville problem was presented by Annaby et al. [10], and the sampling theory for the same problem was constructed by Annaby and Hassan [11]. Fractional Hahn calculus was investigated in [12–14], and Taylor theory based on Hahn's operator was established by Oraby and Hamza [15]. Hıra [16] presented the q, ω-Dirac system.

Different definitions and applications of the q-Laplace transform defined using the q-exponential functions $e_q(x)$ and $E_q(x)$ can be seen in [5,17–20]. The answer to the question of how to apply the q-Laplace transform to the q-differential equations was given in [6,21,22]. A few studies on the q-Sumudu transform associated with the q-Laplace transform can be cited as [23–25] (also cited therein). Researchers can find studies on fractional calculus on the timescale and its q-analog in [26] and [27], respectively. We also refer to the study by Sheng and Zhang [28] for solving some fractional q-differential equations, by Hajiseyedazizi et al. [29] q-integrodifferential equations, by Samei et al. [30] for singular fractional q-differential equations, and by

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Sheng and Zhang [31] for numerical solutions of the q-fractional boundary value problems. As it is known, there are some relations between the beta and the gamma functions, and some properties of the Laplace transform are obtained from the gamma function. Similarly, q-analogs of these functions were investigated in [2,3,32].

This study deals with the q, ω-analogs of the Laplace transform, gamma, and beta functions. First, we define integral representations of q, ω-gamma function $\Gamma_{q,\omega}(t)$ (Definition 1) and q, ω-beta function $B_{q,\omega}(t, s)$ (Definition 2) and obtain some of their properties and relations. Then, we define q, ω-Laplace transform $F_{q,\omega}(s)$ (Definition 3) for suitable function f. As an alternative method, we give q, ω-Laplace transform of f function with the help of q, ω-Taylor series expansion (Theorem 2), then obtain the q, ω-analogs of the basic properties of the Laplace transform, such as linearity, scaling, shifting, and convolution. Finally, we give some examples for solving q, ω-initial value problems via the q, ω-Laplace transform.

Basic definitions and features in q-calculus and q, ω-calculus mentioned throughout this article can be found in [3,7,33,34], so we will not repeat them here. Since the Hahn difference operator is a combination of the q-difference operator D_q and the forward difference operator Δ_ω (see [1,3,4,35,36]), the definitions and properties given in the following sections for q, ω-calculus are reduced to their q-analogs as $\omega \to 0$, and their classical equivalents as $\omega \to 0$ and $q \to 1$.

Shehata et al. [37] investigated the Laplace transform associated with the general quantum difference operator D_β (see resources there for details). From the difference operator D_β, the difference operator $D_{q,\omega}$ is obtained by a special selection of some functions. Therefore, the study may appear to include our study. However, in both studies, different integral definitions are made, and the proof methods are also different. The limits of the integral representations for the q, ω-gamma function and the q, ω-Laplace transform in our study are consistent with those in the q-calculus for $\omega \to 0$ and the classical forms for $\omega \to 0$, $q \to 1$.

2 q, w-gamma and q, w-beta functions

In [2,3, 36,38], the q-analogs of beta and gamma functions were introduced as the infinite product and then the integral representations of them were given in [38–40]. One of them is given as follows (see [39]):

$$\Gamma_q(t) := \int_0^{[\infty]_q} x^{t-1} E_q(-qx) d_q x, \tag{2}$$

and

$$B_q(t, s) := \int_0^1 x^{t-1} (1 - qx)_q^{s-1} d_q x, \tag{3}$$

where $[\alpha]_q = \frac{1-q^\alpha}{1-q}$, $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ for $\alpha = n \in \mathbb{N}$, and $[\infty]_q = \frac{1}{1-q}$ for $q \in (0, 1)$.

Now, we define q, ω-gamma and q, ω-beta functions based on the q, ω-exponential function $E_{q,\omega}(x)$. Then, we obtain q, ω-analogs of some properties.

Definition 1. For any $t > 0$,

$$\Gamma_{q,\omega}(t) := \int_{\omega_0}^{\omega_0+[\infty]_q} (x - \omega_0)^{t-1} E_{q,-\omega}(-(qx + \omega)) d_{q,\omega} x \tag{4}$$

is called the q, ω-gamma function.

Remark 1. If $u = x - \omega_0$ is changed in equation (4), then $\Gamma_{q,\omega}(t) \equiv \Gamma_q(t)$ is obtained. This is equivalent to the reduction of equation (4) to the q-analog in equation (2) for $\omega \to 0$ (in this case, $u = x$).

From equation (4) and using the q, ω-integration by parts, we have

$$\begin{aligned}\Gamma_{q,\omega}(t+1) &= \int_{\omega_0}^{\omega_0+[\infty]_q} (x-\omega_0)^t E_{q,-\omega}(-(qx+\omega))d_{q,\omega}x\\ &= -\int_{\omega_0}^{\omega_0+[\infty]_q} (x-\omega_0)^t D_{q,\omega}E_{q,-\omega}(-x)d_{q,\omega}x\\ &= -((x-\omega_0)^t E_{q,-\omega}(-x))|_{\omega_0}^{\omega_0+[\infty]_q} + \int_{\omega_0}^{\omega_0+[\infty]_q} E_{q,-\omega}(-(qx+\omega))D_{q,\omega}(x-\omega_0)^t d_{q,\omega}x\\ &= [t]_q \int_{\omega_0}^{\omega_0+[\infty]_q} (x-\omega_0)^{t-1} E_{q,-\omega}(-(qx+\omega))d_{q,\omega}x,\end{aligned}$$

