Research Article

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Ruled real hypersurfaces in the complex hyperbolic quadric

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Abstract: In this article, we introduce a new family of real hypersurfaces in the complex hyperbolic quadric $Q^{n*} = SO^o_{2,n}/SO_2SO_n$, namely, the ruled real hypersurfaces foliated by complex hypersurfaces. Berndt described an example of such a real hypersurface in Q^{n*} as a homogeneous real hypersurface generated by a $\mathfrak{A}$-principal horocycle in a real form $\mathbb{R}H^n$. So, in this article, we compute a detailed expression of the shape operator for ruled real hypersurfaces in Q^{n*} and investigate their characterizations in terms of the shape operator and the integrable distribution $\mathcal{C} = \{X \in TM | X \perp \xi\}$. Then, by using these observations, we give two kinds of classifications of real hypersurfaces in Q^{n*} satisfying η-parallelism under either η-commutativity of the shape operator or integrability of the distribution $\mathcal{C}$. Moreover, we prove that the unit normal vector field of a real hypersurface with η-parallel shape operator in Q^{n*} is $\mathfrak{A}$-principal. On the other hand, it is known that all contact real hypersurfaces in Q^{n*} have a $\mathfrak{A}$-principal normal vector field. Motivated by these results, we give a characterization of contact real hypersurfaces in Q^{n*} in terms of η-parallel shape operator.

Keywords: ruled real hypersurface, η-parallel shape operator, η-commuting shape operator, singular vector fields, complex hyperbolic quadric

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1 Introduction

In the class of Hermitian symmetric spaces with rank 2 of noncompact type, we can consider the example of complex hyperbolic quadric $Q^{n*} = SO^o_{2,n}/SO_2SO_n$, which is a simply connected Riemannian manifold whose curvature tensor is the negative of the curvature tensor of the complex quadric $Q^n = SO_{n+2}/SO_2SO_n$ (see [1–5]). The complex hyperbolic quadric Q^{n*} can be regarded as a kind of real Grassmann manifold of noncompact type with rank 2. Accordingly, Q^{n*} admits two important geometric structures, a complex conjugation (or real structure) C, and a Kähler structure (or complex structure) J, which anti-commute with each other, i.e., $CJ = -JC$. Then, for $n \geq 2$, the triple (Q^{n*}, J, g) is a Hermitian symmetric space of noncompact type, and its minimal sectional curvature is equal to -4 (see [6–8]).

In particular, Kimura-Ortega [9] and Montiel-Romero [10] proved that Q^{n*} can be immersed in the indefinite complex hyperbolic space $\mathbb{C}H^{n+1}_1(-c)$, $c > 0$, by interchanging the Kähler metric with its opposite. Indeed, if we change the Kähler metric of $\mathbb{C}P^{n+1}_{n-s}$ by its opposite, we have that Q^n_{n-s} endowed with its opposite metric $g' = -g$ is also an Einstein hypersurface of $\mathbb{C}H^{n+1}_{s+1}(-c)$. In the case of $s = 0$, $(Q^n_n, g' = -g)$ can be regarded as $Q^{n*} = SO^o_{2,n}/SO_2SO_n$, which is immersed in the indefinite complex hyperbolic space $\mathbb{C}H^{n+1}_1(-c)$, $c > 0$ as a complex Einstein hypersurface.

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Apart from the complex structure J, there is another distinguished geometric structure on Q^{n*}, namely a parallel rank 2 vector bundle $\mathfrak{A}$, which contains an S^1-bundle of real structures on the tangent spaces of Q^{n*}, i.e., $\mathfrak{A} = \{\lambda C | \lambda \in S^1\}$. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^{n*}.

In this article, we consider a classification problem of real hypersurfaces in the complex hyperbolic quadric Q^{n*}, $n \geq 3$. Let ζ be a unit normal vector field of a real hypersurface M in Q^{n*}. As a typical classification of real hypersurfaces in Q^{n*}, we introduce the following result, which was given by Suh [11].

Theorem A. *Let M be a complete real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$, with commuting shape operator. Then, M is locally congruent to a tube over a complex hyperbolic space $\mathbb{C}H^k$ in Q^{2k*}, $n = 2k$ or a horosphere.*

Here, if the structure tensor ϕ commutes with the shape operator A of M, i.e., $A\phi = \phi A$, we say that M has the commuting shape operator (i.e., M has isometric Reeb flow). This result motivates us to study the weaker notion of η-commuting property of the shape operator. So, we define η-commuting property and η-parallelism of the shape operator A of M as follows:

Definition. If the shape operator A of M satisfies

$$g((A\phi - \phi A)X, Y) = 0$$

for any $X, Y \in \mathcal{C}$, we say that A is *η-commuting*. Here, ϕ is the structure tensor of M, which is given as the tangential part of $JX = \phi X + g(X, \xi)\zeta$ for any $X \in TM$. Moreover, the shape operator A of M is said to be *η-parallel* if it satisfies

$$g((\nabla_X A)Y, Z) = 0$$

for any $X, Y, Z \in \mathcal{C}$, where $\mathcal{C}$ denotes the orthogonal complement of the Reeb vector field $\xi = -J\zeta$ of M in TM.

A complete classification of real hypersurfaces in the complex quadric Q^n with such two notions for shape operator was given in Kimura et al. [12]. By virtue of this classification, a new characterization of ruled real hypersurfaces foliated by complex totally geodesic hyperplanes Q^{n-1} in Q^n was given in the same article. For the complex projective space $\mathbb{C}P^n$, Kimura [13] and Loknherr and Reckziegel [14] gave some examples of ruled real hypersurfaces. The characterizations of ruled real hypersurfaces in $\mathbb{C}P^n$ were investigated in [15–18] and so on. Recently, the ruled real hypersurfaces in the indefinite complex projective space $\mathbb{C}P^n_p$ have been introduced by Moruz et al. [19]. Moreover, they gave a classification of all minimal ruled real hypersurfaces in $\mathbb{C}P^n_p$.

Motivated by these results, in this article, we will give a classification of real hypersurfaces in the complex hyperbolic quadric Q^{n*} regarding η-parallel and η-commuting shape operator. When the Reeb vector field ξ of M in Q^{n*} is principal, a real hypersurface M is said to be *Hopf*. As another kind of real hypersurfaces in Q^{n*}, we deal with a family of ruled real hypersurfaces in Q^{n*}, which are not Hopf. Indeed, a *ruled real hypersurface* is foliated by totally geodesic complex hypersurfaces Q^{n-1*} in Q^{n*}. More details on this family are given in Section 4. Then, by virtue of Theorems A, 4.2, and 6.4, we assert the following theorem:

Theorem 1.1. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$, with η-parallel and η-commuting shape operator. Then, M is locally congruent to a ruled real hypersurface in Q^{n*}.*

Remark 1.2. In Section 5, we prove that the unit normal vector field ζ on a real hypersurface with η-parallel shape operator in Q^{n*}, $n \geq 3$, is $\mathfrak{A}$-principal (see Lemmas 5.1 and 5.2). Lemma 5.5 shows that the shape operator of a ruled real hypersurface in Q^{n*}, $n \geq 3$, is η-parallel. Consequently, we can assert that the unit normal vector field of a ruled real hypersurface is $\mathfrak{A}$-principal (see Proposition 5.6).

Now, let us consider the notion of *integrability* of the holomorphic distribution $\mathcal{C}$ of a real hypersurface M in the complex hyperbolic quadric Q^{n*}, where $\mathcal{C}$ is given by $\mathcal{C} = \{X \in TM \mid X \perp \xi\}$. Kimura and Maeda [15] considered this notion for a real hypersurface in the complex projective space $\mathbb{C}P^n$. They gave a characterization of ruled real hypersurface in $\mathbb{C}P^n$. Motivated by such a result, for Q^{n*}, we obtain the following theorem:

Theorem 1.3. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$. Then, the shape operator of M is η-parallel and the holomorphic distribution $\mathcal{C} = \{X \in TM \mid X \perp \xi\}$ is integrable if and only if M is locally congruent to a ruled real hypersurface in Q^{n*}.*

As will be discussed in detail in Section 6, we know that if the shape operator of a real hypersurface M in Q^{n*} satisfies the conditions of η-commutativity and η-parallelism, then M is either Hopf or ruled (see Lemma 6.2). Now, let us focus our attention on the case that M is Hopf. Then, the η-commuting property is equivalent to the Reeb flow being isometric. By using this fact, we obtain a characterization of ruled real hypersurfaces in Q^{n*} (see Theorem 1.1). From this point of view, it is necessary to consider Hopf real hypersurfaces with η-parallel shape operator. So, as a final result, we want to give a complete classification of Hopf real hypersurfaces in Q^{n*} with η-parallel shape operator as follows:

Theorem 1.4. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$. Then, the shape operator of M is η-parallel if and only if M is locally congruent to an open part of one of the following contact real hypersurfaces in Q^{n*}:*

- ($\mathrm{T}^*_{\mathrm{B}_1}$) *a tube of radius $r > 0$ around the complex hyperbolic quadric Q^{n-1*}, which is embedded in Q^{n*} as a totally geodesic complex hypersurface,*
- ($\mathrm{T}^*_{\mathrm{B}_2}$) *a tube of radius $r > 0$ around the k-dimensional real hyperbolic space $\mathbb{R}H^k$, which is embedded in Q^{n*} as a real space form of Q^{n*}, $n = 2k$,*
- ($\mathscr{H}^*_{\mathrm{B}}$) *a horosphere in Q^{n*} whose center at infinity is the equivalence class of a $\mathfrak{A}$-principal geodesic in Q^{n*}.*

Remark 1.5. For a Hopf real hypersurface in the complex hyperbolic space $\mathbb{C}H^n$ with η-parallel shape operator, Suh [20] proved that such a real hypersurface in $\mathbb{C}H^n$ is locally congruent to one of types A_0, A_1, A_2 or of type B in $\mathbb{C}H^n$. From this and our result, Theorem 1.4, there is a difference between the theory of real hypersurfaces in $\mathbb{C}H^n$ and that of real hypersurfaces in Q^{n*}.

2 The complex hyperbolic quadric

In this section, we introduce the complex hyperbolic quadric Q^{n*}. This section is due to Klein and Suh (see [11,21]).

The n-dimensional complex hyperbolic quadric Q^{n*} is the noncompact dual of the n-dimensional complex quadric Q^n, i.e., the simply connected Riemannian symmetric space whose curvature tensor is the negative of the curvature tensor of Q^n. It cannot be realized as a homogeneous complex hypersurface of the complex hyperbolic space $\mathbb{C}H^{n+1}$. In fact, Smyth [3, Theorem 3(ii)] has shown that every homogeneous complex hypersurface in $\mathbb{C}H^{n+1}$ is totally geodesic. This is in marked contrast to the situation for the complex quadric Q^n, which can be realized as a homogeneous complex hypersurface of the complex projective space $\mathbb{C}P^{n+1}$ in such a way that the shape operator for any unit normal vector to Q^n has a real structure on the corresponding tangent space of Q^n (see [8,21,22]). Another related result by Smyth, [4, Theorem 1], which states that any complex hypersurface in $\mathbb{C}H^{n+1}$ for which the square of the shape operator has constant eigenvalues (counted with multiplicity) is totally geodesic, also precludes the possibility of a model of Q^{n*} as a complex hypersurface of $\mathbb{C}H^{n+1}$ with the analogous property for the shape operator. Therefore, we realize the complex hyperbolic quadric Q^{n*} as the quotient manifold $SO^o_{2,n}/SO_2SO_n$.

As Q^{1*} is isomorphic to the real hyperbolic space $\mathbb{R}H^2 = SO^o_{2,1}/SO_2$, and Q^{2*} is isomorphic to the Hermitian product of complex hyperbolic spaces $\mathbb{C}H^1 \times \mathbb{C}H^1$, we suppose $n \geq 3$ in the sequel and throughout this article. Let $G := SO_{2,n}$ be the transvection group of Q^{n*} and $K := SO_2SO_n$ be the isotropy group of Q^{n*} at the "origin" $p_0 := eK \in Q^{n*}$. Then,

$$\sigma : G \to G,\ g \mapsto sgs^{-1} \quad \text{with} \quad s := \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

is an involutive Lie group automorphism of G with $\mathrm{Fix}(\sigma)_0 = K$, and therefore, $Q^{n*} = G/K$ is a Riemannian symmetric space. The center of the isotropy group K is isomorphic to SO_2, and therefore, Q^{n*} is in fact a Hermitian symmetric space.

