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Research Article

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On Berezin norm and Berezin number inequalities for sum of operators

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Abstract: The aim of this study is to obtain several inequalities involving the Berezin number and the Berezin norm for various combinations of operators acting on a reproducing kernel Hilbert space. First, we present some bounds regarding the Berezin number associated with $W^*Q + W^*Q'$, where W, Q, and Q' are three bounded linear operators. Next, several Berezin norm and Berezin number inequalities for the sum of n operators are established.

Keywords: Berezin number, Berezin norm, sum, inequality

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1 Introduction

Many articles related to numerical radius of an operator or Berezin number of an operator on a reproducing kernel Hilbert space (RKHS) have been studied for their many applications in engineering, quantum computing, quantum mechanics, numerical analysis, and differential equations [1–9]. There are important results for the upper and lower bounds of Berezin number in the literature (see, e.g., [10–19]).

To characterize the Berezin number and the Berezin norm, we first present some concepts and properties of the bounded linear operators on a Hilbert space.

Let $\mathcal{E}$ be a complex Hilbert space, endowed with the inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$. Let $\mathbb{L}(\mathcal{E})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{E}$, with identity I. An operator $Q \in \mathbb{L}(\mathcal{E})$ is called positive if $\langle Qx, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and then, we write $Q \geq 0$. If a bounded linear operator Q on $\mathcal{E}$ is positive, then there exists a unique positive bounded linear operator denoted by $Q^{\frac{1}{2}}$ such that $Q = (Q^{\frac{1}{2}})^2$. An important operator used is the absolute value of Q, denoted by $|Q|$, which is defined by $|Q| = (Q^*Q)^{\frac{1}{2}}$. It is easy to see that $|Q| \geq 0$. For $Q \in \mathbb{L}(\mathcal{E})$, we have the following numerical values: the operator norm $\|Q\|$ given by $\|Q\| = \sup_{x\in\mathcal{E},\|x\|=1}\|Qx\|$ and the numerical radius of the operator Q defined by $\omega(Q) = \sup_{x\in\mathcal{E},\|x\|=1}|\langle Qx, x\rangle|$. We remark that $\omega(Q) \leq \|Q\|$. When $Q^*Q = QQ^*$, we say that Q is a normal operator, which, in fact, shows us that $|Q^*| = |Q|$ and $\omega(Q) = \|Q\|$. It is important to mention that the operator norm and the numerical radius norm are equivalent, because we have the inequalities: $\frac{\|Q\|}{2} \leq \omega(Q) \leq \|Q\|$ for every $Q \in \mathbb{L}(\mathcal{E})$. We also have $\omega(Q) \leq \omega(|Q|)$. For some properties of the numerical radius, see, for example, [20–22] and references therein.

Let Θ be a non-empty set and $\mathcal{F}(\Theta, \mathbb{C})$ be the set of all functions from Θ to $\mathbb{C}$, where $\mathbb{C}$ is the field of the complex numbers. A set $\mathcal{E}_\Theta$ included in $\mathcal{F}(\Theta, \mathbb{C})$ is called a reproducing kernel Hilbert space (RKHS) on Θ if $\mathcal{E}_\Theta$

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is a Hilbert space (with identity I_Θ), and for every $\lambda \in \Theta$, the linear evaluation functional $E_\lambda : \mathcal{E}_\Theta \to \mathbb{C}$ given by $E_\lambda(f) = f(\lambda)$ is bounded. Using the Riesz representation theorem, we show that for each $\lambda \in \Theta$, there exists a unique vector $k_\lambda \in \mathcal{E}_\Theta$ such that $f(\lambda) = E_\lambda(f) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{E}_\Theta$. Here, the function k_λ is called the reproducing kernel for the element λ, and the set $\{k_\lambda; \lambda \in \Theta\}$ is called the reproducing kernel of $\mathcal{E}_\Theta$. When $k_\lambda \neq 0$, we denote $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ for $\lambda \in \Theta$ as the normalized reproducing kernel of $\mathcal{E}_\Theta$ and we note that the set $\{\widehat{k}_\lambda : \lambda \in \Theta\}$ is a total set in $\mathcal{E}_\Theta$. For $Q \in \mathbb{L}(\mathcal{E}_\Theta)$, the Berezin symbol (or Berezin transform) of Q, which has been first introduced by Berezin [23,24], is the bounded function $\widetilde{Q} : \Theta \to \mathbb{C}$ defined by $\widetilde{Q}(\lambda) = \langle Q\widehat{k}_\lambda, \widehat{k}_\lambda \rangle$. If the operator Q is self-adjoint ($Q = Q^*$), then $\widetilde{Q}(\lambda) \in \mathbb{R}$, and if the operator Q is positive, then $\widetilde{Q}(\lambda) \geq 0$.

The Berezin symbol has been investigated in detail for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it is widely applied in the various questions of analysis and uniquely determines the operator (i.e., for all $\lambda \in \Theta$, $\widetilde{Q}(\lambda) = \widetilde{W}(\lambda)$ implies $Q = W$). For further information about the Berezin symbol, we refer previously published articles [25–28] and references therein.

The Berezin set and the Berezin number of an operator Q are, respectively, defined by

$$\mathrm{Ber}(Q) := \{\widetilde{Q}(\lambda); \lambda \in \Theta\} \quad \text{and} \quad \mathrm{ber}(Q) := \sup_{\lambda \in \Theta} |\widetilde{Q}(\lambda)| = \sup_{\lambda \in \Theta} |\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda \rangle|.$$

By some simple calculations, we obtain $0 \leq \mathrm{ber}(Q) \leq \omega(Q) \leq \|Q\|$ for all $Q \in \mathbb{L}(\mathcal{E}_\Theta)$. Karaev [1] showed that $\frac{\|Q\|}{2} \leq \mathrm{ber}(Q)$ does not hold for every $Q \in \mathbb{L}(\mathcal{E}_\Theta)$.

It is easy to see that $\mathrm{ber}(I_\Theta) = 1$ and $|\langle Qk_\lambda, k_\lambda \rangle| \leq \mathrm{ber}(Q)\|k_\lambda\|^2$, for all reproducing kernels k_λ.