where $(1-a)_q^n \coloneqq (a;q)_n \coloneqq \prod_{i=0}^{n-1}(1-aq^i)$ and $(1-a)_q^0 \coloneqq (a;q)_0 = 1$; $E_{q,\omega}(x) \coloneqq \sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}(x-\omega_0)^n}{[n]_q!} = (1+(x(1-q)-\omega))_q^{\infty}$; $E_{q,-\omega}(-\omega_0)=1$, $E_{q,-\omega}(-(\omega_0+[\infty]_q))=0$, $D_{q,\omega}E_{q,\omega}(x)=E_{q,\omega}(qx+\omega)$, $[n]_q! = [1]_q[2]_q\ldots[n]_q$, and $D_{q,\omega}(x-\omega_0)^n = [n]_q(x-\omega_0)^{n-1}$. Hence, for any $t>0$, we obtain

$$\Gamma_{q,\omega}(t+1) = [t]_q\Gamma_{q,\omega}(t), \tag{5}$$

and for $t = n \in \mathbb{N}$, we have

$$\Gamma_{q,\omega}(n+1) = [n]_q!, \quad \Gamma_{q,\omega}(1) = 1. \tag{6}$$

Definition 2. For any $t, s > 0$,

$$B_{q,\omega}(t,s) \coloneqq \int_{\omega_0}^{\omega_0+1} (x-\omega_0)^{t-1}(1-q(x-\omega_0))_q^{s-1} d_{q,\omega}x \tag{7}$$

is called the q,ω-beta function.

Theorem 1. *The two relations between the q,ω-gamma and the q,ω-beta functions are given as follows:*

$$\Gamma_{q,\omega}(t) = \frac{B_{q,\omega}(t,\infty)}{(1-q)^t}, \tag{8}$$

$$B_{q,\omega}(t,s) = \frac{\Gamma_{q,\omega}(t)\Gamma_{q,\omega}(s)}{\Gamma_{q,\omega}(t+s)}. \tag{9}$$

Proof. If we put $s=\infty$ in Definition 2, use the definition $E_{q,\omega}(x) = (1+(x(1-q)-\omega))_q^{\infty}$ and change of variable $x-\omega_0 = (1-q)(u-\omega_0)$, we obtain

$$\begin{aligned}B_{q,\omega}(t,\infty) &= \int_{\omega_0}^{\omega_0+1} (x-\omega_0)^{t-1}(1-q(x-\omega_0))_q^{\infty} d_{q,\omega}x\\ &= (1-q)^t \int_{\omega_0}^{\omega_0+[\infty]_q} (u-\omega_0)^{t-1} E_{q,-\omega}(-(qu+\omega))d_{q,\omega}u\\ &= (1-q)^t\Gamma_{q,\omega}(t),\end{aligned}$$

which proves equation (8).

From equation (7), we obtain the following recurrence relations for $B_{q,\omega}(t,s)$:

$$
\begin{aligned}
B_{q,\omega}(t+1,s) &= \int_{\omega_0}^{\omega_0+1} (x-\omega_0)^t (1-q(x-\omega_0))_q^{s-1} d_{q,\omega}x \\
&= -\frac{1}{[s]_q} \int_{\omega_0}^{\omega_0+1} (x-\omega_0)^t D_{q,\omega}(1-(x-\omega_0))_q^s d_{q,\omega}x \\
&= -\frac{1}{[s]_q}((x-\omega_0)^t(1-(x-\omega_0))_q^s)\Big|_{\omega_0}^{\omega_0+1} + \frac{1}{[s]_q}\int_{\omega_0}^{\omega_0+1}(1-(qx+\omega-\omega_0))_q^s D_{q,\omega}(x-\omega_0)^t d_{q,\omega}x \\
&= \frac{[t]_q}{[s]_q}\int_{\omega_0}^{\omega_0+1}(x-\omega_0)^{t-1}(1-q(x-\omega_0))_q^s d_{q,\omega}x \\
&= \frac{[t]_q}{[s]_q} B_{q,\omega}(t,s+1),
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
B_{q,\omega}(t,s+1) &= \int_{\omega_0}^{\omega_0+1} (x-\omega_0)^{t-1}(1-q(x-\omega_0))_q^s d_{q,\omega}x \\
&= \int_{\omega_0}^{\omega_0+1} (x-\omega_0)^{t-1}(1-q(x-\omega_0))_q^{s-1}(1-q^s(x-\omega_0)) d_{q,\omega}x \\
&= \int_{\omega_0}^{\omega_0+1} (x-\omega_0)^{t-1}(1-q(x-\omega_0))_q^{s-1} d_{q,\omega}x - q^s \int_{\omega_0}^{\omega_0+1} (x-\omega_0)^t (1-q(x-\omega_0))_q^{s-1} d_{q,\omega}x \\
&= B_{q,\omega}(t,s) - q^s B_{q,\omega}(t+1,s).
\end{aligned} \tag{11}
$$