The Lie algebra $\mathfrak{g} := \mathfrak{so}_{2,n}$ of G is given as follows:

$$\mathfrak{g} = \{X \in \mathfrak{gl}(n+2, \mathbb{R}) | X^t s = -sX\}$$

(see [23, p. 59]). In the sequel, we will write members of $\mathfrak{g}$ as block matrices with respect to the decomposition $\mathbb{R}^{n+2} = \mathbb{R}^2 \oplus \mathbb{R}^n$, i.e., in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where X_{11}, X_{12}, X_{21}, and X_{22} are real matrices of dimension 2×2, $2 \times n$, $n \times 2$, and $n \times n$, respectively. Then,

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \;\middle|\; X_{11}^t = -X_{11},\ X_{12}^t = X_{21},\ X_{22}^t = -X_{22} \right\}.$$

The linearization $\sigma_L = \mathrm{Ad}(s) : \mathfrak{g} \to \mathfrak{g}$ of the involutive Lie group automorphism σ induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where the Lie subalgebra

$$\begin{aligned} \mathfrak{k} &= \mathrm{Eig}(\sigma_*, 1) = \{X \in \mathfrak{g} \mid sXs^{-1} = X\} \\ &= \left\{ \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \;\middle|\; X_{11}^t = -X_{11},\ X_{22}^t = -X_{22} \right\} \\ &\cong \mathfrak{so}_2 \oplus \mathfrak{so}_n \end{aligned}$$

is the Lie algebra of the isotropy group K, and the $2n$-dimensional linear subspace

$$\mathfrak{m} = \mathrm{Eig}(\sigma_*, -1) = \{X \in \mathfrak{g} \mid sXs^{-1} = -X\} = \left\{ \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \;\middle|\; X_{12}^t = X_{21} \right\}$$

is canonically isomorphic to the tangent space $T_{p_0}Q^{n*}$. Under the identification $T_{p_0}Q^{n*} \cong \mathfrak{n}$, the Riemannian metric g of Q^{n*} (where the constant factor of the metric is chosen so that the formulae become as simple as possible) is given as follows:

$$g(X, Y) = \frac{1}{2}\,\mathrm{tr}(Y^t X) = \mathrm{tr}(Y_{12}X_{21}) \quad \text{for} \quad X, Y \in \mathfrak{m},$$

where g is clearly $\mathrm{Ad}(K)$-invariant and therefore corresponds to an $\mathrm{Ad}(G)$-invariant Riemannian metric on Q^{n*}. The complex structure J of the Hermitian symmetric space is given as follows:

$$JX = \mathrm{Ad}(j)X \quad \text{for } X \in \mathfrak{m}, \quad \text{where } j := \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \in K.$$

As j is in the center of K, the orthogonal linear map J is $\mathrm{Ad}(K)$-invariant and thus defines an $\mathrm{Ad}(G)$-invariant Hermitian structure on Q^{n*}. By identifying the multiplication by the unit complex number i with the application of the linear map J, the tangent spaces of Q^{n*} thus become n-dimensional complex linear spaces, and we will adopt this point of view in the sequel.

As for the complex quadric (again compare [8] with [21] and [11]), there is another important structure on the tangent bundle of the complex quadric besides the Riemannian metric and the complex structure, namely an S^1-bundle $\mathfrak{A}$ of real structures. The situation in this case is distinct from that of the complex quadric, as the real structures in $\mathfrak{A}$ cannot be construed as the shape operator of a complex hypersurface in a complex space form, but as the following considerations will show, $\mathfrak{A}$ still plays a fundamental role in the description of the geometry of Q^{n*}.

Let

$$a_0 := \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Note that we have $a_0 \notin K$, but only $a_0 \in O_2\ SO_n$. However, $\mathrm{Ad}(a_0)$ still leaves $\mathfrak{m}$ invariant and therefore defines an $\mathbb{R}$-linear map C_0 on the tangent space $\mathfrak{m} \cong T_{p_0}Q^{n*}$. C_0 turns out to be an involutive orthogonal map with $C_0 \circ J = -J \circ C_0$ (i.e., C_0 is anti-linear with respect to the complex structure of $T_{p_0}Q^{n*}$), and hence a real structure on $T_{p_0}Q^{n*}$. But C_0 commutes with $\mathrm{Ad}(g)$ not for all $g \in K$, but only for $g \in SO_n \subset K$. More specifically, for $g = (g_1, g_2) \in K$ with $g_1 \in SO_2$ and $g_2 \in SO_n$, say $g_1 = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ with $t \in \mathbb{R}$ (so that $\mathrm{Ad}(g_1)$ corresponds to multiplication with the complex number $\mu := e^{it}$), we have

$$C_0 \circ \mathrm{Ad}(g) = \mu^{-2}\mathrm{Ad}(g) \circ C_0.$$

This equation shows that the object that is $\mathrm{Ad}(K)$-invariant and therefore geometrically relevant is not the real structure C_0 by itself but rather the "circle of real structures"

$$\mathfrak{A}_{p_0} := \{\lambda C_0 \mid \lambda \in S^1\}.$$

$\mathfrak{A}_{p_0}$ is $\mathrm{Ad}(K)$-invariant and therefore generates an $\mathrm{Ad}(G)$-invariant S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{n*})$, consisting of real structures on the tangent spaces of Q^{n*}. For any $CV \in \mathfrak{A}$, the tangent line to the fiber of $\mathfrak{A}$ through C is spanned by JC.

For any $p \in Q^{n*}$ and $C \in \mathfrak{A}_p$, the complex conjugation (real structure) C induces a splitting

$$T_pQ^{n*} = V(C) \oplus JV(C)$$

into two orthogonal, maximal totally real subspaces of the tangent space T_pQ^{n*}. Here,$V(C)$ respectively $JV(C)$ are the $(+1)$-eigenspace respectively the (-1)-eigenspace of C. For every unit vector $Z \in T_pQ^{n*}$, there exist $t \in [0, \frac{\pi}{4}]$, $C \in \mathfrak{A}_p$, and orthonormal vectors $X, Y \in V(C)$ so that

$$Z = \cos(t)X + \sin(t)JY$$

holds (see [8, Proposition 3]). Here, t is uniquely determined by Z. The vector Z is singular, i.e., contained in more than one maximal flat in Q^{n*} if and only if either $t = 0$ or $t = \frac{\pi}{4}$ holds. The vectors with $t = 0$ are called $\mathfrak{A}$-*principal*, whereas the vectors with $t = \frac{\pi}{4}$ are called $\mathfrak{A}$-*isotropic*. If Z is regular, i.e., $0 < t < \frac{\pi}{4}$ holds, then also C, X, and Y are also uniquely determined by Z.

The Riemannian curvature tensor $\bar{R}$ of Q^{n*} can be fully described in terms of the "fundamental geometric structures" g, J, and $\mathfrak{A}$ as follows:

$$\begin{aligned}\bar{R}(X,Y)Z = &-g(Y,Z)X + g(X,Z)Y - g(JY,Z)JX + g(JX,Z)JY + 2g(JX,Y)JZ - g(CY,Z)CX + g(CX,Z)CY \\ &- g(JCY,Z)JCX + g(JCX,Z)JCY\end{aligned} \tag{2.1}$$

for arbitrary $C \in \mathfrak{A}$. Therefore, the curvature of Q^{n*} is the negative sign of that of the complex quadric Q^n, compare [8, Theorem 1]. This confirms that the symmetric space Q^{n*}, which we have constructed here, is indeed the noncompact dual of the complex quadric.

It is well known that Q^{n*} becomes a Kähler manifold, i.e., the complex structure J is parallel, $\bar{\nabla}J = 0$, where $\bar{\nabla}$ is the Levi-Civita connection of Q^{n*}. Finally, because the S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{m*})$ is $\mathrm{Ad}(G)$-invariant, it is also parallel with respect to the same covariant derivative $\bar{\nabla}$ induced by $\bar{\nabla}$ on $\mathrm{End}(TQ^{n*})$. Because the tangent line of the fiber of $\mathfrak{A}$ through some $C_p \in \mathfrak{A}$ is spanned by JC_p, this means precisely that, for any section C of $\mathfrak{A}$, there exists a real-valued 1-form $q : TQ^{n*} \to \mathbb{R}$ so that

$$\bar{\nabla}_X C = q(X)JC_p \quad \text{holds for } \; p \in Q^{n*}, \; X \in T_pQ^{n*}. \tag{2.2}$$

3 Some general equations

Let M be a real hypersurface in the complex hyperbolic quadric Q^{n*} and ζ be a local unit normal vector field of M. Any vector field X tangent to M satisfies

$$JX = \phi X + \eta(X)\zeta. \tag{3.1}$$

The tangential component of equation (3.1) defines on M as a skew-symmetric tensor field ϕ of type (1,1), named the structure tensor. The structure vector field ξ is defined by $\xi = -J\zeta$ and is called the Reeb vector field. The 1-form η is given by $\eta(X) = g(\xi, X)$ for any vector field X tangent to M. So, on M, an almost contact metric structure (ϕ, ξ, η, g) is defined. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

We assume that M is a Hopf hypersurface. Then, the Reeb vector field $\xi = -J\zeta$ satisfies the following:

$$A\xi = \alpha\xi,$$

where A denotes the shape operator of the real hypersurface M for a smooth function $\alpha = g(A\xi, \xi)$ on M. Now, we consider the equation of Codazzi:

$$\begin{aligned} g((\nabla_X A)Y - (\nabla_Y A)X, Z) = &-\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y) - g(X, C\zeta)g(CY, Z) \\ &+ g(Y, C\zeta)g(CX, Z) - g(X, C\xi)g(JCY, Z) + g(Y, C\xi)g(JCX, Z). \end{aligned} \tag{3.2}$$

Putting $Z = \xi$ in equation (3.2), we obtain

$$\begin{aligned} g((\nabla_X A)Y - (\nabla_Y A)X, \xi) = &\, 2g(\phi X, Y) - g(X, C\zeta)g(Y, C\xi) + g(Y, C\zeta)g(X, C\xi) + g(X, C\xi)g(JY, C\xi) \\ &- g(Y, C\xi)g(JX, C\xi). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} g((\nabla_X A)Y - (\nabla_Y A)X, \xi) &= g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yield

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, C\zeta)g(Y, C\xi) - 2g(Y, C\zeta)g(\xi, C\xi).$$

Reinserting this into the previous equation yields

$$\begin{aligned} g((\nabla_X A)Y - (\nabla_Y A)X, \xi) = &\, 2g(\xi, C\zeta)g(X, C\xi)\eta(Y) - 2g(X, C\zeta)g(\xi, C\xi)\eta(Y) \\ &- 2g(\xi, C\zeta)g(Y, C\xi)\eta(X) + 2g(Y, C\zeta)g(\xi, C\xi)\eta(X) \\ &+ \alpha g((\phi A + A\phi)X, Y) - 2g(A\phi AX, Y). \end{aligned}$$

Altogether, this implies

$$\begin{aligned}0 = &2g(A\phi AX, Y) - \alpha g((\phi A + A\phi)X, Y) + 2g(\phi X, Y)\\ &- g(X, C\zeta)g(Y, C\xi) + g(Y, C\zeta)g(X, C\xi)\\ &+ g(X, C\xi)g(JY, C\xi) - g(Y, C\xi)g(JX, C\xi)\\ &- 2g(\xi, C\zeta)g(X, C\xi)\eta(Y) + 2g(X, C\zeta)g(\xi, C\xi)\eta(Y)\\ &+ 2g(\xi, C\zeta)g(Y, C\xi)\eta(X) - 2g(Y, C\zeta)g(\xi, C\xi)\eta(X).\end{aligned}$$

At each point $z \in M$, we can choose $C \in \mathfrak{A}_z$ such that

$$\zeta = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(C)$ and $0 \le t \le \frac{\pi}{4}$ (see Proposition 3 in [8]). Note that t is a function on M. First of all, since $\xi = -J\zeta$, we have

$$\begin{aligned}C\zeta &= \cos(t)Z_1 - \sin(t)JZ_2,\\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1,\\ C\xi &= \sin(t)Z_2 + \cos(t)JZ_1.\end{aligned} \tag{3.3}$$

This implies $g(\xi, C\zeta) = 0$ and hence

$$\begin{aligned}0 = &2g(A\phi AX, Y) - \alpha g((\phi A + A\phi)X, Y) + 2g(\phi X, Y)\\ &- g(X, C\zeta)g(Y, C\xi) + g(Y, C\zeta)g(X, C\xi)\\ &+ g(X, C\xi)g(JY, C\xi) - g(Y, C\xi)g(JX, C\xi)\\ &+ 2g(X, C\zeta)g(\xi, C\xi)\eta(Y) - 2g(Y, C\zeta)g(\xi, C\xi)\eta(X).\end{aligned}$$

4 Ruled real hypersurfaces

In this section, we define a ruled real hypersurface in the complex hyperbolic quadric Q^{n*} and give the form of its shape operator. From this fact, we give some characterizations of ruled real hypersurfaces M in Q^{n*}. Moreover, we will introduce the example due to Berndt [24].