For every $Q, W \in \mathbb{L}(\mathcal{E}_\Theta)$, we have the following properties:

(i) $\mathrm{ber}(\alpha Q) = |\alpha|\mathrm{ber}(Q)$, for all $\alpha \in \mathbb{C}$;

(ii) $\mathrm{ber}(Q + W) \leq \mathrm{ber}(Q) + \mathrm{ber}(W)$.

From the aforementioned considerations regarding the Berezin number, it follows that ber $(\cdot)$ is a norm on $\mathbb{L}(\mathcal{E}_\Theta)$. Similarly, for $\mathrm{ber}(Q)$, the following concept can be found in [14,29]: $c(Q) := \inf_{\lambda \in \Theta}\{|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda \rangle|\}$.

For $Q \in \mathbb{L}(\mathcal{E}_\Theta)$, the Berezin norm of Q is given by

$$\|Q\|_{\mathrm{ber}} := \sup\{|\langle Q\widehat{k}_\lambda, \widehat{k}_\mu \rangle|; \lambda, \mu \in \Theta\},$$

where $\widehat{k}_\lambda, \widehat{k}_\mu$ are the two normalized reproducing kernels of the space $\mathcal{E}_\Theta$ (see [30,31]). It is easy to see that $|\langle Q\widehat{k}_\lambda, \widehat{k}_\mu \rangle| \leq \|Q\|_{\mathrm{ber}}$ for all $\widehat{k}_\lambda, \widehat{k}_\mu$. It is important to note that $\|\cdot\|_{\mathrm{ber}}$ does not verify, in general, the submultiplicativity property (see [32]). It is important to mention that

$$\mathrm{ber}(Q) \leq \|Q\|_{\mathrm{ber}} \leq \|Q\|, \quad \forall Q \in \mathbb{L}(\mathcal{E}_\Theta). \tag{1.1}$$

It should be mentioned here that Inequalities (1.1) are in general strict. However, Bhunia et al. proved in [33] that if $Q \in \mathbb{L}(\mathcal{E}_\Theta)$ is a positive operator, then

$$\mathrm{ber}(Q) = \|Q\|_{\mathrm{ber}}. \tag{1.2}$$

Remark 1.1. It is very crucial to mention that Relation (1.2) may not be true, in general, for self-adjoint operators (see [33]).

The objective of this study is to obtain several characterizations of Berezin numbers for various combinations of operators. To this end, this article is organized in the following manner: in Section 2, we collect a few results that are required to state and prove the results in the subsequent section. Section 3 contains our main results and presents several Berezin number inequalities regarding the Berezin number associated with the $W^*Q + W^*Q'$ operator. Next, we study the Berezin norm and Berezin number inequalities for sum of operators.

2 Useful lemmas

The aim of this section is to collect some well-known lemmas that will be useful in proving our results. Throughout this article, $(\mathcal{E}, \langle\cdot,\cdot\rangle)$ denotes a complex Hilbert space. The norm associated with $\langle\cdot,\cdot\rangle$ will be denoted by $\|\cdot\|$.

Our starting lemma is proved in [34] and provides an improvement of the well-known Cauchy-Schwarz inequality.

Lemma 2.1. *Let $x, y \in \mathcal{E}$ and $\nu \in [0,1]$. Then,*

$$|\langle x,y\rangle|^2 \leq \nu(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2) + |\langle x,y\rangle|^{2\nu}\|x\|^{2-2\nu}\|y\|^{2-2\nu} \leq \|x\|^2\|y\|^2,$$

for any $x, y \in \mathcal{E}$.

Another result of the aforementioned type is given by Alomari [35], namely:

Lemma 2.2. *Let $x, y \in \mathcal{E}$ and $\nu \in [0,1]$. Then,*

$$|\langle x,y\rangle|^2 \leq \nu\|x\|^2\|y\|^2 + (1-\nu)|\langle x,y\rangle|\|x\|\|y\| \leq \|x\|^2\|y\|^2, \tag{2.1}$$

for any $x, y \in \mathcal{E}$.

For $\nu = \frac{1}{2}$ in Inequality (2.1), we obtain, after rearranging the terms, the following inequality:

$$|\langle x,y\rangle|^2 \leq |\langle x,y\rangle|\|x\|\|y\| + \frac{1}{2}(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2) \leq \|x\|^2\|y\|^2,$$

for all $x, y \in \mathcal{E}$. This inequality is proved by Kittaneh and Moradi in [21], which is another refinement of the Cauchy-Schwarz inequality.

Remark 2.1. It follows from Inequality (2.1) that

$$|\langle x,y\rangle|^2 \leq \frac{1}{3}\|x\|^2\|y\|^2 + \frac{2}{3}|\langle x,y\rangle|\|x\|\|y\|,$$

for all $x, y \in \mathcal{E}$.

The classical Schwarz inequality for a positive operator $Q \in \mathbb{L}(\mathcal{E})$ is given as:

$$|\langle Qx,y\rangle|^2 \leq \langle Qx,x\rangle\langle Qy,y\rangle, \tag{2.2}$$

for any vectors $x, y \in \mathcal{E}$. Kato [36] established a companion of the Schwarz Inequality (2.2), which asserts

$$|\langle Qx,y\rangle|^2 \leq \langle |Q|^{2\theta}x,x\rangle\langle |Q^*|^{2(1-\theta)}y,y\rangle,$$

for every operator $Q \in \mathbb{L}(\mathcal{E})$, for any vectors $x, y \in \mathcal{E}$, and $\theta \in [0,1]$. For $\theta = \frac{1}{2}$ we obtain a result attributed to Halmos [37, pp. 75–76]:

$$|\langle Qx,y\rangle| \leq \sqrt{\langle |Q|x,y\rangle\langle |Q^*|y,y\rangle}, \tag{2.3}$$

for every $Q \in \mathbb{L}(\mathcal{E})$ and for all $x, y \in \mathcal{E}$.

The inequality in the following lemma deals with positive operators and is known as the McCarthy inequality.