From equations (10) and (11), we obtain

$$B_{q,\omega}(t,s+1) = \frac{[s]_q}{[t+s]_q} B_{q,\omega}(t,s). \tag{12}$$

By substituting equation (11) in equation (10), we arrive at

$$B_{q,\omega}(t+1,s) = \frac{[t]_q}{[t+s]_q} B_{q,\omega}(t,s). \tag{13}$$

By comparing equations (5) and (13), we have that equation (9) holds for any $t, s > 0$. □

3 q, w-Laplace transform

In the literature, several different definitions have been made for the q-Laplace transform using $e_q(x)$ and $E_q(x)$ (see [6,17–21]). In the study by Kobachi [18], the q-Laplace transform of f is defined as follows:

$$L_q(f)(s) = \frac{1}{s}\int_0^{[\infty]_q} E_q(-qx) f\left(\frac{x}{s}\right) d_q x, \tag{14}$$

and if the function f has a series expansion of $f(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{[n]_q!}$, then its q-Laplace transform is given by $L_q(f)(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}}$.

Now we define q, ω-analog of the Laplace transform and investigate some of its basic properties. Let $\mathbf{V} \coloneqq \mathbf{V}([\omega_0, \omega_0 + [\infty]_q), \mathbb{C})$ be the set of q, ω-integrable functions compact on the subintervals of $[\omega_0, \omega_0 + [\infty]_q)$.

Definition 3. Let $f \in \mathbf{V}$. Then,

$$L_{q,\omega}(f)(s) := F_{q,\omega}(s) := \frac{1}{s-\omega_0} \int_{\omega_0}^{\omega_0+[\infty]_q} E_{q,-\omega}(-(qx+\omega)) f\left(\frac{x}{s-\omega_0}\right) d_{q,\omega}x \tag{15}$$

is called the q, ω-Laplace transform of f.

Example 1. For $f(x) = (x-\omega_0)^n$, $n \in \mathbb{N}$, we have

$$\begin{aligned} L_{q,\omega}((x-\omega_0)^n) &= \frac{1}{s-\omega_0} \int_{\omega_0}^{\omega_0+[\infty]_q} E_{q,-\omega}(-(qx+\omega)) \left(\frac{x-\omega_0}{s-\omega_0}\right)^n d_{q,\omega}x \\ &= \frac{1}{(s-\omega_0)^{n+1}} \int_{\omega_0}^{\omega_0+[\infty]_q} E_{q,-\omega}(-(qx+\omega))(x-\omega_0)^n d_{q,\omega}x. \end{aligned}$$

From equations (4) and (6), we obtain

$$L_{q,\omega}((x-\omega_0)^n) = \frac{\Gamma_{q,\omega}(n+1)}{(s-\omega_0)^{n+1}} = \frac{[n]_q!}{(s-\omega_0)^{n+1}}. \tag{16}$$

The q, ω-Taylor expansion of a function f defined on I was established in the study by Oraby and Hamza [15]. Therefore, let $f(x) = \sum_{n=0}^{\infty} \frac{a_n(x-\omega_0)^n}{[n]_q!}$. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r(<\infty)$, then $f(x)$ is convergent in $|x-\omega_0| < \frac{[\infty]_q}{r}$. That is, if $s - \omega_0 > r$, then $f\left(\frac{x}{s-\omega_0}\right)$ is convergent in $|x-\omega_0| < \frac{\omega_0+[\infty]_q}{r}$. Inspired by the work of Kobachi [18], we can give the following theorem, which is more useful than Definition 3 for calculating the Laplace transform of functions.

Theorem 2. *Let* $f(x) = \sum_{n=0}^{\infty} \frac{a_n(x-\omega_0)^n}{[n]_q!}$, $|x-\omega_0| < \frac{[\infty]_q}{r}$. *Then, the* q, ω*-Laplace transform of* f *is given as follows:*

$$L_{q,\omega}(f)(s) := F_{q,\omega}(s) := \sum_{n=0}^{\infty} \frac{a_n}{(s-\omega_0)^{n+1}}. \tag{17}$$

Proof. From equation (16), we have

$$L_{q,\omega}(f)(s) = \sum_{n=0}^{\infty} \frac{a_n}{[n]_q!} L_{q,\omega}((x-\omega_0)^n) = \sum_{n=0}^{\infty} \frac{a_n}{[n]_q!} \frac{[n]_q!}{(s-\omega_0)^{n+1}} = \sum_{n=0}^{\infty} \frac{a_n}{(s-\omega_0)^{n+1}}. \square$$

Example 2. For the q, ω-exponential function $e_{q,\omega}(x) = \sum_{n=0}^{\infty} \frac{(x-\omega_0)^n}{[n]_q!}$, $|x-\omega_0| < [\infty]_q$ (see [7, 19]), using Definition 3, we have

$$\begin{aligned} L_{q,\omega}(e_{q,\omega}(x)) &= \frac{1}{s-\omega_0} \int_{\omega_0}^{\omega_0+[\infty]_q} E_{q,-\omega}(-(qx+\omega)) \sum_{n=0}^{\infty} \frac{(x-\omega_0)^n}{[n]_q!(s-\omega_0)^n} d_{q,\omega}x \\ &= \sum_{n=0}^{\infty} \frac{1}{[n]_q!(s-\omega_0)^{n+1}} \int_{\omega_0}^{\omega_0+[\infty]_q} E_{q,-\omega}(-(qx+\omega))(x-\omega_0)^n d_{q,\omega}x, \end{aligned}$$