Let M be a real hypersurface in the complex hyperbolic quadric Q^{n*}. If the Reeb vector field $\xi = -J\zeta$ of M is principal, M is said to be Hopf. Now, let us introduce another kind of real hypersurfaces, *ruled real hypersurfaces* in the complex hyperbolic quadric Q^{n*}, which are not Hopf, as follows:

Definition 4.1.
(a) Let $\mathcal{C}$ be the distribution given by $\mathcal{C} = \{X \in TM \mid X \perp \xi\}$. It is called the *holomorphic distribution* of M.
(b) If $[X, Y] \in \mathcal{C}$ for any vector fields $X, Y \in \mathcal{C}$, then $\mathcal{C}$ is said to be *integrable*.
(c) A real hypersurface M is said to be *ruled* if the holomorphic distribution $\mathcal{C}$ is integrable and each of its leaves is locally congruent to a totally geodesic complex hyperplane Q^{n-1*} in Q^{n*}.

Note. The above (c) can be rewritten as follows: when M is foliated by the integrable totally geodesic complex hyperplane Q^{n-1*} in Q^{n*}, then M can be given by $M = \{p \in Q^{n-1*}(t) \mid t \in I\}$. In such a case, we say that M is a *ruled real hypersurface* in Q^{n*}.

Theorem 4.2. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \ge 3$. Then, M is locally congruent to a ruled real hypersurface in Q^{n*} if and only if the shape operator A satisfies $g(AX, Y) = 0$ for any vector fields X and $Y \in \mathcal{C}$.*

Proof. Assume that M is ruled. Let L be a totally geodesic leaf of $\mathcal{C}$ in Q^{n*}, that is, L is an integral manifold of $\mathcal{C}$. For any L, we call ∇^L its Levi-Civita connection. Then, we obtain $\bar{\nabla}_X Y = \nabla^L_X Y$ for any vector fields $X, Y \in TL$, which implies $\bar{\nabla}_X Y \in TL$. As $T_pL = \mathcal{C}_p$ for any point p of L, we obtain

$$g(\bar{\nabla}_X Y, \zeta) = 0 \tag{4.1}$$

for any $X, Y \in TL$. On the other hand, the Gauss formula of M in Q^{n*} is given as follows:

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)\zeta, \tag{4.2}$$

where $\nabla_X Y$ denotes the tangential part of $\bar{\nabla}_X Y$. By taking the inner product of equation (4.2) with the unit normal vector field ζ and using equation (4.1), it follows that $g(AX, Y) = 0$ for any $X, Y \in C$.

Conversely, suppose that the shape operator A of M satisfies $g(AX, Y) = 0$ for any $X, Y \in C$. Let us show that the holomorphic distribution C of M is integrable. In order to do this, we first show that $\bar{\nabla}_X Y$ is tangent to M and is orthogonal to ξ, i.e., $\bar{\nabla}_X Y \in C$ for any $X, Y \in C$. In fact, by virtue of the Weingarten formula $\bar{\nabla}_X \zeta = -AX$, our assumption assures

$$0 = g(AX, Y) = -g(\bar{\nabla}_X \zeta, Y) = g(\zeta, \bar{\nabla}_X Y)$$

for any $X, Y \in C$. It means that $\bar{\nabla}_X Y$ is tangent to M. On the other hand, it is known that $\phi Y \in C$ for any $Y \in TM$, because $\phi\xi = 0$. So, our assumption $g(AX, Y) = 0$ for any $X, Y \in C$ gives $g(AX, \phi Y) = 0$ for any $X, Y \in C$. From this, together with the Gauss formula and the formula $\nabla_X \xi = \phi AX$, we obtain

$$g(\bar{\nabla}_X Y, \xi) = -g(Y, \bar{\nabla}_X \xi) = -g(Y, \nabla_X \xi) - g(AX, \xi)g(Y, \zeta) = -g(Y, \phi AX) = g(\phi Y, AX) = 0.$$

It means that the tangent vector field $\bar{\nabla}_X Y$ of M is orthogonal to the Reeb vector field ξ, i.e., $\bar{\nabla}_X Y \in C$. Similarly, we obtain that $\bar{\nabla}_Y X \in C$. Thus, for any $X, Y \in C$,

$$[X, Y] = \bar{\nabla}_X Y - \bar{\nabla}_Y X \in C.$$

Hence, we can assert that the distribution C of M is integrable.

Next, let us see that the leaves of C are totally geodesic. Take L as one leaf of them, i.e., L is a submanifold of Q^{n*} such that $T_pL = C_p$ for any point $p \in L$. Let ∇^L and σ be the Levi-Civita connection on L and the second fundamental form of L in Q^{n*}, respectively. Then, we may write the Gauss equation of L in Q^{n*} as follows:

$$\bar{\nabla}_X Y = \nabla^L_X Y + \sigma(X, Y) \tag{4.3}$$

for any $X, Y \in T_pL$, $p \in L$. As the result was proven above, it holds that $\bar{\nabla}_X Y \in C$. Also, it holds $\nabla^L_X Y \in TL$ for any $X, Y \in TL$. From these facts and $C = TL$, equation (4.3) gives $\sigma(X, Y) = 0$. It follows that

$$\bar{\nabla}_X Y = \nabla^L_X Y$$

for any $X, Y \in C$. Hence, we assert that the leaf L of C is totally geodesic. □

From this result, we can compute a detailed description of the shape operator A of a ruled real hypersurface M in Q^{n*}. In fact, it can be seen that this property is also true on ruled real hypersurfaces of nonflat complex space forms and complex quadric Q^n (see [12,13,25]). So, as a characterization of ruled real hypersurfaces in Q^{n*}, we have:

Theorem 4.3. *The expression of the shape operator A of a ruled real hypersurface M in Q^{n*} is given as follows:*

$$A\xi = \alpha\xi + \beta U, \quad AU = \beta\xi, \quad AX = 0$$

for any vector field $X \perp \xi$, and U, where U is a unit vector field in C, which is orthogonal to the Reeb vector field ξ. Here, the functions $\alpha = g(A\xi, \xi)$ and $\beta = g(A\xi, U)$ are smooth and the function β does not vanish on a neighborhood of a point $p \in M$.

Proof. As mentioned above, the assumption of M being ruled means that M is not Hopf. So, we may write

$$A\xi = \alpha\xi + \beta U,$$

where the unit vector field $U \in C$ is orthogonal to the Reeb vector field ξ and the smooth function $\beta = g(A\xi, U)$ is nonvanishing on a neighborhood of a point $p \in M$.

Now, we take

$$\mathcal{B} = \{e_1 = \xi, \underbrace{e_2 = U, e_3 = \phi U, e_4, e_5 = \phi e_4, \cdots, e_{2n-2}, e_{2n-1} = \phi e_{2n-2}}_{\in C}\}$$

as a basis of TM. Then, by virtue of Theorem 4.2, we obtain $g(AU, e_i) = 0$ for any $i = 2, 3, \ldots, 2n-1$. Therefore, it gives

$$AU = \sum_{i=1}^{2n-1} g(AU, e_i)e_i = g(AU, e_1)e_1 + \sum_{i=2}^{2n-1} g(AU, e_i)e_i = g(AU, \xi)\xi.$$

Moreover, by using the facts $A\xi = \alpha\xi + \beta U$ and $\xi \perp U$, it becomes

$$AU = g(U, A\xi)\xi = \beta\xi.$$

Let us consider AX for any tangent vector field X which is orthogonal to ξ and U. In fact, by using Theorem 4.2, $g(AX, Y) = 0$ for any $X, Y \in C$, and the expression of $\mathcal{B}$, we obtain

$$AX = g(AX, \xi)\xi = g(X, \alpha\xi + \beta U)\xi = 0$$

for any $X \in C$ orthogonal to the unit vector field U, finishing the proof. □

It holds that $g(\nabla_X Y, \xi) = -g(Y, \nabla_X \xi) = -g(Y, \phi AX) = g(\phi Y, AX)$ for any $X, Y \in C$. By virtue of Theorem 4.2, it implies that $\nabla_X Y \in C$. From this, we assert that the shape operator A of a ruled real hypersurface M is η-parallel, i.e., $g((\nabla_X A)Y, Z) = 0$ for any $X, Y, Z \in C$. By linearization, it becomes $g((\nabla_X A)X, X) = 0$ for any $X \in C$. Then, this is equivalent to the constancy of $g(A\gamma', \gamma')^2 = \bar{g}(\bar{\nabla}_{\gamma'}\gamma', \bar{\nabla}_{\gamma'}\gamma')$, where γ is a geodesic on M. Here, $\bar{g}$ and $\bar{\nabla}$ denote, respectively, the Riemannian metric and the Riemannian connection of the complex hyperbolic quadric Q^{n*}. This means that every geodesic $\gamma\colon I \to M$ in Q^{n*}, which is orthogonal to the Reeb vector field ξ, i.e., $\gamma'(0) \perp \xi_p$, and $\gamma(0) = p$, has constant first curvature.

Remark 4.4. Let M be a ruled real hypersurface in the complex hyperbolic quadric Q^{n*}. Of course, the shape operator A is η-parallel. Moreover, by Theorem 4.3, we obtain $A\phi U = 0$. If the Reeb function $\alpha = g(A\xi, \xi) = 0$, the function $\beta = g(A\xi, U)$ is a nonvanishing constant, and the vector field U is parallel, i.e., $\nabla_\xi U = 0$, along the integral curve (horocycle) of the Reeb vector field ξ, respectively, then the unit normal vector field $\zeta = J\xi$ becomes singular.

In fact, let us use the equation of Codazzi for $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi$. Then, it follows that

$$\begin{aligned} g(\bar{R}(X, Y)\xi, \zeta) &= g((\nabla_X A)Y - (\nabla_Y A)X, \xi) \\ &= g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ &= \mathrm{d}\alpha(X)\eta(Y) - \mathrm{d}\alpha(Y)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y) + (X\beta)g(U, Y) \\ &\quad - (Y\beta)g(U, X) + \beta\{g(\nabla_X U, Y) - g(\nabla_Y U, X)\}. \end{aligned} \tag{4.4}$$

By putting $X = \xi$ into equation (4.4) and using the assumption for ruled hypersurfaces in Q^{n*}, we have

$$\begin{aligned} g(\bar{R}(\xi, \zeta)\zeta, JY) &= g(\bar{R}(JY, J\xi)\zeta, \xi) = g(\bar{R}(\xi, Y)\xi, \zeta) \\ &= \mathrm{d}\alpha(\xi)\eta(Y) - \mathrm{d}\alpha(Y) + \alpha\beta g(\phi U, Y) + (\xi\beta)g(U, Y) + \beta g(\nabla_\xi U, Y) = 0, \end{aligned} \tag{4.5}$$

where we have used $A\phi U = 0$ in the third equality. This implies that the normal Jacobi operator $\bar{R}_\zeta$ satisfies

$$\bar{R}_\zeta\xi = \bar{R}(\xi, \zeta)\zeta = c\xi$$

for $c \in \mathbb{R}$. Then, by a result due to Berndt and Suh (see Proposition 3.1, [26]), we know that the unit normal vector field ζ is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. But, in Lemma 5.2, we will see that there does not exist any real hypersurface in Q^{n*} with η-parallel shape operator and $\mathfrak{A}$-isotropic unit normal vector field. Accordingly, among these two types of singular normal vector fields, Remark 1.2 gives us that the normal vector field ζ is $\mathfrak{A}$-principal.