Lemma 2.3. [38, Theorem 1.4] *Let $Q \in \mathbb{L}(\mathcal{E})$ be a positive operator and $x \in \mathcal{E}$ be such that $\|x\| = 1$. Then, for all $r \geq 1$, we have*

$$\langle Qx,x\rangle^r \leq \langle Q^r x,x\rangle.$$

If $0 \leq r \leq 1$, then the aforementioned inequality is reversed.

The interesting inequality in the following lemma is proved by Buzano in [39].

Lemma 2.4. *Let* $x, y, e \in \mathcal{E}$ *be such that* $\|e\| = 1$. *Then,*

$$|\langle x, e\rangle\langle e, y\rangle| \le \frac{1}{2}(\|x\|\|y\| + |\langle x, y\rangle|).$$

In the next lemma, we recall a known inequality due to Bohr (see e.g. [40]).

Lemma 2.5. *Let* a_k *be the positive real numbers for all* $i \in \{1, 2, \ldots, d\}$. *Then, for all* $r \ge 1$, *we have*

$$\left(\sum_{k=1}^{d} a_k\right)^r \le d^{r-1}\sum_{k=1}^{d} a_k^r.$$

Our final lemma is given by Dragomir in [41].

Lemma 2.6. *For any* $x, y, z \in \mathcal{E}$, *we have*

$$|\langle x, y\rangle|^2 + |\langle x, z\rangle|^2 \le \|x\|^2(\max\{\|y\|^2, \|z\|^2\} + |\langle y, z\rangle|).$$

3 Main results

In this section, we denote by $(\mathcal{E}_\Theta, \langle\cdot,\cdot\rangle)$ a RKHS on a set Θ with associated norm $\|\cdot\|$.

Our first result in this article reads as follows.

Theorem 3.1. *Let* $Q, Q', W \in \mathbb{L}(\mathcal{E}_\Theta)$, *and* $\nu \in (0, 1)$. *Then, we have*

$$\begin{aligned}\mathrm{ber}^2(W^*Q + W^*Q') \le{}& \frac{2\nu(1-\nu)}{1+\nu-\nu^2}(\mathrm{ber}(W^*Q)\||Q|^2 + |W|^2\|_{\mathrm{ber}} + \mathrm{ber}(W^*Q')\||Q'|^2 + |W|^2\|_{\mathrm{ber}})\\ &+ \frac{1-\nu+\nu^2}{1+\nu-\nu^2}\||Q|^4 + |Q'|^4 + 2\,|W|^4\|_{\mathrm{ber}}.\end{aligned}$$

Proof. We take the first inequality from Lemma 2.1:

$$|\langle x, y\rangle|^2 \le \frac{\nu}{1+\nu}\|x\|^2\|y\|^2 + \frac{1}{1+\nu}\,|\langle x, y\rangle|^{2\nu}\|x\|^{2-2\nu}\|y\|^{2-2\nu}, \tag{3.1}$$

for any $x, y \in \mathcal{E}$ and $\nu \in [0, 1]$. If we replace x and y by $Q\hat{k}_\lambda$ and $W\hat{k}_\lambda$, respectively, in (3.1), and we note that $\|\hat{k}_\lambda\| = 1$, then we have

$$\left|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^2 \le \frac{\nu}{1+\nu}\|Q\hat{k}_\lambda\|^2\|W\hat{k}_\lambda\|^2 + \frac{1}{1+\nu}\,|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2\nu}\|Q\hat{k}_\lambda\|^{2-2\nu}\|W\hat{k}_\lambda\|^{2-2\nu}.$$

Using the same idea as in [34], for the aforementioned inequality, we deduce

$$\begin{aligned}|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 &\le \frac{\nu}{1+\nu}\langle|Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle|W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle + \frac{1}{1+\nu}\,|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2\nu}(\langle|Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle|W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{1-\nu}\\ &= \frac{\nu}{1+\nu}\langle|Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle|W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle + \frac{1}{1+\nu}\,|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{\nu}\langle|Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1-\nu}|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{\nu}\langle|W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1-\nu}.\end{aligned}$$

By applying the well-known Young inequality

$$a^\nu b^{1-\nu} \le \nu a + (1-\nu)b, \quad a, b \ge 0,\ \nu \in (0, 1)$$

to the aforementioned relation for $a = |\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|$ and $b = \langle |Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle$, we obtain the following inequality:

$$\begin{aligned}
&|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2\\
&\le \frac{\nu}{1+\nu}\langle |Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle\\
&\quad + \frac{1}{1+\nu}(\nu|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle| + (1-\nu)\langle |Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle)(\nu|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle| + (1-\nu)\langle |W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle)\\
&= \frac{\nu}{1+\nu}\langle |Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle + \frac{\nu^2}{1+\nu}|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2\\
&\quad + \frac{\nu(1-\nu)}{1+\nu}|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle (|Q|^2+|W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle + \frac{(1-\nu)^2}{1+\nu}\langle |Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&(1+\nu-\nu^2)|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2\\
&\le (1-\nu+\nu^2)\langle |Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle + \nu(1-\nu)|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle (|Q|^2+|W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle\\
&\le \frac{1-\nu+\nu^2}{2}\langle (|Q|^4+|W|^4)\hat{k}_\lambda, \hat{k}_\lambda\rangle + \nu(1-\nu)|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle (|Q|^2+|W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
2(1+\nu-\nu^2)|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 &\le (1-\nu+\nu^2)\langle (|Q|^4+|W|^4)\hat{k}_\lambda, \hat{k}_\lambda\rangle\\
&\quad + 2\nu(1-\nu)|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle (|Q|^2+|W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle.
\end{aligned} \tag{3.2}$$

Therefore, from the algebraic inequality $|z_1+z_2|^2 \le 2(|z_1|^2+|z_2|^2)$, we deduce that

$$\begin{aligned}
&(1+\nu-\nu^2)|\langle (W^*Q+W^*Q')\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2\\
&\overset{(3.2)}{\le} 2(1+\nu-\nu^2)|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + 2(1+\nu-\nu^2)|\langle W^*Q'\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2\\
&\le (1-\nu+\nu^2)\langle (|Q|^4+|Q'|^4+2|W|^4)\hat{k}_\lambda, \hat{k}_\lambda\rangle\\
&\quad + 2\nu(1-\nu)[|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle (|Q|^2+|W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle + |\langle W^*Q'\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle (|Q'|^2+|W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle].
\end{aligned}$$