and from equations (4) and (6), we obtain

$$\begin{aligned} L_{q,\omega}(e_{q,\omega}(x)) &= \sum_{n=0}^{\infty} \frac{\Gamma_{q,\omega}(n+1)}{[n]_q!(s-\omega_0)^{n+1}} = \sum_{n=0}^{\infty} \frac{[n]_q!}{[n]_q!(s-\omega_0)^{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{1}{(s-\omega_0)^{n+1}} = \frac{1}{s-\omega_0} \sum_{n=0}^{\infty} \left(\frac{1}{s-\omega_0}\right)^n \\ &= \frac{1}{s-\omega_0} \frac{1}{1-\frac{1}{s-\omega_0}} = \frac{1}{s-\omega_0-1}, \end{aligned}$$

provided that $|s - \omega_0| > 1$.

If we use Theorem 2 for $a_n = 1$, similarly we obtain a geometric series with

$$L_{q,\omega}(e_{q,\omega}(x)) = \sum_{n=0}^{\infty} \frac{1}{(s - \omega_0)^{n+1}} = \frac{1}{s - \omega_0 - 1},$$

which converges as $|s - \omega_0| > 1$.

Let $f(x) = \sum_{n=0}^{\infty} \frac{a_n (x - \omega_0)^n}{[n]_q!}$, $|x - \omega_0| < \frac{[\infty]_q}{r_1}$, and $g(x) = \sum_{n=0}^{\infty} \frac{b_n (x - \omega_0)^n}{[n]_q!}$, $|x - \omega_0| < \frac{[\infty]_q}{r_2}$ such that $\{a_n\}$ and $\{b_n\}$ satisfy $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r_1(<\infty)$ and $\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = r_2(<\infty)$. The q,ω-Laplace transform of these functions be $L_{q,\omega}(f)(s) \coloneqq F_{q,\omega}(s)$ and $L_{q,\omega}(g)(s) \coloneqq G_{q,\omega}(s)$. The following theorem shows that the q,ω-Laplace transform satisfies the linearity property.

Theorem 3. (Linearity property) *Let $f, g \in \mathbf{V}$ and c_1, c_2 be constant. Then,*

$$L_{q,\omega}(c_1 f(x) + c_2 g(x)) = c_1 F_{q,\omega}(s) + c_2 G_{q,\omega}(s). \tag{18}$$

Proof. From $c_1 f(x) + c_2 g(x) = \sum_{n=0}^{\infty} \frac{(c_1 a_n + c_2 b_n)(x - \omega_0)^n}{[n]_q!}$, we have

$$L_{q,\omega}(c_1 f(x) + c_2 g(x)) = \sum_{n=0}^{\infty} \frac{(c_1 a_n + c_2 b_n)}{(s - \omega_0)^{n+1}} = c_1 \sum_{n=0}^{\infty} \frac{a_n}{(s - \omega_0)^{n+1}} + c_2 \sum_{n=0}^{\infty} \frac{b_n}{(s - \omega_0)^{n+1}} = c_1 F_{q,\omega}(s) + c_2 G_{q,\omega}(s). \square$$

Theorem 4. (Scaling property) *Let $f \in \mathbf{V}$. Then,*

$$L_{q,\omega}(f(\lambda x)) = \frac{1}{\lambda} F_{q,\omega}\left(\frac{s}{\lambda}\right), |s - \omega_0| > |\lambda|. \tag{19}$$

Proof. From $f(\lambda x) = \sum_{n=0}^{\infty} \frac{a_n \lambda^n (x - \omega_0)^n}{[n]_q!}$, we have

$$\begin{aligned} L_{q,\omega}(f(\lambda x)) &= \sum_{n=0}^{\infty} \frac{a_n \lambda^n}{(s - \omega_0)^{n+1}} = \frac{1}{\lambda} \sum_{n=0}^{\infty} a_n \left(\frac{\lambda}{s - \omega_0}\right)^{n+1} \\ &= \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{a_n}{\left(\frac{s - \omega_0}{\lambda}\right)^{n+1}} = \frac{1}{\lambda} F_{q,\omega}\left(\frac{s}{\lambda}\right). \end{aligned}$$

□

Example 3. From $e_{q,\lambda\omega}(\lambda x) = \sum_{n=0}^{\infty} \frac{\lambda^n (x - \omega_0)^n}{[n]_q!}$, $\cos_{q,\lambda\omega}(\lambda x) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{2n} (x - \omega_0)^{2n}}{[2n]_q!}$, and $\sin_{q,\lambda\omega}(\lambda x) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{2n+1} (x - \omega_0)^{2n+1}}{[2n+1]_q!}$, we have

$$\begin{aligned} L_{q,\omega}(e_{q,\lambda\omega}(\lambda x)) &= \frac{1}{s - \omega_0 - \lambda}, |s - \omega_0| > |\lambda|, \\ L_{q,\omega}(\cos_{q,\lambda\omega}(\lambda x)) &= \frac{s - \omega_0}{(s - \omega_0)^2 + \lambda^2}, s - \omega_0 > \lambda, \\ L_{q,\omega}(\sin_{q,\lambda\omega}(\lambda x)) &= \frac{\lambda}{(s - \omega_0)^2 + \lambda^2}, s - \omega_0 > \lambda. \end{aligned}$$