Example 4.5. (The minimal homogeneous ruled real hypersurface in Q^{n*}) According to Berndt's research [24] and Remark 4.4, it is known that the unit normal vector field ζ of a ruled real hypersurface in Q^{n*} is

$\mathfrak{A}$-principal. So, there exists a real structure C on Q^{n*} so that $C\zeta = \zeta$. The real structure C is unique up to sign. Let $V(C)$ be the (+1)-eigenspace of the real structure. Then, $JV(C)$ is the (−1)-eigenspace of the real structure. Since $\zeta \in V(C)$, we have $\xi \in JV(C)$. There exists a real hyperbolic space $\mathbb{R}H^n$, embedded in Q^{n*} as a real form (i.e., an n-dimensional totally geodesic totally real submanifold) with $o \in \mathbb{R}H^n$ and $T_o\mathbb{R}H^n = JV(C)$. Then, $\xi \in T_o\mathbb{R}H^n$ determines a horocycle γ in $\mathbb{R}H^n$. The orthogonal complement of $\mathbb{R}\xi$ in $T_o\mathbb{R}H^n$ determines a totally geodesic $\mathbb{R}H^{n-1} \subset \mathbb{R}H^n$. This $\mathbb{R}H^{n-1} \subset \mathbb{R}H^n$ determines a totally geodesic $Q^{n-1*} \subset Q^{n*}$ by complexification such that $(X, JX) \in T_oQ^{n-1*}$ for $X \in T_o\mathbb{R}H^{n-1}$. By parallel translation of T_oQ^{n-1*} along the horocycle γ, we obtain a one-parameter family of totally geodesic complex hyperbolic hyperplanes, which is the ruling of the real hypersurface M in Q^{n*}.

This example explains how the homogeneous real hypersurface $M = S \cdot o$ in the complex hyperbolic quadric Q^{n*} can be viewed as a ruled hypersurface. Here, the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of the Lie algebra $\mathfrak{g}$ of the complex hyperbolic quadric Q^{n*} is used, where S denotes the Lie group corresponding to the Lie algebra $\mathfrak{s}$. The Lie algebra $\mathfrak{s}$ is defined as $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathbb{R}\zeta)$ for each unit vector $\zeta \in \mathfrak{g}_{\alpha_2}$, where $\mathfrak{a}$ denotes the maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{n}$ denotes a nilpotent subalgebra of $\mathfrak{g}$ given by $\mathfrak{n} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}$.

The shape operator A_ζ of M in Q^{n*} can be defined as follows:

$$A_\zeta X = \frac{1}{2}[\zeta - \theta(\zeta), X]_{\mathfrak{s}},$$

where $[\cdot]_{\mathfrak{s}}$ is the orthogonal projection onto $\mathfrak{s}$ and $\theta \in \mathrm{Aut}(\mathfrak{g})$ denotes the Cartan involution on $\mathfrak{g}$. Then, by a calculation due to Berndt [24], we have

$$A_\zeta \xi = \frac{1}{\sqrt{2n}} U \quad \text{and} \quad A_\zeta U = \frac{1}{\sqrt{2n}} \xi.$$

Here, the Reeb vector field ξ is defined as follows:

$$\xi = \frac{1}{\sqrt{2n}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathfrak{g}_{\alpha_1+\alpha_2}$$

and the orthogonal unit vector field U is defined as follows:

$$U = \frac{1}{\sqrt{2n}} \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1 \oplus 2\alpha_2}.$$

Berndt [24] has proved the following fact: the homogeneous ruled real hypersurface M in Q^{n*}, i.e., generated by an $\mathfrak{A}$-principal horocycle in Q^{n*}, has three distinct constant principal curvatures 0, $\sqrt{2}$, and $-\sqrt{2}$ with multiplicities $2n - 3$, 1, and 1, respectively. In particular, M is a minimal real hypersurface in (Q^{n*}, g).

5 η-parallel shape operator and key results

In this section, we will show that the unit normal vector field ζ of a ruled real hypersurface in the complex hyperbolic quadric Q^{n*} is $\mathfrak{A}$-principal. In order to do this, we will use the notion of η-parallelism, i.e., $g((\nabla_X A)Y, Z) = 0$ for any $X, Y, Z \in \mathcal{C}$, where $\mathcal{C} = \{X \in TM \mid X \perp \xi\}$ denotes the orthogonal complement of the Reeb vector field ξ on M in Q^{n*}.

By the Gauss equation of a real hypersurface M in Q^{n*}, the curvature tensor $R(X, Y)Z$ on M induced from the curvature tensor $\bar{R}$ of Q^{n*} can be described in terms of the complex structure J and the complex conjugation $C \in \mathfrak{A}$ as follows:

$$\begin{aligned} R(X,Y)Z = & -g(Y,Z)X + g(X,Z)Y - g(\phi Y,Z)\phi X + g(\phi X,Z)\phi Y + 2g(\phi X,Y)\phi Z \\ & - g(CY,Z)(CX)^{\top} + g(CX,Z)(CY)^{\top} - g(JCY,Z)(JCX)^{\top} \\ & + g(JCX,Z)(JCY)^{\top} + g(AY,Z)AX - g(AX,Z)AY \end{aligned}$$

for any $X, Y, Z \in TM$. Here, $(\,\cdot\,)^{\top}$ denotes the tangential component of $(\,\cdot\,)$.

Now let us put

$$CX = BX + \rho(X)\zeta \quad \text{and} \quad \rho(X) = g(CX, \zeta),$$

for any vector field $X \in TM$, where BX and $\rho(X)\zeta$ denote the tangential and normal components of the vector field $CX \in TQ^{n*}$, respectively. Then, together with $\rho(\xi) = g(C\xi, \zeta) = 0$, it follows that

$$C\xi = B\xi + \rho(\xi)\zeta = B\xi \tag{5.1}$$

and

$$C\zeta = CJ\xi = -JC\xi = -J(B\xi + \rho(\xi)\zeta) = -\phi B\xi - \eta(B\xi)\zeta = -\phi C\xi - \eta(C\xi)\zeta. \tag{5.2}$$

Indeed, equation (5.1) means that the vector field $C\xi$ is tangent to M, i.e., $C\xi \in TM$. Taking the covariant derivative of $C\xi$, together with the Gauss formula and equation (2.2), it follows

$$\begin{aligned} \nabla_X(C\xi) &= \bar{\nabla}_X(C\xi) - g(AX, C\xi)\zeta \\ &= (\bar{\nabla}_X C)\xi + C(\bar{\nabla}_X \xi) - g(AX, C\xi)\zeta \\ &= q(X)JC\xi + C(\nabla_X \xi + g(AX, \xi)\zeta) - g(AX, C\xi)\zeta \\ &= q(X)(\phi C\xi + g(C\xi, \xi)\zeta) + C\phi AX + g(AX, \xi)C\zeta - g(AX, C\xi)\zeta \\ &= q(X)(\phi C\xi + g(C\xi, \xi)\zeta) + B\phi AX + g(C\phi AX, \zeta)\zeta \\ &\quad - g(AX, \xi)\phi C\xi - g(AX, \xi)g(C\xi, \xi)\zeta - g(AX, C\xi)\zeta, \end{aligned}$$

where $\bar{\nabla}$ denotes the Levi-Civita connection of Q^{n*}. Then, by comparing the tangential and the normal components of the above equation, together with equation (5.2) and $\phi^2 X = -\phi X + \eta(X)\xi$, we obtain

$$\nabla_X(C\xi) = q(X)\phi C\xi + B\phi AX - g(AX, \xi)\phi C\xi \tag{5.3}$$

and

$$\begin{aligned} q(X)g(A\xi, \xi) &= -g(C\phi AX, \zeta) + g(AX, \xi)g(C\xi, \xi) + g(AX, C\xi) \\ &= g(\phi AX, \phi C\xi) + g(AX, \xi)g(C\xi, \xi) + g(AX, C\xi) \\ &= 2g(AX, C\xi). \end{aligned} \tag{5.4}$$

Moreover, it is well known that the complex structure J and the real structure C of Q^{n*} satisfy the anti-commuting property, which is given by $JC = -CJ$. From this and $J\zeta = -\xi$, we have

$$JCX = J(BX + \rho(X)\zeta) = \phi BX + \eta(BX)\zeta + \rho(X)J\zeta = \phi BX + \eta(BX)\zeta - \rho(X)\xi. \tag{5.5}$$

In addition, from the property of $C^2 = I$ and (5.2), we obtain

$$B^2 X = X - g(\phi C\xi, X)\phi C\xi, \quad B\phi C\xi = g(C\xi, \xi)\phi C\xi \tag{5.6}$$

for any tangent vector field X on M. Then, we assert the following:

Lemma 5.1. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$. If the shape operator A of M is η-parallel, then the unit normal vector field ζ of M in Q^{n*} is singular. That is, ζ is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Proof. By using equations (3.2), (5.2), and (5.5), our assumption of the shape operator A being η-parallel yields

$$0 = g(X, \phi C\xi)g(BY, Z) - g(Y, \phi C\xi)g(BX, Z) - g(X, C\xi)g(\phi BY, Z) + g(Y, C\xi)g(\phi BX, Z)$$

for any tangent vector fields X, Y, and Z belonging to the distribution $\mathcal{C} = \{X \in TM \mid X \perp \xi\}$. It can be rearranged as follows:

$$g(g(X, \phi C\xi)BY - g(Y, \phi C\xi)BX - g(X, C\xi)\phi BY + g(Y, C\xi)\phi BX, Z) = 0 \tag{5.7}$$

for any tangent vector fields $X, Y, Z \in \mathcal{C}$.

Now, let us consider that for any $X, Y \in \mathcal{C}$,

$$W_{X,Y} = g(X, \phi C\xi)BY - g(Y, \phi C\xi)BX - g(X, C\xi)\phi BY + g(Y, C\xi)\phi BX. \tag{5.8}$$

As $W_{X,Y} \in TM$, without loss of generality, it can be expressed as follows:

$$W_{X,Y} = \sum_{i=1}^{2n-1} g(W_{X,Y}, e_i)e_i = \sum_{i=1}^{2n-2} g(W, e_i)e_i + g(W, \xi)\xi$$

for any basis $\{\underbrace{e_1, \cdots, e_{2n-2}}_{\in \mathcal{C}}, e_{2n-1} = \xi\}$ of TM.

On the other hand, since $W_{X,Y}$ satisfies equation (5.7), it consequently becomes

$$W_{X,Y} = g(W_{X,Y}, \xi)\xi.$$

Its inner product with $C\xi$ implies

$$g(W_{X,Y}, C\xi) = g(W_{X,Y}, \xi)g(\xi, C\xi). \tag{5.9}$$

By using equations (5.1) and (5.6), we obtain

$$g(W_{X,Y}, C\xi) = g(C\xi, \xi)\{g(X, C\xi)g(\phi C\xi, Y) - g(Y, C\xi)g(\phi C\xi, X)\}$$

and

$$g(W_{X,Y}, \xi) = g(X, \phi C\xi)g(Y, C\xi) - g(Y, \phi C\xi)g(X, C\xi)$$

for any $X, Y \in \mathcal{C}$. From these two equations, equation (5.9) gives

$$g(C\xi, \xi)\{g(X, C\xi)g(\phi C\xi, Y) - g(Y, C\xi)g(\phi C\xi, X)\} = 0 \tag{5.10}$$

for any $X, Y \in \mathcal{C}$. So, we consider the following two cases.

Case 1. $g(C\xi, \xi) = 0$

From equation (3.3), we obtain $g(C\xi, \xi) = -\cos(2t)$, $t \in [0, \frac{\pi}{4}]$. Thus, the assumption $g(C\xi, \xi) = 0$ provides $t = \frac{\pi}{4}$. From this, the unit vector field ζ can be expressed as follows:

$$\zeta = \cos\left(\frac{\pi}{4}\right)Z_1 + \sin\left(\frac{\pi}{4}\right)JZ_2 = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for some $Z_1, Z_2 \in V(C)$. Here, $V(C)$ is the (+1)-eigenspace of C, i.e., $V(C) = \{Z \in TQ^{n*} \mid CZ = Z\}$. It means that the unit normal vector field ζ of M in Q^{n*} is $\mathfrak{A}$-isotropic.

Case 2. $g(C\xi, \xi) \neq 0$

With regard to equation (5.10), the assumption $g(C\xi, \xi) \neq 0$ indicates that

$$g(g(X, C\xi)\phi C\xi - g(X, \phi C\xi)C\xi, Y) = 0 \quad \text{for any } X, Y \in \mathcal{C}. \tag{5.11}$$

From this, the tangent vector field $U_X := g(X, C\xi)\phi C\xi - g(X, \phi C\xi)C\xi$ of M is expressed as follows:

$$U_X = \sum_{i=1}^{2n-2} g(U_X, e_i)e_i + g(U_X, \xi)\xi = g(U_X, \xi)\xi \tag{5.12}$$

for any $X \in \mathcal{C}$. Taking the inner product of equation (5.12) with $C\xi$ gives

$$g(U_X, C\xi) = g(U_X, \xi)g(\xi, C\xi). \tag{5.13}$$

By a straight calculation, together with $C^2 = I$, the vector field U_X satisfies

$$g(U_X, C\xi) = -g(X, \phi C\xi) \quad \text{and} \quad g(U_X, \xi) = -g(X, \phi C\xi)g(C\xi, \xi).$$

From these equations, equation (5.13) becomes

$$\{1 - g(C\xi, \xi)^2\}g(X, \phi C\xi) = 0 \quad \text{for any } X \in \mathcal{C}. \tag{5.14}$$

Taking $\phi C\xi \in \mathcal{C}$ instead of X in equation (5.14), together with $g(\phi C\xi, \phi C\xi) = 1 - g(C\xi, \xi)^2$, it yields

$$\{1 - g(C\xi, \xi)^2\}^2 = 0,$$

which implies $1 - g(C\xi, \xi)^2 = 0$. From this, we have $g(C\xi, \xi) = \pm 1$. Since $g(C\xi, \xi) = -\cos(2t)$, $2t \in \left[0, \frac{\pi}{2}\right)$, consequently, we have $t = 0$. From this, the unit normal vector field ζ satisfies

$$\zeta = \cos(0)Z_1 + \sin(0)JZ_2 = Z_1 \in V(C).$$

It implies that ζ is $\mathfrak{A}$-principal.