Taking the supremum over $\lambda \in \Theta$ in the aforementioned inequality and taking into account that the operators $|Q|^4+|Q'|^4+2|W|^4$, $|Q|^2+|W|^2$, and $|Q'|^2+|W|^2$ are positive, we obtain the inequality

$$\begin{aligned}
&(1+\nu-\nu^2)\mathrm{ber}^2(W^*Q+W^*Q')\\
&\le (1-\nu+\nu^2)\||Q|^4+|Q'|^4+2|W|^4\|_{\mathrm{ber}}\\
&\quad + 2\nu(1-\nu)(\mathrm{ber}(W^*Q)\||Q|^2+|W|^2\|_{\mathrm{ber}} + \mathrm{ber}(W^*Q')\||Q'|^2+|W|^2\|_{\mathrm{ber}}).
\end{aligned}$$

Hence, we have proved the inequality of the statement. □

Remark 3.1. For $\nu \in \{0,1\}$ in Inequality (3.1), we have $|\langle x, y\rangle|^2 \le \|x\|^2\|y\|^2$, and so $|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 \le \langle |Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle$. Therefore, from the algebraic inequality $|z_1+z_2|^2 \le 2(|z_1|^2+|z_2|^2)$, we obtain

$$\begin{aligned}
|\langle (W^*Q+W^*Q')\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 &\le 2|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + 2|\langle W^*Q'\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2\\
&\le \langle (|Q|^4+|Q'|^4+2|W|^4)\hat{k}_\lambda, \hat{k}_\lambda\rangle.
\end{aligned}$$

Thus, we deduce the following:

$$\mathrm{ber}^2(W^*Q+W^*Q') \le \||Q|^4+|Q'|^4+2|W|^4\|_{\mathrm{ber}}. \tag{3.3}$$

Theorem 3.2. *Let $Q, Q', W \in \mathbb{L}(\mathcal{E}_\Theta)$ and $\nu \in [0,1]$. Then,*

$$\begin{aligned}
\mathrm{ber}^2(W^*Q+W^*Q') &\le \nu\||Q|^4+|Q'|^4+2|W|^4\|_{\mathrm{ber}} + (1-\nu)\mathrm{ber}(W^*Q)\||Q|^2+|W|^2\|_{\mathrm{ber}}\\
&\quad + (1-\nu)\mathrm{ber}(W^*Q')\||Q'|^2+|W|^2\|_{\mathrm{ber}}.
\end{aligned}$$

Proof. We consider the first inequality from Lemma 2.2:

$$|\langle x, y\rangle|^2 \leq \nu\|x\|^2\|y\|^2 + (1-\nu)|\langle x, y\rangle|\|x\|\|y\|,$$

for any $x, y \in \mathcal{E}$ and $\nu \in [0, 1]$. The Hölder-McCarthy inequality can only be applied to positive operators, and the operators $|Q|^2$ and $|W|^2$ are positive. Now, we replace x and y by $Q\hat{k}_\lambda$ and $W\hat{k}_\lambda$, in the aforementioned inequality, and we know that $\|\hat{k}_\lambda\| = 1$, then

$$\begin{aligned}
&|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2\\
&\leq \nu\|Q\hat{k}_\lambda\|^2\|W\hat{k}_\lambda\|^2 + (1-\nu)|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\|Q\hat{k}_\lambda\|\|W\hat{k}_\lambda\|\\
&\leq \nu\langle Q\hat{k}_\lambda, Q\hat{k}_\lambda\rangle\langle W\hat{k}_\lambda, W\hat{k}_\lambda\rangle + (1-\nu)|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\sqrt{\langle Q\hat{k}_\lambda, Q\hat{k}_\lambda\rangle\langle W\hat{k}_\lambda, W\hat{k}_\lambda\rangle}\\
&= \nu\langle |Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle + (1-\nu)|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\sqrt{\langle |Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle}\\
&\leq \frac{\nu}{4}(\langle |Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle)^2 + \frac{1-\nu}{2}|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|(\langle |Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle)\\
&\leq \frac{\nu}{2}(\langle |Q|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + \langle |W|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^2) + \frac{1-\nu}{2}|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|(\langle (|Q|^2 + |W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle)\\
&\overset{\text{McCarthy}}{\leq} \frac{\nu}{2}(\langle |Q|^4\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |W|^4\hat{k}_\lambda, \hat{k}_\lambda\rangle) + \frac{1-\nu}{2}|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|(\langle (|Q|^2 + |W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle)\\
&= \frac{\nu}{2}(\langle (|Q|^4 + |W|^4)\hat{k}_\lambda, \hat{k}_\lambda\rangle) + \frac{1-\nu}{2}|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|(\langle (|Q|^2 + |W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle).
\end{aligned}$$

Therefore, the inequality

$$\begin{aligned}
2\,|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 \leq \nu(\langle (|Q|^4 + |W|^4)\hat{k}_\lambda, \hat{k}_\lambda\rangle)\\
+ (1-\nu)|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|(\langle (|Q|^2 + |W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle)
\end{aligned} \tag{3.4}$$

is true. Consequently, from the algebraic inequality $|z_1 + z_2|^2 \leq 2(|z_1|^2 + |z_2|^2)$, we deduce

$$\begin{aligned}
&|\langle W^*Q + W^*Q'\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2\\
&\leq 2\,|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + 2\,|\langle W^*Q'\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2\\
&\overset{(3.4)}{\leq} \nu\langle (|Q|^4 + |Q'|^4 + 2\,|W|^4)\hat{k}_\lambda, \hat{k}_\lambda\rangle + (1-\nu)|\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle (|Q|^2 + |W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle\\
&\quad + (1-\nu)|\langle W^*Q'\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle (|Q'|^2 + |W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle.
\end{aligned}$$

Taking the supremum over $\lambda \in \Theta$ in the aforementioned inequality and taking into account that the operators $|Q|^4 + |Q'|^4 + 2\,|W|^4$, $|Q|^2 + |W|^2$ and $|Q'|^2 + |W|^2$ are positive, we obtain the inequality of the statement. □