Theorem 5. (Shifting property) *Let $f \in \mathbf{V}$, $\{b\}_q = \prod_{k=0}^{n-1} q^k b$, $(a + \{b\}_q)^n = \prod_{k=0}^{n-1}(a + q^k b) = a^n \left(-\frac{b}{a}; q\right)_n$, and $R = \max\left\{r, \frac{[\infty]_q}{r}\right\}$. If $|x - \omega_0| < R$, $s - \omega_0 > R$, then*

$$L_{q,\omega}(e_{q,\lambda\omega}(\lambda x) f(x)) = L_{q,\omega}(f(x))|_{s=s-\{\lambda\}_q} = F_{q,\omega}(s - \{\lambda\}_q). \tag{20}$$

Proof. From

$$e_{q,\lambda\omega}(\lambda x)f(x) = \left(\sum_{n=0}^{\infty}\frac{\lambda^n(x-\omega_0)^n}{[n]_q!}\right)\left(\sum_{k=0}^{\infty}\frac{a_k(x-\omega_0)^k}{[k]_q!}\right) = \left(\sum_{n=0}^{\infty}\frac{(x-\omega_0)^n}{[n]_q!}\right)\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\lambda^{n-k}a_k\right),$$

where $\begin{bmatrix}n\\k\end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, we have

$$\begin{aligned}
L_{q,\omega}(e_{q,\lambda\omega}(\lambda x)f(x)) &= \sum_{n=0}^{\infty}\frac{1}{(s-\omega_0)^{n+1}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\lambda^{n-k}a_k\\
&= \sum_{n=0}^{\infty}\frac{1}{(s-\omega_0)^{n+1}}\sum_{k=0}^{n}\begin{bmatrix}n\\n-k\end{bmatrix}_q\lambda^{n-k}a_k\\
&= \sum_{k=0}^{\infty}\frac{a_k}{(s-\omega_0)^{k+1}}\sum_{n=0}^{\infty}\begin{bmatrix}n+k\\n\end{bmatrix}_q\left(\frac{\lambda}{s-\omega_0}\right)^n\\
&= \sum_{k=0}^{\infty}\frac{a_k}{(s-\omega_0)^{k+1}}\sum_{n=0}^{\infty}\frac{(q^{k+1};q)_n}{(q;q)_n}\left(\frac{\lambda}{s-\omega_0}\right)^n\\
&= \sum_{k=0}^{\infty}\frac{a_k}{(s-\omega_0)^{k+1}\left(\frac{\lambda}{s-\omega_0};q\right)_{k+1}}\\
&= \sum_{k=0}^{\infty}\frac{a_k}{(s-\omega_0-\{\lambda\}_q)^{k+1}} = F_{q,\omega}(s-\{\lambda\}_q).
\end{aligned}$$

□

Theorem 6. (Differentiation property) *Let $f \in \mathbf{V}$. If any $n \in \mathbb{N}$, then*

$$L_{q,\omega}(D_{q,\omega}^{n}f(x)) = (s-\omega_0)^nF_{q,\omega}(s) - \sum_{i=1}^{n}(s-\omega_0)^{n-i}D_{q,\omega}^{i-1}f(\omega_0). \tag{21}$$

Proof. From $D_{q,\omega}f(x) = \sum_{n=1}^{\infty}\frac{a_n(x-\omega_0)^{n-1}}{[n-1]_q!} = \sum_{n=0}^{\infty}\frac{a_{n+1}(x-\omega_0)^n}{[n]_q!}$, we have

$$\begin{aligned}
L_{q,\omega}(D_{q,\omega}f(x)) &= \sum_{n=0}^{\infty}\frac{a_{n+1}}{(s-\omega_0)^{n+1}} = (s-\omega_0)\left\{\sum_{n=0}^{\infty}\frac{a_n}{(s-\omega_0)^{n+1}} - \frac{a_0}{s-\omega_0}\right\}\\
&= (s-\omega_0)F_{q,\omega}(s) - a_0\\
&= (s-\omega_0)F_{q,\omega}(s) - f(\omega_0).
\end{aligned}$$

Therefore, relation (21) is true for $n = 1$. Assume that relation (21) is true for $n = k \in \mathbb{N}$

$$L_{q,\omega}(D_{q,\omega}^{k}f(x)) = (s-\omega_0)^kF_{q,\omega}(s) - \sum_{i=1}^{k}(s-\omega_0)^{k-i}D_{q,\omega}^{i-1}f(\omega_0).$$

Then,

$$\begin{aligned}
L_{q,\omega}(D_{q,\omega}^{k+1}f(x)) &= L_{q,\omega}(D_{q,\omega}(D_{q,\omega}^{k}f(x)))\\
&= (s-\omega_0)L_{q,\omega}(D_{q,\omega}^{k}f(x)) - D_{q,\omega}^{k}f(\omega_0)\\
&= (s-\omega_0)\left\{(s-\omega_0)^kF_{q,\omega}(s) - \sum_{i=1}^{k}(s-\omega_0)^{k-i}D_{q,\omega}^{i-1}f(\omega_0)\right\} - D_{q,\omega}^{k}f(\omega_0)\\
&= (s-\omega_0)^{k+1}F_{q,\omega}(s) - \sum_{i=1}^{k}(s-\omega_0)^{k+1-i}D_{q,\omega}^{i-1}f(\omega_0) - D_{q,\omega}^{k}f(\omega_0)\\
&= (s-\omega_0)^{k+1}F_{q,\omega}(s) - \sum_{i=1}^{k+1}(s-\omega_0)^{k+1-i}D_{q,\omega}^{i-1}f(\omega_0).
\end{aligned}$$