Combining the above two cases, Cases 1 and 2, we can assert that the unit normal vector field ζ of M is singular. □

By virtue of Lemma 5.1, let us consider the case of ζ being $\mathfrak{A}$-isotropic. Then, we have the following:

Lemma 5.2. *There does not exist any real hypersurface in Q^{n*}, $n \geq 3$, with η-parallel shape operator and $\mathfrak{A}$-isotropic normal vector field ζ.*

Proof. Let us assume that M is a real hypersurface with η-parallel shape operator in Q^{n*}, $n \geq 3$. That is, the shape operator A of M satisfies the following condition:

$$g((\nabla_X A)Y, Z) = 0 \tag{*}$$

for any tangent vector field $X, Y, Z \in \mathcal{C}$, where $\mathcal{C}$ denotes the orthogonal complement of the Reeb vector field ξ on M in Q^{n*}. From this, together with the equation of Codazzi (3.2) and (5.8), it yields the following for any X, $Y \in \mathcal{C}$,

$$W_{X,Y} = g(W, \xi)\xi, \tag{*}$$

where $W_{X,Y}$ is as above.

Now, since ζ is $\mathfrak{A}$-isotropic, equations (3.3) and (5.2) imply that

$$g(C\zeta, \zeta) = -g(C\xi, \xi) = 0 \quad \text{and} \quad C\zeta = -\phi C\xi \in \mathcal{C}.$$

Taking $C\zeta = -\phi C\xi$ instead of Y in (*) and using $g(C\xi, \xi) = 0$ and $B\phi C\xi = g(C\xi, \xi)\phi C\xi = 0$, we have

$$BX = g(\phi C\xi, \phi C\xi)BX = W_{X,C\zeta} = g(W_{X,C\zeta}, \xi)\xi = g(BX, \xi)\xi = g(X, C\xi)\xi$$

for any tangent vector field $X \in \mathcal{C}$. From this, applying the symmetric operator B, together with equations (5.1) and (5.6), it follows that

$$X - g(\phi C\xi, X)\phi C\xi = B^2X = g(X, C\xi)B\xi = g(X, C\xi)C\xi,$$

which implies

$$X = g(X, \phi C\xi)\phi C\xi + g(X, C\xi)C\xi \in \mathcal{C}.$$

This means $\dim_{\mathbb{R}} \; \mathcal{C} = 2$. But, in fact, any vector field $X \in \mathcal{C}$ is expressed as:

$$X = \sum_{k=1}^{2n-2} g(X, e_k)e_k$$

with respect to the basis $\{C\zeta = -\phi C\xi, C\xi, e_1, e_2, \cdots, e_{2n-4}\}$ of the distribution $\mathcal{C}$. So, we obtain $\dim_{\mathbb{R}} \mathcal{C} = 2n - 2$, $n \geq 3$, which gives a contradiction. From this, we give a complete proof of our lemma. □

Consequently, summing up Lemmas 5.1 and 5.2, we obtain the following proposition:

Proposition 5.3. *Let M be a real hypersurface in Q^{n*}, $n \geq 3$. If the shape operator A of M is η-parallel, then the unit normal vector field ζ of M in Q^{n*} is $\mathfrak{A}$-principal.*

On the other hand, as introduced in Theorem A, a tube $(\mathcal{T}^*_A)$ and a horosphere $(\mathcal{H}^*_A)$ are given as the model spaces of real hypersurfaces with $\mathfrak{A}$-isotropic normal vector field in Q^{n*}, $n \geq 3$. Here, $(\mathcal{T}^*_A)$ and $(\mathcal{H}^*_A)$, respectively, denote a tube over a complex hyperbolic space $\mathbb{C}H^k$ in Q^{2k*} and a horosphere whose center at infinity is the equivalence class of $\mathfrak{A}$-isotropic singular geodesics in Q^{n*}. We will give a proof of Theorem 1.1 in Section 6. In order to do this, we need the following proposition:

Proposition 5.4. *The shape operators of type $(\mathcal{T}^*_A)$ and $(\mathcal{H}^*_A)$ real hypersurfaces in Q^{n*} are not η-parallel.*

Proof. Let a tube $(\mathcal{T}^*_A)$ and a horosphere $(\mathcal{H}^*_A)$ in the complex hyperbolic quadric Q^{n*} be denoted as M_A. Then, the unit normal vector field ζ of M_A is $\mathfrak{A}$-isotropic, and the shape operator A of M_A commutes with the structure tensor ϕ (see Suh [11]).

Now, let us assume that the shape operator A of M_A is η-parallel, i.e., A satisfies

$$g((\nabla_X A)Y, Z) = 0 \quad \text{for any } X, Y, Z \in \mathcal{C}.$$

From this, for the case X, $Z \in \mathcal{Q} = TM_A \ominus (\text{span}\{\xi\} \oplus T_\beta)$ and $Y \in T_\beta$, where $T_\beta = \{Y \in TM_A \mid AY = \beta Y = 0\} = \text{span}\{C\xi, \phi C\xi\}$, we know that $AY = 0$ for $Y \in T_\beta$, which implies $(\nabla_X A)Y = -A(\nabla_X Y)$. Then, the inner product with $Z \in \mathcal{Q}$ gives

$$g((\nabla_X A)Y, Z) = -g(A(\nabla_X)Y, Z) = -g(\nabla_X Y, AZ) = -\sigma g(\nabla_X Y, Z), \tag{5.16}$$

where the constant principal curvature σ is given by

$$\sigma = \begin{cases} \lambda = \tanh(r) & \text{for } Z \in T_\lambda = T(\mathbb{C}H^k) \ominus (\text{span}\{\xi\} \oplus T_\beta), \\ \mu = \coth(r) & \text{for } Z \in T_\mu = \nu(\mathbb{C}H^k) \ominus \mathbb{C}(\nu \mathcal{T}^*_A), \\ \quad 1 & \text{for } Z \in T(\mathcal{H}^*_A) \ominus (\text{span}\{\xi\} \oplus T_\beta), \end{cases} \tag{5.17}$$

respectively.

On the other hand, we may put

$$\nabla_X Y = g(\nabla_X Y, \xi)\xi + g(\nabla_X Y, C\xi)C\xi + g(\nabla_X Y, \phi C\xi)\phi C\xi + g(\nabla_X Y, W)W \tag{5.18}$$

for some vector field $W \in \mathcal{Q}$. Since M_A satisfies $A\phi = \phi A$, we obtain $A\phi Y = \phi AY = 0$ for any $Y \in T_\beta$. Also, M_A has a $\mathfrak{A}$-isotropic unit normal vector field ζ, which means that $\eta(C\xi) = g(C\xi, \xi) = 0$. From these facts, together with equation (5.3) and $\phi^2 C\xi = -C\xi + \eta(C\xi)\xi = -C\xi$, we obtain

$$\begin{aligned} g(\nabla_X Y, \xi) &= -g(Y, \nabla_X \xi) = -g(Y, \phi AX) = g(A\phi Y, X) = 0, \\ g(\nabla_X Y, C\xi) &= -g(Y, \nabla_X C\xi) = -g(Y, q(X)\phi C\xi + B\phi AX) \\ &= -q(X)g(Y, \phi C\xi) - g(Y, B\phi AX), \end{aligned}$$

and

$$\begin{aligned} g(\nabla_X Y, \phi C\xi) &= -g(Y, \nabla_X(\phi C\xi)) = -g(Y, (\nabla_X \phi)C\xi) - g(Y, \phi(\nabla_X C\xi)) \\ &= -g(Y, \eta(C\xi)AX - g(AX, C\xi)\xi) + g(\phi Y, q(X)\phi C\xi + B\phi AX) \\ &= q(X)g(Y, C\xi) + g(\phi Y, B\phi AX) \end{aligned}$$

for $X \in Q$ and $Y \in T_\beta$. From the above three equations, equation (5.18) can be arranged as follows:

$$\nabla_X Y = \{-q(X)g(Y, \phi C\xi) - g(Y, B\phi AX)\}C\xi + \{q(X)g(Y, C\xi) + g(\phi Y, B\phi AX)\}\phi C\xi + g(\nabla_X Y, W)W, \quad (5.19)$$

which gives

$$g(\nabla_X Y, Z) = g(\nabla_X Y, W)g(W, Z)$$

for $X, Z \in Q$ and $Y \in T_\beta$. From this, (5.16) becomes

$$g((\nabla_X A)Y, Z) = -\sigma g(\nabla_X Y, W)g(W, Z) \quad \forall X, Z \in Q, \quad Y \in T_\beta. \quad (5.20)$$

- On the tube $(\mathcal{T}^*_A)$

Since $Q = T_\lambda \oplus T_\mu \subset T(\mathcal{T}^*_A)$, we put $W = W_1 + W_2$ for some two vectors W_1 and W_2 such that $W_1 \in T_\lambda$ and $W_2 \in T_\mu$. So, equation (5.20) is rearranged as follows:

$$g((\nabla_X A)Y, Z) = -\sigma\{g(\nabla_X Y, W_1)g(W_1, Z) + g(\nabla_X Y, W_2)g(W_1, Z) + g(\nabla_X Y, W_1)g(W_2, Z) + g(\nabla_X Y, W_2)g(W_2, Z)\},$$

and our assumption of A being η-parallel implies

$$-\sigma\{g(\nabla_X Y, W_1)g(W_1, Z) + g(\nabla_X Y, W_2)g(W_1, Z) + g(\nabla_X Y, W_1)g(W_2, Z) + g(\nabla_X Y, W_2)g(W_2, Z)\} = 0 \quad (5.21)$$

for any $X, Z \in Q$ and $Y \in T_\beta$.

On the other hand, from equation (5.17), we see that $\lambda = \tanh(r) \neq 0$ and $\mu = \coth(r) \neq 0$ for $r \in \mathbb{R}^+$. Hence, equation (5.21) yields that for any $X, Z \in Q$ and $Y \in T_\beta$

$$g(\nabla_X Y, W_1)g(W_1, Z) + g(\nabla_X Y, W_2)g(W_1, Z) + g(\nabla_X Y, W_1)g(W_2, Z) + g(\nabla_X Y, W_2)g(W_2, Z) = 0,$$

which gives a contradiction. So, we claim that $(\mathcal{T}^*_A)$ does not have η-parallel shape operator.