Remark 3.2. Through various particular cases of ν in Theorem 3.2, we obtain some known results; thus, for $\nu = 1$ in Theorem 3.2, we deduce Inequality (3.3), and for $\nu = 0$, we find

$$\mathrm{ber}^2(W^*Q + W^*Q') \leq \mathrm{ber}(W^*Q)\||Q|^2 + |W|^2\|_{\mathrm{ber}} + \mathrm{ber}(W^*Q')\||Q'|^2 + |W|^2\|_{\mathrm{ber}}.$$

In particular, if $Q = Q'$ in the last relation, we deduce the following inequality:

$$\mathrm{ber}(W^*Q) \leq \frac{1}{2}\||Q|^2 + |W|^2\|_{\mathrm{ber}}. \tag{3.5}$$

Theorem 3.3. *Let* $Q, W \in \mathbb{L}(\mathcal{E}_\Theta)$ *and* $\nu \in [0, 1]$. *Then,*

$$\begin{aligned}
\mathrm{ber}^2(Q + W) \leq \nu\||Q|^2 + |W|^2 + |Q^*|^2 + |W^*|^2\|_{\mathrm{ber}} + (1-\nu)\mathrm{ber}(Q)\||Q| + |Q^*|\|_{\mathrm{ber}}\\
+ (1-\nu)\mathrm{ber}(W)\||W| + |W^*|\|_{\mathrm{ber}}.
\end{aligned}$$

Proof. Applying Inequality (2.3), we deduce the following relation:

$$|\langle Q\hat{k}_\lambda, \hat{k}_\lambda\rangle| \leq \nu\sqrt{\langle |Q|\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |Q^*|\hat{k}_\lambda, \hat{k}_\lambda\rangle} + (1-\nu)|\langle Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|,$$

for any $\nu \in [0,1]$. Multiplying by $\sqrt{\langle |Q|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle\langle |Q^*|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}$ and taking into account that

$$|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 \le |\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\sqrt{\langle |Q|x, x\rangle\langle |Q^*|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle},$$

we find the following inequality:

$$|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 \le \nu\langle |Q|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle\langle |Q^*|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + (1-\nu)|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\sqrt{\langle |Q|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle\langle |Q^*|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}. \tag{3.6}$$

Because the operators $|Q|$ and $|Q^*|$ are positive, we can apply the McCarthy inequality; thus, $\langle |Q|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^2 \le \langle |Q|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle$, $\langle |Q^*|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^2 \le \langle |Q^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle$. Therefore, Inequality (3.6) becomes

$$\begin{aligned}|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 &\le \frac{\nu}{4}(\langle |Q|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + \langle |Q^*|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle)^2 + \frac{1-\nu}{2}|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\langle (|Q| + |Q^*|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle \\ &\le \frac{\nu}{2}(\langle |Q|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^2 + \langle |Q^*|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^2) + \frac{1-\nu}{2}|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\langle (|Q| + |Q^*|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle \\ &\le \frac{\nu}{2}\langle (|Q|^2 + |Q^*|^2)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + \frac{1-\nu}{2}|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\langle (|Q| + |Q^*|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle.\end{aligned}$$

Hence, we deduce that

$$2\,|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 \le \nu\langle (|Q|^2 + |Q^*|^2)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + (1-\nu)|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\langle (|Q| + |Q^*|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle.$$

Consequently, we have

$$\begin{aligned}&|\langle (Q+W)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 \le 2\,|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + 2\,|\langle W\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 \\ &\le \nu\langle (|Q|^2 + |Q^*|^2)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + (1-\nu)|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\langle (|Q| + |Q^*|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle \\ &+ \nu\langle (|W|^2 + |W^*|^2)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + (1-\nu)|\langle W\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\langle (|W| + |W^*|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle.\end{aligned}$$

Since $\|\widehat{k}_\lambda\| = 1$ and taking the supremum over all $\lambda \in \Theta$ and taking into account that the operators $|Q|^2 + |Q^*|^2$, $|Q| + |Q^*|$, $|W|^2 + |W^*|^2$, and $|W| + |W^*|$ are positive, the inequality of the statement is true. □

Remark 3.3. If we take $\nu = \frac{1}{2}$ in Theorem 3.3, then we obtain the following sequence of inequalities:

$$2\mathrm{ber}^2(Q+W) \le \||Q|^2 + |W|^2 + |Q^*|^2 + |W^*|^2\|_{\mathrm{ber}} + \mathrm{ber}(Q)\||Q| + |Q^*|\|_{\mathrm{ber}} + \mathrm{ber}(W)\||W| + |W^*|\|_{\mathrm{ber}}.$$

In all that follows, for any arbitrary operator $Q \in \mathbb{L}(\mathcal{E}_\Theta)$, we write

$$\mathfrak{R}(Q) := \frac{Q + Q^*}{2} \quad \text{and} \quad \mathfrak{I}(Q) := \frac{Q - Q^*}{2i}.$$

Thus, we have $Q = \mathfrak{R}(Q) + i\mathfrak{I}(Q)$ and $|\mathfrak{R}(Q)|^2 + |\mathfrak{I}(Q)|^2 = |Q|^2 + |Q^*|^2$.

Theorem 3.4. *If $Q \in \mathbb{L}(\mathcal{E}_\Theta)$, then the inequality*

$$\mathrm{ber}(\mathfrak{R}(Q) + \mathfrak{I}(Q)) \le \sqrt{2}\,\mathrm{ber}(Q)$$

holds.

Proof. It is easy to see that

$$|\langle (\mathfrak{R}(Q) + \mathfrak{I}(Q))\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| \le \sqrt{2}\,|\langle (\mathfrak{R}(Q) + i\mathfrak{I}(Q))\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| = \sqrt{2}\,|\langle (Q)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|.$$

Taking the supremum over all $\lambda \in \Theta$, we find that the inequality of the statement holds. □

Remark 3.4. If $Q \in \mathbb{L}(\mathcal{E}_\Theta)$, then from Relation (3.5) and knowing that $\mathfrak{I}(Q)^* = \mathfrak{I}(Q)$, we find

$$\mathrm{ber}(\mathfrak{I}(Q)\mathfrak{R}(Q)) \le \frac{1}{2}\||Q|^2 + |Q^*|^2\|_{\mathrm{ber}}.$$

Theorem 3.5. *If $Q, Q_1, W, W_1 \in \mathbb{L}(\mathcal{E}_\Theta)$, then the inequality*

$$\|W_1^*(Q+W)Q_1\|_{\mathrm{ber}} \le \frac{\sqrt{2}}{2}(\mathrm{ber}(Q_1^*(|Q|+i|W|)Q_1)+\mathrm{ber}(W_1^*(|Q^*|+i|W^*|)W_1))$$

holds.