Therefore, relation (21) is true for $n = k + 1$, and by induction, it is true for any $n \in \mathbb{N}$. □

Theorem 7. *Let $f \in \mathbf{V}$. If any $n \in \mathbb{N}$, then*

$$L_{q,\omega}((x-\omega_0)^n f(x)) = \frac{(-1)^n}{q^n} D^n_{q^{-1},-\omega q^{-1}} F_{q,\omega}(s). \tag{22}$$

Proof. For $n = 1$, from $(x-\omega_0)f(x) = \sum_{n=0}^{\infty} \frac{a_n (x-\omega_0)^{n+1}}{[n]_q!} = \sum_{n=0}^{\infty} \frac{[n+1]_q a_n (x-\omega_0)^{n+1}}{[n+1]_q!}$, we have

$$L_{q,\omega}((x-\omega_0)f(x)) = \sum_{n=0}^{\infty} \frac{[n+1]_q a_n}{(s-\omega_0)^{n+2}}. \tag{23}$$

On the other hand, using equation (1), we have

$$\begin{aligned} D_{q^{-1},-\omega q^{-1}} F_{q,\omega}(s) &= \frac{F_{q,\omega}(q^{-1}s - \omega q^{-1}) - F_{q,\omega}(s)}{(q^{-1}s - \omega q^{-1}) - s} \\ &= \frac{\sum_{n=0}^{\infty} \frac{a_n}{(q^{-1}s - \omega q^{-1} - \omega_0)^{n+1}} - \sum_{n=0}^{\infty} \frac{a_n}{(s-\omega_0)^{n+1}}}{q^{-1}(1-q)(s-\omega_0)} \\ &= \sum_{n=0}^{\infty} \frac{a_n}{(s-\omega_0)^{n+1}} \left(\frac{q^{n+1}-1}{1-q} \right) \frac{q}{(s-\omega_0)} \\ &= -q \sum_{n=0}^{\infty} \frac{[n+1]_q a_n}{(s-\omega_0)^{n+2}}. \end{aligned} \tag{24}$$

By comparing equations (23) and (24), equation (22) is true for $n = 1$. Similarly, for $n = 2$, from $(x-\omega_0)^2 f(x) = \sum_{n=0}^{\infty} \frac{a_n (x-\omega_0)^{n+2}}{[n]_q!} = \sum_{n=0}^{\infty} \frac{[n+1]_q [n+2]_q a_n (x-\omega_0)^{n+2}}{[n+2]_q!}$, we obtain

$$L_{q,\omega}((x-\omega_0)^2 f(x)) = \sum_{n=0}^{\infty} \frac{[n+1]_q [n+2]_q a_n}{(s-\omega_0)^{n+3}}. \tag{25}$$

Using equations (1) and (24) and comparing the results,

$$\begin{aligned} D^2_{q^{-1},-\omega q^{-1}} F_{q,\omega}(s) &= D_{q^{-1},-\omega q^{-1}} \left(-q \sum_{n=0}^{\infty} \frac{[n+1]_q a_n}{(s-\omega_0)^{n+2}} \right) \\ &= \frac{\sum_{n=0}^{\infty} \frac{-q[n+1]_q a_n}{(sq^{-1} - \omega q^{-1} - \omega_0)^{n+2}} - \sum_{n=0}^{\infty} \frac{-q[n+1]_q a_n}{(s-\omega_0)^{n+2}}}{sq^{-1} - \omega q^{-1} - s} \\ &= \sum_{n=0}^{\infty} \frac{q[n+1]_q a_n}{(s-\omega_0)^{n+2}} \left(\frac{1-q^{n+2}}{1-q} \right) \frac{q}{(s-\omega_0)} \\ &= q^2 \sum_{n=0}^{\infty} \frac{[n+1]_q [n+2]_q a_n}{(s-\omega_0)^{n+3}}. \end{aligned} \tag{26}$$

By comparing equations (25) and (26), equation (22) is true for $n = 2$. Continuing this process, we obtain that equation (22) is true for any $n \in \mathbb{N}$. □

Remark 2. It can be easily seen from equation (1) that $(D_{q,\omega} f)\left(\frac{x-\omega}{q}\right) = D_{q^{-1},-\omega q^{-1}} f(x)$. Let $h(x) = qx + \omega, h^n(x) = q^n x + \omega[n]_q$ and $h^{-n}(x) = \frac{x - \omega[n]_q}{q^n}$ (see [7] for further details). Using these connections, equation (22) can be written as follows:

$$L_{q,\omega}((x-\omega_0)^n f(x)) = \frac{(-1)^n}{q^n} (D^n_{q,\omega} F_{q,\omega})(h^{-n}(s)), \tag{27}$$

which is more convenient when the q, ω-Laplace transform is applied to differential equations with variable coefficients.