- On the horosphere $(\mathcal{H}^*_A)$

On $Q \subset T(\mathcal{H}^*_A)$, the principal curvature σ is given by 1 in equation (5.17). So, by equation (5.20) and the assumption of A being η-parallel, we obtain $g(\nabla_X Y, W)g(W, Z) = 0$ for any $Z \in Q$. So, putting $Z = W$ follows $g(\nabla_X Y, W) = 0$. From this fact and equation (5.19), we obtain

$$\nabla_X Y = \{-q(X)g(Y, \phi C\xi) - g(Y, B\phi AX)\}C\xi + \{q(X)g(Y, C\xi) + g(\phi Y, B\phi AX)\}\phi C\xi.$$

Taking $Y = C\xi \in T_\beta$, together with $BC\xi = \xi$ and $B\phi C\xi = g(C\xi, \xi)\phi C\xi = 0$, becomes

$$\nabla_X C\xi = q(X)\phi C\xi.$$

Combining this formula and equation (5.3) and using $AX = X$ for $X \in Q$, we obtain $B\phi X = 0$. Applying the symmetric operator B to this formula and using equation (5.6), together with $\phi^2 = -I + \eta \otimes \xi$, we obtain $\phi X = 0$, which means that $X = 0$ for any $X \in Q$. It means that the dimension of Q is 0, i.e., $\dim Q = 0$. But, by virtue of Proposition A in [27], we obtain $\dim Q = 2n - 4$. It makes a contradiction for $n \geq 3$. So the shape operator A of the horosphere $(\mathcal{H}^*_A)$ is not η-parallel. It gives a complete proof of our proposition. □

Now, as a characterization of a ruled real hypersurface in Q^{n*}, $n \geq 3$, we can assert the following lemma:

Lemma 5.5. *Let M be a ruled real hypersurface in Q^{n*}, $n \geq 3$. Then, the shape operator A of M is η-parallel.*

Proof. As mentioned in Introduction, the expression of the shape operator A of M in Q^{n*} is given as follows:

$$\begin{cases} A\xi = \alpha\xi + \beta U, \\ AU = \beta\xi, \\ AX = 0 \quad \text{for any } X \perp \xi, U, \end{cases} \quad (5.22)$$

where U is some unit vector field in $\mathcal{C} = \{X \in TM \mid X \perp \xi\}$ and $\beta := g(A\xi, U)$ is a nonzero function on M. From this, we obtain

$$g(AX, Y) = 0 \quad \text{for any } X, Y \in \mathcal{C}. \quad (5.23)$$

Let Y be any tangent vector field of M such that $Y \in \mathcal{C}$, i.e., $g(Y, \xi) = 0$. Taking the covariant derivative of this formula with $X \in \mathcal{C}$ and using equation (5.23), we obtain

$$g(\nabla_X Y, \xi) = -g(Y, \nabla_X \xi) = -g(Y, \phi AX) = g(\phi Y, AX) = 0, \tag{5.24}$$

i.e., it assures that $\nabla_X Y \in \mathcal{C}$ for any $X, Y \in \mathcal{C}$.

On the other hand, taking the covariant derivative of equation (5.23) with $Z \in \mathcal{C}$ and using equation (5.24), it follows that

$$0 = g((\nabla_Z A)X, Y) + g(A\nabla_Z X, Y) + g(AX, \nabla_Z Y) = g((\nabla_Z A)X, Y)$$

for any $X, Y, Z \in \mathcal{C}$. Hence, we can assert that the shape operator A of M is η-parallel. □

By virtue of Proposition 5.3 and Lemma 5.5, we obtain the following proposition:

Proposition 5.6. *The unit normal vector field ζ of a ruled real hypersurface in Q^{n*}, $n \geq 3$, is $\mathfrak{A}$-principal.*

6 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 from the Introduction. By the notions of η-parallel and η-commuting shape operator, we give a complete classification of real hypersurfaces in the complex hyperbolic quadric Q^{n*} with these properties. To do so, unless otherwise specified, we assume that M is a real hypersurface in the complex hyperbolic quadric Q^{n*} for $n \geq 3$, and the shape operator A of M satisfies η-parallelism and η-commutativity. Since in Proposition 5.6 we have proved that the unit normal vector field ζ of a ruled real hypersurface in Q^{n*} is $\mathfrak{A}$-principal, we remarked in Theorem 1.1 that the unit normal ζ of ruled real hypersurfaces in the complex hyperbolic quadric Q^{n*} is $\mathfrak{A}$-principal.

Lemma 6.1. *Let M be a real hypersurface in Q^{n*}, $n \geq 3$, with η-parallel and η-commuting shape operator. Then, for any $X, Y, Z \in \mathcal{C}$, we have*

$$\begin{aligned} 0 = {} & g(Y, C\zeta)g(CX, Z) + g(\phi Z, C\zeta)g(CX, \phi Y) - g(Y, C\xi)g(CX, \phi Z) \\ & + g(\phi Z, C\xi)g(CX, Y) - \eta(A\phi Z)g(Y, AX) + g(X, V)g(Y, AZ) + g(Y, V)g(X, AZ). \end{aligned}$$

where $\mathcal{C}$ denotes the orthogonal complement of the Reeb vector field ξ and V is given by $\phi A\xi$.

Proof. The notion of η-commuting shape operator gives

$$g((A\phi - \phi A)Y, Z) = 0$$

for any $Y, Z \in \mathcal{C}$. By differentiating this, we have

$$\begin{aligned} g((\nabla_X A)Y, \phi Z) + g((\nabla_X A)Z, \phi Y) = {} & \eta(AY)g(X, AZ) + \eta(AZ)g(Y, AX) + g(X, A\phi Y)g(Z, V) \\ & + g(X, A\phi Z)g(Y, V). \end{aligned} \tag{6.1}$$

Then, let us consider cyclic formulas with respect X, Y, and Z as follows:

$$\begin{aligned} g((\nabla_Y A)Z, \phi X) + g((\nabla_Y A)X, \phi Z) = {} & \eta(AZ)g(Y, AX) + \eta(AX)g(Z, AY) + g(Y, A\phi Z)g(X, V) \\ & + g(Y, A\phi X)g(Z, V) \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} g((\nabla_Z A)X, \phi Y) + g((\nabla_Z A)Y, \phi X) = {} & \eta(AX)g(Z, AY) + \eta(AY)g(X, AZ) + g(Z, A\phi X)g(Y, V) \\ & + g(Z, A\phi Y)g(X, V). \end{aligned} \tag{6.3}$$

Then, let us subtract equation (6.3) from the summing up of equations (6.1) and (6.2). From this, by using the equation of Codazzi (3.2), it follows that

$$\begin{aligned} & g((\nabla_X A)Y, \phi Z) + g((\nabla_Y A)X, \phi Z) + g((\nabla_X A)Z - (\nabla_Z A)X, \phi Y) + g((\nabla_Y A)Z - (\nabla_Z A)Y, \phi X) \\ &= 2\eta(AZ)g(Y, AX) + 2g(X, V)g(Y, A\phi Z) + 2g(Y, V)g(X, A\phi Z) \\ &= 2g((\nabla_X A)Y, \phi Z) + \{g(X, C\zeta)g(CY, \phi Z) - g(Y, C\zeta)g(CX, \phi Z) + g(X, C\xi)g(JCY, \phi Z) \\ &\quad - g(Y, C\xi)g(JCX, \phi Z)\} - \{g(X, C\zeta)g(CZ, \phi Y) - g(Z, C\zeta)g(CX, \phi Y) + g(X, C\xi)g(JCZ, \phi Y) \\ &\quad - g(Z, C\xi)g(JCX, \phi Y)\} - \{g(Y, C\zeta)g(CZ, \phi X) - g(Z, C\zeta)g(CY, \phi X) \\ &\quad + g(Y, C\xi)g(JCZ, \phi X) - g(Z, C\xi)g(JCY, \phi X)\}. \end{aligned} \tag{6.4}$$

Then, by using the η-commuting property in equation (6.4) and using the following:

$$g(JCY, \phi Z) = -g(CY, J\phi Z) = g(CY, Z),$$

we have

$$\begin{aligned} & g((\nabla_X A)Y, \phi Z) - g(Y, C\zeta)g(CX, \phi Z) + g(Z, C\zeta)g(CX, \phi Y) - g(Y, C\xi)g(CX, Z) + g(Z, C\xi)g(CX, Y) \\ &= \eta(AZ)g(Y, AX) + g(X, V)g(Y, A\phi Z) + g(Y, V)g(X, A\phi Z) \end{aligned} \tag{6.5}$$

for any $X, Y, Z \in \mathcal{C}$. Then, by replacing Z with ϕZ in equation (6.5), we have

$$\begin{aligned} g((\nabla_X A)Y, Z) = {} & g(Y, C\zeta)g(CX, Z) + g(\phi Z, C\zeta)g(CX, \phi Y) - g(Y, C\xi)g(CX, \phi Z) + g(\phi Z, C\xi)g(CX, Y) \\ & - \eta(A\phi Z)g(Y, AX) + g(X, V)g(Y, AZ) + g(Y, V)g(X, AZ). \end{aligned} \tag{6.6}$$

This gives a complete proof of our Lemma. □

By virtue of Proposition 5.3, we see that the unit normal vector field ζ of M in Q^{n*} is $\mathfrak{A}$-principal, i.e., $C\zeta = \zeta$ and $C\xi = -\xi$. Thus, by using $V = \phi A\xi$, Lemma 6.1 gives

$$g(X, V)g(Y, AZ) + g(Y, V)g(Z, AX) + g(Z, V)g(X, AY) = 0 \tag{6.7}$$

for any vector fields X, Y, and $Z \in \mathcal{C}$. Now, let us put $A\xi = \alpha\xi + \beta U$ in equation (6.7). Then, we assert the following lemma:

Lemma 6.2. *Let M be a complete real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$, with η-parallel and η-commuting shape operator. Then,*

$$\beta = 0 \quad \text{or} \quad g(AY, Z) = 0$$

for any vector fields $Y, Z \in \mathcal{C}$, where $\mathcal{C}$ denotes the orthogonal distribution of the Reeb vector field ξ.

Proof. Let us put $Z = V = \phi A\xi$ in equation (6.7) and use $A\xi = \alpha\xi + \beta U$ for some $U \in \mathcal{C}$. Then, it follows that

$$\begin{aligned} 0 &= g(AX, Y)||V||^2 + g(AY, V)g(X, V) + g(AV, X)g(Y, V) \\ &= g(AX, Y)||V||^2 + \beta^2 g(AY, \phi U)g(X, \phi U) + \beta^2 g(A\phi U, X)g(Y, \phi U). \end{aligned} \tag{6.8}$$

Then, for any $X, Y \in \mathcal{C}$, which are orthogonal to ϕU, the formula (6.8) gives $g(AX, Y) = 0$. Now, we put $X = Y = \phi U$ in equation (6.8). Then, it follows that

$$0 = g(A\phi U, \phi U)||V||^2 + 2\beta^2 g(A\phi U, \phi U) = 3\beta^2 g(A\phi U, \phi U), \tag{6.9}$$

where we have used $||V||^2 = g(\phi A\xi, \phi A\xi) = \beta^2$. Then, (6.9) gives that the function $\beta = 0$ or $g(A\phi U, \phi U) = 0$. Now, let us consider the case that $\beta \neq 0$ on the open subset $\mathcal{U}$ in M, i.e., $\mathcal{U} = \{p \in M \mid \beta(p) \neq 0\}$. Then, $g(A\phi U, \phi U) = 0$ on $\mathcal{U}$. From this, together with putting $Y = \phi U$ in equation (6.8), we have, for any $X \in \mathcal{C}$,

$$0 = g(A\phi U, X)||V||^2 + \beta^2 g(A\phi U, X) = 2\beta^2 g(A\phi U, X). \tag{6.10}$$

Hence, it follows that $g(A\phi U, X) = 0$ on $\mathcal{U}$ for any $X \in \mathcal{C}$. From this, together with $g(AX, Y) = 0$ for any $X, Y \in \mathcal{C}$ orthogonal to ϕU, we can assert the latter part of Lemma 6.2. From this, we give a complete proof of Lemma 6.2. □

If M is Hopf, i.e., the Reeb vector field ξ is principal for the shape operator A of a real hypersurface M in Q^{n*}, then we obtain $0 = \phi A\xi = A\phi\xi$. From this, together with the η-commuting shape operator, $g((A\phi - \phi A)X, Y) = 0$ for any $X, Y \in \mathcal{C}$, it naturally gives that the structure tensor ϕ commutes with the shape operator A, i.e., $A\phi = \phi A$. Then, by Theorem A we assert the following proposition:

Proposition 6.3. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$, with η-parallel and η-commuting shape operator. Then, M is locally congruent to a tube of radius r over a totally geodesic complex submanifold $\mathbb{C}H^k$ in Q^{2k*}, $n = 2k$, or a horosphere.*

Moreover, in Proposition 5.4, we have mentioned that the shape operator of a tube over $\mathbb{C}H^k$ in Q^{2k*} or a horosphere does not satisfy η-parallelism. Then, combining Propositions 6.3 and 5.4, we assert the following theorem:

Theorem 6.4. *There does not exist any Hopf real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$, with η-parallel and η-commuting shape operator.*

Then, by Lemma 6.2 and Theorem 6.4, we have only the case $g(AY, Z) = 0$ for any vector fields Y and Z in the distribution $\mathcal{C}$. Hence, by Theorem 4.2, we can assert Theorem 1.1. Moreover, by virtue of Proposition 5.6, the unit normal vector field of a ruled real hypersurface in Q^{n*} is $\mathfrak{A}$-principal. This completes the proof of Theorem 1.1.