Proof. While proving a result from [42], we found the inequality

$$|\langle (Q+W)x,y\rangle| \le \frac{\sqrt{2}}{2}(|\langle (|Q|+i|W|)x,x\rangle| + |\langle (|Q^*|+i|W^*|)y,y\rangle|),$$

where Q and W are the operators in a Hilbert space $\mathcal{E}$. Now, we take $x = Q_1\hat{k}_\lambda$, $y = W_1\hat{k}_\mu$, where $\lambda, \mu \in \mathbb{C}$; then, we obtain the following inequality:

$$|\langle W_1^*(Q+W)Q_1\hat{k}_\lambda,\hat{k}_\mu\rangle| \le \frac{\sqrt{2}}{2}(|\langle Q_1^*(|Q|+i|W|)Q_1\hat{k}_\lambda,\hat{k}_\lambda\rangle| + |\langle W_1^*(|Q^*|+i|W^*|)W_1\hat{k}_\mu,\hat{k}_\mu\rangle|),$$

for four operators Q, Q_1, W, W_1 in a RKHS H_Θ. Taking the supremum over $\lambda, \mu \in \mathbb{C}$, we obtain the inequality of the statement. □

Corollary 3.1. *If $Q \in \mathbb{L}(\mathcal{E}_\Theta)$, then the inequality*

$$\|Q\|_{\mathrm{ber}} \le \frac{\sqrt{2}}{2}(\mathrm{ber}(|\mathfrak{R}(Q)|+i|\mathfrak{I}(Q)|)+\mathrm{ber}(|\mathfrak{R}(Q)^*|+i|\mathfrak{I}(Q)^*|))$$

holds.

Proof. It is easy to see that $|W| = |iW|$. In Theorem 3.5, we take $Q_1 = W_1 = I$, $Q = \mathfrak{R}(Q)$ and $W = i\mathfrak{I}(Q)$, to obtain the relation of the statement. □

Theorem 3.6. *If $Q, T, W \in \mathbb{L}(\mathcal{E}_\Theta)$, then the inequality*

$$\mathrm{ber}^2 T^*(Q+W)T \le \mathrm{ber}(T^*Q^2T) + \mathrm{ber}(T^*W^2T) + \frac{\sqrt{2}}{2}\mathrm{ber}(T^*(|Q|^2+|W|^2+i(|Q^*|^2+|W^*|^2))T)$$

holds.

Proof. Another inequality in the proof from [42] is the following:

$$|\langle (Q+W)x,x\rangle|^2 \le |\langle Q^2x,x\rangle| + |\langle W^2x,x\rangle + \frac{\sqrt{2}}{2}|\langle (\langle (|Q|^2+|W|^2)+i(|Q^*|^2+|W^*|^2)x,x\rangle|)|,$$

where Q and W are the operators in a Hilbert space $\mathcal{E}$. If we take $x = T\hat{k}_\lambda$, where $\lambda \in \mathbb{C}$, then we deduce

$$\begin{aligned}|\langle T^*(Q+W)T\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2 \le{} & |\langle T^*Q^2T\hat{k}_\lambda,\hat{k}_\lambda\rangle| + |\langle T^*W^2T\hat{k}_\lambda,\hat{k}_\lambda\rangle \\ & + \frac{\sqrt{2}}{2}|(\langle T^*(|Q|^2+|W|^2)+i(|Q^*|^2+|W^*|^2)T\hat{k}_\lambda,\hat{k}_\lambda\rangle|)|,\end{aligned}$$

for three operators Q, T, and W in a RKHS H_Θ. Consequently, taking the supremum over $\lambda \in \mathbb{C}$, we obtain the inequality of the statement. □

Corollary 3.2. *If $Q \in \mathbb{L}(\mathcal{E}_\Theta)$, then the inequality*

$$\mathrm{ber}^2(Q) \le \mathrm{ber}(\mathfrak{R}(Q)^2) + \mathrm{ber}(\mathfrak{I}(Q)^2) + \frac{\sqrt{2}}{2}\mathrm{ber}(|\mathfrak{R}(Q)|^2+|\mathfrak{I}(Q)|^2+i(|\mathfrak{R}(Q)^*|^2+|\mathfrak{I}(Q)^*|^2))$$

holds.

Proof. In Theorem 3.6, we take $T = I$, $Q = \Re(Q)$, and $W = i\Im(Q)$, to prove the relation of the statement. □

Now, we will prove several results related to the sum of operators.

Theorem 3.7. *Let $Q_k \in \mathbb{L}(\mathcal{E}_\Theta)$ for all $k \in \{1, 2, \dots, d\}$ with $d \in \mathbb{N}^*$. Then, for every $r \geq 1$, we have*

$$\operatorname{ber}^{2r}\left(\sum_{k=1}^{d} Q_k\right) \leq \frac{d^{2r-1}}{\sqrt{2}} \sum_{k=1}^{d} \operatorname{ber}(|Q_k|^{2r} + i\,|Q_k^*|^{2r}).$$

Proof. Let $\lambda \in \Theta$ and $\widehat{k}_\lambda$ be the normalized reproducing kernel of the space $\mathcal{E}_\Theta$. Using Lemmas 2.3 and 2.5, and Inequality (2.3), we see that

$$\begin{aligned}\left|\left\langle\left(\sum_{k=1}^{d} Q_k\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^{2r} &\leq \left(\sum_{k=1}^{d} |\langle Q_k\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\right)^{2r}\\ &\leq d^{2r-1}\sum_{k=1}^{d} |\langle Q_k\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^{2r}\\ &\leq d^{2r-1}\sum_{k=1}^{d} \langle |Q_k|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{r}\langle |Q_k^*|\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^{r}\\ &\leq d^{2r-1}\sum_{k=1}^{d} \langle |Q_k|^{r}\widehat{k}_\lambda, \widehat{k}_\lambda\rangle\langle |Q_k^*|^{r}\widehat{k}_\lambda, \widehat{k}_\lambda\rangle.\end{aligned}$$