Theorem 8. (Integration property) *Let* $f \in \mathbf{V}$. *Then,*

(*i*)

$$L_{q,\omega}\left(\int_{\omega_0}^{x} f(t)d_{q,\omega}t\right) = \frac{1}{s-\omega_0}F_{q,\omega}(s). \tag{28}$$

(*ii*) *If* $f(\omega_0) = 0$, *then*

$$L_{q,\omega}\left(\frac{f(x)}{x-\omega_0}\right) = \int_{s}^{\infty} F_{q,\omega}(t)d_{q,\omega}t. \tag{29}$$

Proof. (i) From the fundamental theorems of q, ω-integral (see [7]) and the q, ω-Taylor expansion of f, we have

$$\begin{aligned}\int_{\omega_0}^{x} f(t)d_{q,\omega}t &= \int_{\omega_0}^{x}\sum_{n=0}^{\infty}\frac{a_n(t-\omega_0)^n}{[n]_q!}d_{q,\omega}t\\ &= \sum_{n=0}^{\infty}\frac{a_n}{[n]_q!}\int_{\omega_0}^{x}(t-\omega_0)^n d_{q,\omega}t\\ &= \sum_{n=0}^{\infty}\frac{a_n(x-\omega_0)^{n+1}}{[n+1]_q!}.\end{aligned}$$

Thus, we obtain

$$L_{q,\omega}\left(\int_{\omega_0}^{x} f(t)d_{q,\omega}t\right) = \sum_{n=0}^{\infty}\frac{a_n}{(s-\omega_0)^{n+2}} = \frac{1}{s-\omega_0}\sum_{n=0}^{\infty}\frac{a_n}{(s-\omega_0)^{n+1}} = \frac{1}{s-\omega_0}F_{q,\omega}(s).$$

(ii) From $f(\omega_0) = 0$, we have $a_0 = 0$. Thus, we obtain

$$\frac{f(x)}{x-\omega_0} = \sum_{n=0}^{\infty}\frac{a_{n+1}(x-\omega_0)^n}{[n+1]_q!}.$$

Therefore, we obtain

$$L_{q,\omega}\left(\frac{f(x)}{x-\omega_0}\right) = \sum_{n=0}^{\infty}\frac{a_{n+1}}{[n+1]_q(s-\omega_0)^{n+1}} = \sum_{n=1}^{\infty}\frac{a_n}{[n]_q(s-\omega_0)^n}. \tag{30}$$

Since we need the improper q, ω-integral of f on $[\omega_0, \infty)$, it is defined as follows (similar to q-analog; see [3], p. 70):

$$\int_{\omega_0}^{\infty} f(x)d_{q,\omega}x = \sum_{k=-\infty}^{\infty}\int_{h^{k+1}(a)}^{h^k(a)} f(x)d_{q,\omega}x = (a-h(a))\sum_{k=-\infty}^{\infty} q^k f(h^k(a)),\ a \in [\omega_0, \infty). \tag{31}$$

From q, ω-integral and equation (31), we can write

$$\begin{aligned}\int_{s}^{\infty} F_{q,\omega}(t)d_{q,\omega}t &= \int_{\omega_0}^{\infty} F_{q,\omega}(t)d_{q,\omega}t - \int_{\omega_0}^{s} F_{q,\omega}(t)d_{q,\omega}t\\ &= (s-h(s))\sum_{k=-\infty}^{0} q^k F_{q,\omega}(h^k(s))\\ &= (s-h(s))\sum_{k=0}^{\infty} q^{-k} F_{q,\omega}(h^{-k}(s)).\end{aligned}$$

Using equation (17), we have

$$\begin{aligned}\int_s^\infty F_{q,\omega}(t)d_{q,\omega}t &= (s-h(s))\sum_{k=0}^{\infty}q^{-k}\left(\sum_{n=0}^{\infty}\frac{a_n}{(h^{-k}(s)-\omega_0)^{n+1}}\right)\\
&= (s-h(s))\sum_{k=0}^{\infty}q^{-k}\left(\sum_{n=0}^{\infty}\frac{a_n(q^k)^{n+1}}{(s-\omega_0)^{n+1}}\right)\\
&= (1-q)(s-\omega_0)\sum_{k=0}^{\infty}q^{-k}\left(\sum_{n=1}^{\infty}\frac{a_n(q^k)^{n+1}}{(s-\omega_0)^{n+1}}\right)\\
&= (1-q)(s-\omega_0)\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{a_n(q^k)^{n}}{(s-\omega_0)^{n+1}}\\
&= (1-q)\sum_{n=1}^{\infty}\frac{a_n}{(s-\omega_0)^{n}}\sum_{k=0}^{\infty}(q^n)^k\\
&= (1-q)\sum_{n=1}^{\infty}\frac{a_n}{(s-\omega_0)^{n}}\frac{1}{1-q^n}\\
&= \sum_{n=1}^{\infty}\frac{a_n}{[n]_q(s-\omega_0)^{n}}\end{aligned} \tag{32}$$

by comparing equations (30) and (32) and thus equation (29) is true. □

Definition 4. The q, ω-analog of convolution of two functions f and g is defined as follows:

$$(f*g)(x) = \int_{\omega_0}^{x} f(t)g(x-\omega_0-qt)d_{q,\omega}t, \tag{33}$$

where $g(x-\omega_0-qt) = (x-\omega_0-\{q(t-\omega_0)\}_q)$.