7 Proof of Theorem 1.3

Let M be a real hypersurface with η-parallel shape operator in the complex hyperbolic quadric Q^{n*}, $n \geq 3$. In this section, we give a complete classification of such real hypersurfaces in Q^{n*} with integrable holomorphic distribution $\mathcal{C} = \{X \in TM \mid X \perp \xi\}$. To do so, let us study the geometric property of $\mathcal{C}$ being integrable as follows:

Lemma 7.1. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$. The following assertions are equivalent*:

(*i*) *The holomorphic distribution $\mathcal{C} = \{X \in TM \mid X \perp \xi\}$ is integrable.*

(*ii*) *The shape operator A is η-anticommuting, i.e., $g((\phi A + A\phi)X, Y) = 0$ for any $X, Y \in \mathcal{C}$.*

Proof. (i) ⇒ (ii): Assume that the holomorphic distribution $\mathcal{C}$ is integrable. Then, we obtain

$$[X, Y] \in \mathcal{C}, \tag{7.1}$$

which implies $g([X, Y], \xi) = 0$ for any $X, Y \in \mathcal{C}$. Since the Levi-Civita connection ∇ of M is torsion-free, it follows that $[X, Y] = \nabla_X Y - \nabla_Y X$. So, equation (7.1) yields

$$g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi) = 0. \tag{7.2}$$

By the differentiation of $g(Y, \xi) = 0$ on M, we obtain $g(\nabla_X Y, \xi) = -g(Y, \nabla_X \xi) = -g(Y, \phi AX)$. From this, equation (7.2) is rewritten as follows:

$$-g(Y, \phi AX) + g(X, \phi AY) = 0.$$

Since the operator ϕA is skew-symmetric, it becomes

$$g((\phi A + A\phi)X, Y) = 0$$

for any $X, Y \in C$. It means that the shape operator A of M is η-anticommuting.

(ii) ⇒ (i): By virtue of the contents above, it is clear (*vice versa*). □

With regard to Theorem 4.2 and Lemma 5.5, we give some characterizations of a ruled real hypersurface in Q^{n*} as follows:

Proposition 7.2. *Let M be a ruled real hypersurface in Q^{n*}, $n \geq 3$. Then, the following statements hold*:
(*a*) *The holomorphic distribution C of M is integrable.*
(*b*) *The shape operator A of M is η-parallel.*

Proof. (b) As shown in Lemma 5.5, the shape operator A of a ruled real hypersurface M in Q^{n*} is η-parallel. So, in the remaining part of this proof, we will show that the holomorphic distribution C of M is integrable.

(a) By virtue of Theorem 4.2, the shape operator A of M satisfies $g(AX, Y) = 0$ for any $X, Y \in C$. Since the tangent vector fields ϕX and ϕY belong to C, this property provides

$$g((\phi A + A\phi)X, Y) = -g(AX, \phi Y) + g(AY, \phi X) = 0$$

for any $X, Y \in C$. That is, M has η-anticommuting shape operator. Hence, by Lemma 7.1, we can assure that the holomorphic distribution C of M is integrable. □

Now, as the converse of Proposition 7.2, we prove:

Proposition 7.3. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$. If the shape operator of M is η-parallel and the holomorphic distribution $C = \{X \in TM \mid X \perp \xi\}$ is integrable, then the shape operator A of M satisfies $g(AX, Y) = 0$ for any vector fields $X, Y \in C$. Furthermore, M is locally congruent to a ruled real hypersurface in Q^{n*}.*

Proof. From Lemma 7.1, the assumption of C being integrable gives

$$g((\phi A + A\phi)X, Y) = 0 \quad \text{for } X, Y \in C. \tag{7.3}$$

Taking the covariant derivative of equation (7.3) with $Z \in C$, we obtain

$$\begin{aligned} &g((\nabla_Z\phi)AX, Y) + g(\phi(\nabla_Z A)X, Y) + g(\phi A(\nabla_Z X), Y) + g(\phi AX, \nabla_Z Y) \\ &\quad + g((\nabla_Z A)\phi X, Y) + g(A(\nabla_Z\phi)X, Y) + g(A\phi(\nabla_Z X), Y) + g(A\phi X, \nabla_Z Y) = 0. \end{aligned} \tag{7.4}$$

Because of $T_pM = \operatorname{span}\{\xi\} \oplus C$ for any point p of M, we may put $\nabla_Z X = (\nabla_Z X)_C + g(\nabla_Z X, \xi)\xi \in TM$, where $(\cdot)_C$ denotes the C-component of any tangent vector field $(\cdot)$ of M. From this, equation (7.4) can be rearranged as follows:

$$\begin{aligned} &g((\nabla_Z\phi)AX, Y) + g(\phi(\nabla_Z A)X, Y) + g(\phi A(\nabla_Z X)_C, Y) \\ &\quad + g(\nabla_Z X, \xi)g(\phi A\xi, Y) + g(\phi AX, (\nabla_Z Y)_C) \\ &\quad + g((\nabla_Z A)\phi X, Y) + g(A(\nabla_Z\phi)X, Y) + g(A\phi(\nabla_Z X)_C, Y) \\ &\quad + g(A\phi X, (\nabla_Z Y)_C) + g(A\phi X, \xi)g(\nabla_Z Y, \xi) = 0. \end{aligned}$$

By our assumption of A being η-parallel and equation (7.3), the previous equation becomes

$$0 = g((\nabla_Z\phi)AX, Y) + g(\nabla_Z X, \xi)g(\phi A\xi, Y) + g(A(\nabla_Z\phi)X, Y) + g(\nabla_Z Y, \xi)g(A\phi X, \xi) \tag{7.5}$$

for any $X, Y, Z \in C$. By the formula $(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi$, we obtain

$$g((\nabla_Z\phi)AX, Y) = \eta(AX)g(AZ, Y) - g(AZ, AX)\eta(Y) = g(AX, \xi)g(AZ, Y) \tag{7.6}$$

and

$$g(A(\nabla_Z\phi)X, Y) = g((\nabla_Z\phi)X, AY) = \eta(X)g(AZ, AY) - g(AZ, X)g(AY, \xi) = -g(AZ, X)g(AY, \xi). \tag{7.7}$$

Substituting equations (7.6) and (7.7) in equation (7.5) yields

$$g(A\xi, X)g(AY, Z) - g(X, \phi AZ)g(\phi A\xi, Y) - g(A\xi, Y)g(AX, Z) - g(Y, \phi AZ)g(A\phi X, \xi) = 0, \tag{7.8}$$

where we have used $g(\nabla_Z X, \xi) = g(X, \nabla_Z \xi) = g(X, \phi AZ)$ for any $X, Y, Z \in \mathcal{C}$.

In Lemma 7.4, we prove that there does not exist any Hopf real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$, satisfying all assumptions given in Proposition 7.3. By virtue of this assertion, we may put $A\xi = \alpha\xi + \beta U$, where β is a nonvanishing smooth function on a neighborhood of a point $p \in M$ and U is a unit vector field in $\mathcal{C}$. From this, equation (7.8) becomes

$$0 = \beta g(U, X)g(AY, Z) + \beta g(X, \phi AZ)g(U, \phi Y) - \beta g(U, Y)g(AX, Z) - \beta g(Y, \phi AZ)g(\phi X, U). \tag{7.9}$$

Putting $X = \phi U \in \mathcal{C}$ and $Y = U \in \mathcal{C}$ in equation (7.9) we obtain $\beta g(A\phi U, Z) = 0$. Since $\beta \neq 0$, it implies $g(A\phi U, Z) = 0$ for any $Z \in \mathcal{C}$. So, we obtain

$$A\phi U = g(A\phi U, \xi)\xi = \alpha g(\phi U, \xi)\xi + \beta g(\phi U, U)\xi = 0. \tag{7.10}$$

Substituting $Y = U$ in equation (7.9) and using equation (7.10), together with $\beta \neq 0$, provide

$$g(U, X)g(AU, Z) - g(AX, Z) = 0. \tag{7.11}$$

Take $X = W \in \mathcal{C}$, where W is any tangent vector field satisfying $W \perp U$. Then, equation (7.11) gives $g(AW, Z) = 0$ for any $Z \in \mathcal{C}$. So, we obtain

$$AW = g(W, A\xi)\xi = \alpha g(W, \xi)\xi + \beta g(W, U)U = 0. \tag{7.12}$$

Now, putting $X = U$ and $Y = \phi U$ in equation (7.3) and using equation (7.10) yield

$$0 = g(\phi AU, \phi U) = g(AU, U) - \eta(U)g(AU, \xi) = g(AU, U).$$

From this fact and $A\xi = \alpha\xi + \beta U$, together with equations (7.10) and (7.12), the tangent vector field AU is expressed as follows:

$$\begin{aligned} AU &= \sum_{i=1}^{2n-1} g(AU, e_i)e_i \\ &= \sum_{i=1}^{2n-4} g(AU, e_i)e_i + g(AU, U)U + g(AU, \phi U)\phi U + g(AU, \xi)\xi \\ &= \sum_{i=1}^{2n-4} g(U, Ae_i)e_i + g(AU, U)U + g(U, A\phi U)\phi U + g(U, A\xi)\xi = \beta\xi \end{aligned}$$

for any basis $\{e_1, e_2, \cdots, e_{2n-4}, e_{2n-3} = U, e_{2n-2} = \phi U, e_{2n-1} = \xi\}$ of TM.

Summing up the above facts, we obtain

$$AX = \begin{cases} \beta\xi & \text{if } X = U \\ 0 & \text{if } X = \phi U \\ 0 & \text{if } X \in \mathcal{C} \ominus \text{span}\{U, \phi U\}, \end{cases}$$

which means that $g(AX, Y) = 0$ for any $X, Y \in \mathcal{C}$. By virtue of Theorem 4.2, we can assert that M is locally congruent to a ruled real hypersurface in Q^{n*}. □

Finally, let us consider the case of $\beta = 0$, which means that M is Hopf, in Proposition 7.3 as follows. By means of Proposition 5.3, we obtain the following lemma:

Lemma 7.4. *There does not exist any Hopf real hypersurface M in the complex hyperbolic quadric Q^{n*}, $n \geq 3$, with η-parallel shape operator and integrable holomorphic distribution $\mathcal{C}$.*

Proof. Since M is Hopf, we may put $A\xi = \alpha\xi$. From this fact and our assumption of $\mathcal{C}$ being integrable, Lemma 7.1 assures $\phi AX + A\phi X = 0$ for all $X \in TM$. That is, we obtain

$$A\phi X = -\phi AX \tag{7.13}$$

for any $X \in TM$. In this case, the shape operator A of M in Q^{n*} is said to be *anti-commuting*.

Now, by the assumption of η-parallelism and Proposition 5.3, the unit normal vector field ζ of M in Q^{n*} is $\mathfrak{A}$-principal. By using this fact and our assumption, we obtain

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= \sum_{i=1}^{2n-2} g((\nabla_X A)Y - (\nabla_Y A)X, e_i)e_i + g((\nabla_X A)Y - (\nabla_Y A)X, \xi)\xi \\ &= g((\nabla_X A)Y - (\nabla_Y A)X, \xi)\xi \\ &= \{g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X)\}\xi \\ &= \{(X\alpha)\eta(Y) + \alpha g(\phi AX, Y) - g(A\phi AX, Y) - (Y\alpha)\eta(X) - \alpha g(\phi AY, X) + g(A\phi AY, X)\}\xi \\ &= \{\alpha g(\phi AX, Y) - g(A\phi AX, Y) - \alpha g(\phi AY, X) + g(A\phi AY, X)\}\xi \\ &= \{\alpha g((\phi A + A\phi)X, Y) - 2g(A\phi AX, Y)\}\xi\end{aligned}$$

for any basis $\{\underbrace{e_1, e_2, \cdots, e_{2n-2}}_{C}, e_{2n-1} = \xi\}$ of T_pM, $p \in M$. Then, from equation (7.13), it becomes

$$(\nabla_X A)Y - (\nabla_Y A)X = -2g(A\phi AX, Y)\xi \quad \text{for any } X, Y \in C. \tag{7.14}$$

On the other hand, the fact of ζ being $\mathfrak{A}$-principal gives $C\xi = -\xi$ and $C\zeta = \zeta$. From these formulas and equation (3.2), we obtain

$$(\nabla_X A)Y - (\nabla_Y A)X = 2g(\phi X, Y)\xi \quad \text{for any } X, Y \in C. \tag{7.15}$$

Combining with equations (7.14) and (7.15) yields

$$g(A\phi AX + \phi X, Y) = 0 \quad \text{for any } X, Y \in C.$$

It follows that $A\phi AX + \phi X = g(A\phi AX + \phi X, \xi)\xi = 0$, i.e.,

$$A\phi AX = -\phi X \quad \text{for any } X \in C. \tag{7.16}$$

By equations (7.13) and (7.16), we obtain $\phi A^2 X = \phi X$ for any $X \in C$. Applying the structure tensor ϕ to this equation and using $\phi^2 = -I + \eta \otimes \xi$, we obtain

$$A^2 X = X \quad \text{for any } X \in C. \tag{7.17}$$

Take $X_0 \in C$ with $AX_0 = \lambda X_0$. Then, from equation (7.17), we obtain $\lambda^2 = 1$, i.e., $\lambda = \pm 1$. It implies that $AX_0 = \pm X_0$. Besides, by virtue of equation (7.13), we obtain $A\phi X_0 = \mp\phi X_0$. By such relations, the expression of the shape operator A of M is given as follows:

$$A = \operatorname{diag}(\alpha, \underbrace{\overbrace{1, 1, \ldots, 1}^{T_1}, \overbrace{-1, -1, \ldots, -1}^{T_{-1}}}_{C}),$$

where T_1 and T_{-1} are the eigenspaces given by $T_1 = \{X \in C \mid AX = X\}$ and $T_{-1} = \{X \in C \mid AX = -X\}$, respectively. Their corresponding multiplicities satisfy $m(T_1) = m(T_{-1}) = n - 1$.