Moreover, since $|Q_k|^r \geq 0$ and $|Q_k^*|^r \geq 0$, using the arithmetic-geometric mean inequality together with Lemma 2.3, we see that

$$\begin{aligned}\left|\left\langle\left(\sum_{k=1}^{d} Q_k\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^{2r} &\leq \frac{d^{2r-1}}{2}\sum_{k=1}^{d} (\langle |Q_k|^{r}\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^2 + \langle |Q_k^*|^{r}\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^2)\\ &\leq \frac{d^{2r-1}}{2}\sum_{k=1}^{d} (\langle |Q_k|^{2r}\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + \langle |Q_k^*|^{2r}\widehat{k}_\lambda, \widehat{k}_\lambda\rangle)\\ &\leq \frac{d^{2r-1}}{\sqrt{2}}\sum_{k=1}^{d} |\langle |Q_k|^{2r}\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + i\langle |Q_k^*|^{2r}\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|,\end{aligned}$$

where we have used in the last inequality the well-known relation: $|\alpha + \beta| \leq \sqrt{2}|\alpha + i\beta|$ for all real numbers α and β. Hence, we infer that

$$\begin{aligned}\left|\left\langle\left(\sum_{k=1}^{d} Q_k\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^{2r} &\leq \frac{d^{2r-1}}{\sqrt{2}}\sum_{k=1}^{d} |\langle(|Q_k|^{2r} + i\,|Q_k^*|^{2r})\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\\ &\leq \frac{d^{2r-1}}{\sqrt{2}}\sum_{k=1}^{d} \operatorname{ber}(|Q_k|^{2r} + i\,|Q_k^*|^{2r}).\end{aligned}$$

Taking the supremum over all $\lambda \in \Theta$ in the last inequality yields the desired result. □

Remark 3.5. By letting $r = d = 1$ in Theorem 3.7, we obtain

$$\operatorname{ber}^2(Q) \leq \frac{1}{\sqrt{2}}\operatorname{ber}(|Q|^2 + i\,|Q^*|^2).$$

In order to establish our next result, we need the following elementary equality

$$\max\{a, b\} = \frac{1}{2}(a + b + |a - b|), \tag{3.7}$$

for every $a, b \in \mathbb{R}$.

Theorem 3.8. *Let $Q, W \in \mathbb{L}(\mathcal{E}_\Theta)$. Then, the inequality*

$$\mathrm{ber}(Q+W) \le \sqrt{\left\| |Q|^2 + |W|^2 \right\|_{\mathrm{ber}} + \frac{1}{2}\mathrm{ber}(|Q|^2 - |W|^2) + 2\mathrm{ber}(W^*Q)}$$

holds.

Proof. Let $\lambda \in \Theta$ and $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{E}_\Theta$. Then, we see that

$$\begin{aligned} |\langle (Q+W)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 &\le (|\langle Q\hat{k}_\lambda, \hat{k}_\lambda\rangle| + |\langle W\hat{k}_\lambda, \hat{k}_\lambda\rangle|)^2 \\ &= |\langle \hat{k}_\lambda, Q\hat{k}_\lambda\rangle|^2 + |\langle \hat{k}_\lambda, W\hat{k}_\lambda\rangle|^2 + 2|\langle Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\,|\langle W\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\ &\overset{\text{(Lemma 2.6)}}{\le} \max\{\|Q\hat{k}_\lambda\|^2, \|W\hat{k}_\lambda\|^2\} + |\langle Q\hat{k}_\lambda, W\hat{k}_\lambda\rangle| + 2|\langle Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\,|\langle W\hat{k}_\lambda, \hat{k}_\lambda\rangle|. \end{aligned}$$

Furthermore, by applying (3.7), we obtain

$$\begin{aligned} |\langle (Q+W)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 &\le \frac{1}{2}(\|Q\hat{k}_\lambda\|^2 + \|W\hat{k}_\lambda\|^2 + |\|Q\hat{k}_\lambda\|^2 - \|W\hat{k}_\lambda\|^2|) + |\langle Q\hat{k}_\lambda, W\hat{k}_\lambda\rangle| + 2|\langle Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\,|\langle W\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\ &= \frac{1}{2}(\langle Q\hat{k}_\lambda, Q\hat{k}_\lambda\rangle + \langle S\hat{k}_\lambda, S\hat{k}_\lambda\rangle + |\langle T\hat{k}_\lambda, T\hat{k}_\lambda\rangle - \langle W\hat{k}_\lambda, W\hat{k}_\lambda\rangle|) \\ &\quad + |\langle Q\hat{k}_\lambda, W\hat{k}_\lambda\rangle| + 2|\langle Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\,|\langle W\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\ &= \frac{1}{2}(\langle (|Q|^2 + |W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle + |\langle (|Q|^2 - |W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle) + |\langle W^*Q\hat{k}_\lambda, \hat{k}_\lambda\rangle| + 2|\langle Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\,|\langle W\hat{k}_\lambda, \hat{k}_\lambda\rangle|, \end{aligned}$$

whence

$$\begin{aligned} |\langle (Q+W)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 &\le \frac{1}{2}\mathrm{ber}(|Q|^2 + |W|^2) + \frac{1}{2}\mathrm{ber}(|Q|^2 - |W|^2) + \mathrm{ber}(W^*Q) + 2|\langle Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\,|\langle \hat{k}_\lambda, W\hat{k}_\lambda\rangle| \\ &= \frac{1}{2}\left\| |Q|^2 + |W|^2 \right\|_{\mathrm{ber}} + \frac{1}{2}\mathrm{ber}(|Q|^2 - |W|^2) + \mathrm{ber}(W^*Q) + 2|\langle Q\hat{k}_\lambda, \hat{k}_\lambda\rangle|\,|\langle \hat{k}_\lambda, W\hat{k}_\lambda\rangle|, \end{aligned}$$