Theorem 9. (q, ω-convolution) *Let $F_{q,\omega}(s)$ and $G_{q,\omega}(s)$ be the q, ω-Laplace transform of f and g, respectively. Then,*

$$L_{q,\omega}((f*g)(x)) = F_{q,\omega}(s)G_{q,\omega}(s). \tag{34}$$

Proof. From equation (33), we have

$$\begin{aligned}(f*g)(x) &= \int_{\omega_0}^{x} f(t)g(x-\omega_0-qt)d_{q,\omega}t\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_nb_k}{[n]_q![k]_q!}\int_{\omega_0}^{x}(t-\omega_0)^n(x-\omega_0-\{q(t-\omega_0)\}_q)^k d_{q,\omega}t.\end{aligned} \tag{35}$$

In equation (35), change of variable $t-\omega_0 = (x-\omega_0)(u-\omega_0)$, and using the equality $(x-\omega_0-\{q(t-\omega_0)\}_q)^k = (x-\omega_0)^k(1-q(u-\omega_0))_q^k$, we obtain

$$(f*g)(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_nb_k(x-\omega_0)^{n+k+1}}{[n]_q![k]_q!}\int_{\omega_0}^{\omega_0+1}(u-\omega_0)^n(1-q(u-\omega_0))_q^k d_{q,\omega}u. \tag{36}$$

From equations (7) and (9), we obtain

$$(f*g)(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_nb_k(x-\omega_0)^{n+k+1}}{[n+k+1]_q!}$$

and then obtain

$$L_{q,\omega}((f*g)(x)) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{a_kb_l}{(s-\omega_0)^{k+l+2}} = \left(\sum_{k=0}^{\infty}\frac{a_k}{(s-\omega_0)^{k+1}}\right)\left(\sum_{l=0}^{\infty}\frac{b_l}{(s-\omega_0)^{l+1}}\right) = F_{q,\omega}(s)G_{q,\omega}(s).$$ □

4 Applications to q, w-differential equations

Let $L_{q,\omega}(y(x)) = Y_{q,\omega}(s)$ be the q,ω-Laplace transform of y function and $L^{-1}_{q,\omega}$ be the inverse q,ω-Laplace transform, i.e., $L^{-1}_{q,\omega}(Y_{q,\omega}(s)) = y(x)$.

Example 4. Let

$$D^2_{q,\omega}y(x) + y(x) = 0, y(\omega_0) = 0, D_{q,\omega}y(\omega_0) = 1. \tag{37}$$

By applying the q,ω-Laplace transform of equation (37), we obtain

$$(s-\omega_0)^2 Y_{q,\omega}(s) - (s-\omega_0)y(\omega_0) - D_{q,\omega}y(\omega_0) + Y_{q,\omega}(s) = 0,$$

and then, we have

$$Y_{q,\omega}(s) = \frac{1}{(s-\omega_0)^2+1}.$$

Thus,

$$y(x) = L^{-1}_{q,\omega}(Y_{q,\omega}(s)) = L^{-1}_{q,\omega}\left(\frac{1}{(s-\omega_0)^2+1}\right) = \sin_{q,\omega}(x)$$

is the solution of equation (37).

Example 5. Let

$$D^2_{q,\omega}y(x) - 3[2]_q D_{q,\omega}y(x) + 9qy(x) = 0, y(\omega_0) = 1, D_{q,\omega}y(\omega_0) = -2. \tag{38}$$

Taking the q,ω-Laplace transform of equation (38), we obtain

$$Y_{q,\omega}(s) = \frac{s-\omega_0-3q-5}{(s-\omega_0)^2 - 3(1+q)(s-\omega_0) + 9q} = \frac{1}{s-\omega_0-3} - \frac{5}{(s-\omega_0-\{3\}_q)^2}.$$

Thus,

$$y(x) = L^{-1}_{q,\omega}(Y_{q,\omega}(s)) = L^{-1}_{q,\omega}\left(\frac{1}{s-\omega_0-3}\right) - 5L^{-1}_{q,\omega}\left(\frac{1}{(s-\omega_0-\{3\}_q)^2}\right).$$

Hence,

$$y(x) = (1-5(x-\omega_0))e_{q,3\omega}(3x),$$

is the solution of equation (38).

Example 6. Consider the q,ω-diffusion equation

$$\frac{\partial_{q,\omega}}{\partial_{q,\omega}t}u(x,t) = \frac{\partial^2_{q,\omega}}{\partial_{q,\omega}x^2}u(x,t), \tag{39}$$

with the initial condition

$$u(x,\omega_0) = e_{q,\omega}(x). \tag{40}$$

Let $L_{q,\omega}(u(x,t)) = U(x,s)$. Taking the q,ω-Laplace transform of both sides of equation (39) with respect to t, we obtain

$$(s-\omega_0)U(x,s) - u(x,\omega_0) = \frac{d^2_{q,\omega}}{d_{q,\omega}x^2}U(x,s).$$

Rearranging and using equation (40), we write

$$\frac{d_{q,\omega}^2}{d_{q,\omega}x^2}U(x,s) - (s-\omega_0)U(x,s) = -e_{q,\omega}(x). \tag{41}$$

Solving the q,ω-differential equation (41) with the initial condition (40) we have

$$U(x,s) = \frac{1}{s-\omega_0-1}e_{q,\omega}(x),$$

and taking the inverse q,ω-Laplace transform, we obtain the following solution:

$$u(x,t) = e_{q,\omega}(x)e_{q,\omega}(t).$$

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