In general, if the unit normal vector field ζ of a Hopf real hypersurface in Q^{n*} is $\mathfrak{A}$-principal, then we obtain

$$CAX = AX - 2g(AX, \xi)\xi = AX - 2\alpha\eta(X)\xi$$

for any tangent vector field X on M (see Lemma 5.1 in [28]). From this fact, we obtain

$$CAX = AX \quad \text{for any } X \in C. \tag{7.18}$$

By the above expression of A, the holomorphic distribution C is given by $C = T_1 \oplus T_{-1}$. Thus, equation (7.18) yields

$$\left.\begin{matrix} CX \\ -CX \end{matrix}\right\} = CAX = AX = \begin{cases} X & \text{for } X \in T_1 \\ -X & \text{for } X \in T_{-1}, \end{cases}$$

i.e., $CX = X$ for all $X \in \mathcal{C}$. So, we have

$$CX = \begin{cases} \zeta & \text{for } X = \zeta \\ -\xi & \text{for } X = \xi \\ X & \text{for } X \in \mathcal{C}. \end{cases}$$

From this, let us calculate the trace $\operatorname{Tr} C$ of C. Then, we obtain for any basis $\{e_1, e_2, \dots, e_{2n-2}, e_{2n-1} = \xi, e_{2n} = \zeta\}$ of TQ^{n*}

$$\operatorname{Tr} C = \sum_{i=1}^{2n} g(Ce_i, e_i) = \sum_{i=1}^{2n-2} g(Ce_i, e_i) + g(C\xi, \xi) + g(C\zeta, \zeta) = 2n - 2, \tag{7.19}$$

which gives a contradiction. In fact, it is well known that the trace of C in Q^{n*} satisfies $\operatorname{Tr} C = 0$. From this, equation (7.19) implies $n = 1$. But, in this lemma, we only consider the case of $n \geq 3$. So, it completes this proof. □

Hence, by using Propositions 7.2 and 7.3, we give a complete proof of Theorem 1.3.

8 Proof of Theorem 1.4

In Section 5, we have focused on the notion of η-parallel shape operator on a real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$. Under this observation, in this section, we will give a classification of Hopf real hypersurfaces with η-parallel shape operator in Q^{n*}, $n \geq 3$.

Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$. By virtue of Proposition 5.3, the unit normal vector field ζ of any real hypersurface in Q^{n*} with η-parallel shape operator is $\mathfrak{A}$-principal. On the other hand, it is known that a Hopf real hypersurface M has $\mathfrak{A}$-principal ζ in Q^{n*} if and only if M is contact with constant mean curvature (see Proposition 5.3 in [28]). Consequently, by virtue of these results and the classification of contact hypersurfaces in Q^{n*} due to Klein and Suh [21], we can assert the following lemma;

Lemma 8.1. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{n*}, $n \geq 3$. If the shape operator A of M is η-parallel, then M is locally congruent to an open part of one of the following contact hypersurfaces in Q^{n*}:*

- ($T^*_{B_1}$) *a tube of radius $r > 0$ around the complex hyperbolic quadric Q^{n-1*}, which is embedded in Q^{n*} as a totally geodesic complex hypersurface,*
- ($T^*_{B_2}$) *a tube of radius $r > 0$ around the k-dimensional real hyperbolic space $\mathbb{R}H^k$, which is embedded in Q^{n*} as a real space form of Q^{n*},*
- ($\mathscr{H}^*_B$) *a horosphere in Q^{n*} whose center at infinity is the equivalence class of a $\mathfrak{A}$-principal geodesic in Q^{n*}.*

For the model spaces mentioned in Lemma 8.1, we give its geometric structures in detail as follows (see also Klein and Suh [21]).

Proposition A. *Let M_B be the tubes $(\mathcal{T}^*_{B_1})$, $(\mathcal{T}^*_{B_2})$ and the horosphere $(\mathcal{H}^*_B)$ in Q^{n*}, $n \geq 3$. For M_B, the following statements hold:*

1. *Every unit normal vector ζ of M_B is $\mathfrak{A}$-principal.*
2. *M_B is a Hopf hypersurface.*
3. *The shape operator A and the structure tensor field ϕ satisfy $A\phi + \phi A = \mu\phi$. In particular, M_B is a contact real hypersurface.*
4. *M_B has constant principal curvatures and, in particular, constant mean curvature. Then, the principal curvatures of M_B with respect to the unit normal vector field ζ and the corresponding principal curvature spaces are given as follows.*

Table 1: Principal curvatures of model spaces of M_B

Type	Eigenvalues	Eigenspace	Multiplicity
$(\mathcal{T}^*_{B_1})$	$\alpha = -\sqrt{2}\coth(\sqrt{2}r)$	$\mathbb{R}J\zeta$	1
	$\lambda = 0$	$JV(C) \cap C = \{X \in C \mid CX = -X\}$	n–1
	$\mu = -\sqrt{2}\tanh(\sqrt{2}r)$	$V(C) \cap C = \{X \in C \mid CX = X\}$	n–1
$(\mathcal{T}^*_{B_2})$	$\alpha = -\sqrt{2}\tanh(\sqrt{2}r)$	$\mathbb{R}J\zeta$	1
	$\lambda = 0$	$JV(C) \cap C = \{X \in C \mid CX = -X\}$	n–1
	$\mu = -\sqrt{2}\coth(\sqrt{2}r)$	$V(C) \cap C = \{X \in C \mid CX = X\}$	n–1
$(\mathcal{H}^*_B)$	$\alpha = \mu = -\sqrt{2}$	$(V(C) \cap C) \oplus \mathbb{R}J\zeta$	n
	$\lambda = 0$	$JV(C) \cap C$	n–1

Now, by using Proposition A, let us check the converse of Lemma 8.1, whether they satisfy η-parallelism, i.e.,

$$g((\nabla_X A)Y, Z) = 0 \quad \text{for any } X, Y, Z \in \mathcal{C}. \tag{*}$$

Let $T_\lambda = \{X \in TM_B \mid CX = -X, \quad X \perp \xi\}$ and $T_\mu = \{X \in TM_B \mid CX = X, \quad X \perp \xi\}$. Then, by Table 1, the holomorphic distribution $\mathcal{C}$ in TM_B is given by $\mathcal{C} = T_\lambda \oplus T_\mu$. In order to show that the shape operator A of M_B is η-parallel, we consider the following four cases, respectively:

- Case 1. $X, Y, Z \in T_\mu$ (or $X, Y, Z \in T_\lambda$)

  Since $Y \in T_\mu \subset TM_B$, we have $AY = \mu Y$ $(\mu \in \mathbb{R})$, where

$$\mu = \begin{cases} -\sqrt{2}\tanh(\sqrt{2}r) & \text{for } Y \in T_\mu \subset T\big(\mathcal{T}^*_{B_1}\big) \\ -\sqrt{2}\coth(\sqrt{2}r) & \text{for } Y \in T_\mu \subset T\big(\mathcal{T}^*_{B_2}\big) \in \mathbb{R}\backslash\{0\}. \\ -\sqrt{2} & \text{for } Y \in T_\mu \subset T(\mathcal{H}^*_B) \end{cases}$$

It gives that $(\nabla_X A)Y = \mu\nabla_X Y - A(\nabla_X Y)$ for any $X, Y \in T_\mu$. Its inner product of $Z \in T_\mu$ becomes

$$g((\nabla_X A)Y, Z) = \mu g(\nabla_X Y, Z) - g(\nabla_X Y, AZ) = (\mu - \mu)g(\nabla_X Y, Z) = 0.$$

So, we assert that the shape operator A of M_B satisfies $g((\nabla_X A)Y, Z) = 0$ for $X, Y, Z \in T_\mu$ (or for $X, Y, Z \in T_\lambda$).

- Case 2. $X \in T_\mu$ and $Y, Z \in T_\lambda$ (or $X \in T_\lambda$ and $Y, Z \in T_\mu$)

  By using the symmetric property of A, it holds that

$$g((\nabla_X A)Y, Z) = g((\nabla_X A)Z, Y) \quad \text{for any } X, Y, Z \in TM_B. \tag{8.1}$$

This fact leads to

$$\begin{aligned} g((\nabla_X A)Y, Z) &= g((\nabla_X A)Z, Y) \\ &= g(\lambda(\nabla_X Z) - A(\nabla_X Z), Y) \\ &= \lambda g(\nabla_X Z, Y) - g(\nabla_X Z, AY) = (\lambda - \lambda)g(\nabla_X Z, Y) = 0, \end{aligned}$$

where $AY = \lambda Y$ and $AZ = \lambda Z$. From this, we conclude that M_B has η-parallel shape operator for this case.

- Case 3. $X, Z \in T_\mu$ and $Y \in T_\lambda$ (or $X, Z \in T_\lambda$ and $Y \in T_\mu$)

  From the fact of ζ being $\mathfrak{A}$-principal, we obtain $C\xi = -\xi$. Then, the equation of Codazzi (3.2) gives

$$g((\nabla_X A)Y, Z) = g((\nabla_Y A)X, Z) \quad \text{for any } X, Y, Z \in \mathcal{C}. \tag{8.2}$$

Since $AX = \mu X$ and $AZ = \mu Z$, equation (8.2) gives

$$\begin{aligned} g((\nabla_X A)Y, Z) &= g((\nabla_Y A)X, Z) \\ &= g(\mu\nabla_Y X - A(\nabla_Y X), Z) \\ &= \mu g(\nabla_Y X, Z) - g(\nabla_Y X, AZ) \\ &= (\mu - \mu)g(\nabla_Y X, Z) = 0, \end{aligned}$$

which implies that (*ast*) holds for this case.

• Case 4. $X, Y \in T_\mu$ and $Z \in T_\lambda$ (or $X, Y \in T_\lambda$ and $Z \in T_\mu$)

Using the above two formulas, equations (8.1) and (8.2), with respect to $X, Y, Z \in C$ provides

$$\begin{aligned} g((\nabla_X A)Y, Z) \underset{\text{by (8.1)}}{=} g((\nabla_X A)Z, Y) \underset{\text{by (8.2)}}{=} g((\nabla_Z A)X, Y) \\ = g((\nabla_Z A)Y, X) = g((\nabla_Y A)Z, X) = g((\nabla_Y A)X, Z) \end{aligned} \tag{8.3}$$

for any $X, Y, Z \in C$.

Now, from $X, Y \in T_\mu$ we know that $AX = \mu X$ and $AY = \mu Y$. With regard to equation (8.3), these facts yield

$$\begin{aligned} g((\nabla_X A)Y, Z) &= g((\nabla_Z A)Y, X) \\ &= g(\mu \nabla_Z Y - A(\nabla_Z Y), X) \\ &= \mu g(\nabla_Z Y, X) - g(\nabla_Z Y, AX) \\ &= (\mu - \mu) g(\nabla_Z Y, X) = 0. \end{aligned}$$

Summing up the above four cases, we can assert that the shape operator of M_B is η-parallel. From this and (1) and (2) in Proposition A, we conclude with the following lemma:

Lemma 8.2. *The model spaces of types $(\mathcal{T}^*_{B_1})$, $(\mathcal{T}^*_{B_2})$, and $(\mathcal{H}^*_B)$ in Q^{n*}, $n \geq 3$, are Hopf real hypersurfaces with $\mathfrak{A}$-principal normal vector field. Furthermore, the shape operators of the above model spaces are η-parallel.*

Then, by virtue of Lemmas 8.1 and 8.2, we give a complete proof of Theorem 1.4 in the Introduction.

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