where we have used (1.2) in the last equality since $|Q|^2 + |W|^2 \ge 0$. Moreover, by applying Lemma 2.4 and the arithmetic-geometric mean inequality, we see that

$$\begin{aligned} |\langle (Q+W)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 &\le \frac{1}{2}\left\| |Q|^2 + |W|^2 \right\|_{\mathrm{ber}} + \frac{1}{2}\mathrm{ber}(|Q|^2 - |W|^2) + \mathrm{ber}(W^*Q) + \|Q\hat{k}_\lambda\|\|W\hat{k}_\lambda\| + |\langle Q\hat{k}_\lambda, W\hat{k}_\lambda\rangle| \\ &\le \frac{1}{2}\left\| |Q|^2 + |W|^2 \right\|_{\mathrm{ber}} + \frac{1}{2}\mathrm{ber}(|Q|^2 - |W|^2) + \mathrm{ber}(W^*Q) + \frac{1}{2}\langle (|Q|^2 \\ &\quad + |W|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle + |\langle Q\hat{k}_\lambda, W\hat{k}_\lambda\rangle| \\ &\le \frac{1}{2}\left\| |Q|^2 + |W|^2 \right\|_{\mathrm{ber}} + \frac{1}{2}\mathrm{ber}(|Q|^2 - |W|^2) + \mathrm{ber}(W^*Q) + \frac{1}{2}\mathrm{ber}(|Q|^2 + |W|^2) + \mathrm{ber}(W^*Q). \end{aligned}$$

So, another application of (1.2) gives

$$|\langle (Q+W)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 \le \left\| |Q|^2 + |W|^2 \right\|_{\mathrm{ber}} + \frac{1}{2}\mathrm{ber}(|Q|^2 - |W|^2) + 2\mathrm{ber}(W^*Q),$$

for every $\lambda, \mu \in \Theta$. Therefore, by taking the supremum over all $\lambda, \mu \in \Theta$ in the aforementioned inequality, we obtain the desired result. □

In order to prove our next result, we need to recall the following inequality:

$$(a_1a_2 + b_1b_2)^2 \le (a_1^2 + b_1^2)(a_2^2 + b_2^2), \quad \forall a_1, a_2, b_1, b_2 \in \mathbb{R}. \tag{3.8}$$

Our next result reads as follows.

Theorem 3.9. *Let $Q, W \in \mathbb{L}(\mathcal{E}_\Theta)$. Then, the following inequality*

$$\mathrm{ber}^4(Q+W) \le 4\min\{\||Q|^2+|W|^2\|_{\mathrm{ber}}\||Q^*|^2+|W^*|^2\|_{\mathrm{ber}}, \||Q|^2+|W^*|^2\|_{\mathrm{ber}}\||Q^*|^2+|W|^2\|_{\mathrm{ber}}\}$$

holds.

Proof. Let $\lambda \in \Theta$ and $\widehat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{E}_\Theta$. Using the convexity of the function $t \mapsto t^2$, we see that

$$\begin{aligned}
|\langle (Q+W)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 &\le (|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| + |\langle W\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|)^2 \\
&\le 2(|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + |\langle W\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2) \\
&= 2(|\langle Q\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\,|\langle \widehat{k}_\lambda, Q^*\widehat{k}_\lambda\rangle| + |\langle W\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\,|\langle \widehat{k}_\lambda, W^*\widehat{k}_\lambda\rangle|) \\
&\le 2(\|Q\widehat{k}_\lambda\|\,\|Q^*\widehat{k}_\lambda\| + \|W\widehat{k}_\lambda\|\,\|W^*\widehat{k}_\lambda\|) \\
&= 2\left(\sqrt{\langle |Q|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}\sqrt{\langle |Q^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle} + \sqrt{\langle |W|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}\sqrt{\langle |W^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}\right).
\end{aligned}$$

This implies that

$$|\langle (Q+W)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^4 \le 4\left(\sqrt{\langle |Q|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}\sqrt{\langle |Q^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle} + \sqrt{\langle |W|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}\sqrt{\langle |W^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}\right)^2.$$

By choosing $a_1 = \sqrt{\langle |Q|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}$, $a_2 = \sqrt{\langle |Q^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}$, $b_1 = \sqrt{\langle |W|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}$, and $b_2 = \sqrt{\langle |W^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}$ in the above inequality and then applying (3.8), we obtain

$$\begin{aligned}
|\langle (Q+W)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^4 &\le 4(\langle |Q|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + \langle |W|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle)(\langle |Q^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + \langle |W^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle) \\
&= 4\langle (|Q|^2+|W|^2)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle\langle (|Q^*|^2+|W^*|^2)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle \\
&\le 4\mathrm{ber}(|Q|^2+|W|^2)\mathrm{ber}(|Q^*|^2+|W^*|^2) \\
&= 4\,\||Q|^2+|W|^2\|_{\mathrm{ber}}\||Q^*|^2+|W^*|^2\|_{\mathrm{ber}},
\end{aligned}$$

where we have used (1.2) in the last equality since we have $|Q|^2+|W|^2 \ge 0$ and $|Q^*|^2+|W^*|^2 \ge 0$. By taking the supremum over all $\lambda \in \Theta$ in the last inequality, we obtain

$$\mathrm{ber}^4(Q+W) \le 4\,\||Q|^2+|W|^2\|_{\mathrm{ber}}\||Q^*|^2+|W^*|^2\|_{\mathrm{ber}}. \tag{3.9}$$

By choosing $a_1 = \sqrt{\langle |Q|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}$, $a_2 = \sqrt{\langle |Q^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}$, $b_1 = \sqrt{\langle |W^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}$ and $b_2 = \sqrt{\langle |W|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}$ and then proceeding as above, we see that

$$\mathrm{ber}^4(Q+W) \le 4\,\||Q|^2+|W^*|^2\|_{\mathrm{ber}}\||Q^*|^2+|W|^2\|_{\mathrm{ber}}. \tag{3.10}$$

Combining (3.9) together with (3.10) yields to the desired result. □

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