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Research Article

Larisa Cheregi and Mohammad Mursaleen*

Approximation of complex q-Beta-Baskakov-Szász-Stancu operators in compact disk

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Abstract: The purpose this study is to present and investigate the q-Beta-Baskakov-Szasz-Stancu operator. The operators are accompanied by Voronovskaja-type consequences, which include both an exact approximation order and a quantitative assessment, specifically within compact disks.

Keywords: q-Beta-Baskakov-Schurer-Szasz-Stancu operator, q-integers, Voronovskaja estimate, approximation in compact disks, simultaneous approximation, overconvergence

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1 Introduction

Over the years, there have been numerous discussions and proposals surrounding the use of complex approximations. The field of approximation theory has seen significant advancements, with noteworthy contributions to its knowledge base. One particular area of research that has garnered attention is the work of Aslan, which centers around their exploration of [1]. Aslan's research delves into the intricate nature of approximation methods that involve operators. Their article provides valuable insights and significant findings regarding the behavior of these operators, ultimately contributing to our overall understanding of their ability to approximate. On the other hand, Agrawal et al.'s research [2] takes a different approach by studying the interrelationships among various types of operators. This enriches the scope of approximation theory and provides valuable resources for effectively approximating functions.

Cicek and İzgi's study [3] is centered on bivariate operators and their ability to estimate functions within triangular regions. The findings of this research carry significant weight for numerous fields and practical uses.

The past few years have seen a proliferation of talks and propositions regarding the effective employment of intricate approximation operators. Q-calculus holds a crucial importance in the realm of mathematical research and has been emphasized in the seminal book by Aral et al. [4]. This work serves as a fundamental reference in the realm of q-calculus, revealing the profound utility of q-calculus in operator theory. Various researchers have introduced and examined several q-operators, focusing on their properties in approximation. One notable contribution to this field came in 1987 from Lupaş [5], who presented the initial q-analogue of the traditional Bernstein polynomials. Additionally, Phillips [6] made a significant contribution by introducing another important innovation. The q-analogue of Bernstein polynomials is a mathematical concept that seeks to extend the traditional Bernstein polynomials to a q-parameterized version. This extension introduces a new

*** Corresponding author: Mohammad Mursaleen,** Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan; Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India,
e-mail: mursaleenm@gmail.com
Larisa Cheregi: Department of Mathematics, Technical University of Cluj-Napoca, Cluj-Napoca 400114, Romania,
e-mail: larisa.cheregi@math.utcluj.ro

level of complexity and versatility to the polynomials, allowing for a broader range of applications and calculations within the realm of mathematical analysis. In 2012, there were two notable studies conducted by Maheshwari and Sharma [7] and Govil and Gupta [8]. These studies focused on examining the approximation properties of q-Baskakov-Beta-Stancu operators and q-Beta-Szász-Stancu operators. The researchers also provided estimations of the moments and successfully established a direct correlationbetween the outcomes in relation to the modulus of continuity and a formula that approaches infinity for the q-operators.

Yüksel and Dinlemez provided a Voronovskaja theorem for the q-analogue of a specific family called Szász-Beta in their work published in [9]. The use of type operators is discussed in the literature. Gupta and Agrawal [10] provide information on the rate at which the Szász-Beta operators converge, using q-integers. Dinlemez [11] explores the approximation properties of q-Szász-Beta-Stancu-type operators and presents a weighted approximation theorem for these operators. Following these studies, further research has been conducted to expand upon these findings. Numerous captivating generalizations concerning the q-calculus of the Beta operator have been presented in various sources, including [12–14].

The main focus of [15] revolves around the q analogue of the widely recognized Stancu-Beta operators. This includes an examination of their moments, recurrence relation, and the presentation of various direct outcomes related to the modulus of continuity of the q-Stancu-Beta operators. In a separate study, Yüksel and Dinlemez introduced a positive linear operator of the q-Baskakov-Schurer-Szasz variety, along with its Stancu generalization [16]. Furthermore, Cheregi explored the complex q-Baskakov-Schurer-Szasz-Stancu operators on compact disks [17]. The current research expands upon prior studies and introduces a modified version of the Beta-type generalization of complex q-Baskakov-Szasz-Stancu operators on compact disks. The focus of this study is specifically on the modified Beta operators, as mentioned in [18]. For recent related work, we refere to [19–23].

In order to address the restricted abilities of q-Beta operators to solely replicate constant functions, we present a resolution to improve the convergence of approximating functions on compact disks. Notwithstanding this constraint, the operators retain their versatility as a result of their capacity to handle intricate variables and adapt to diverse weight functions. This adaptability renders them a valuable instrument for a range of applications.

In programming languages, the aforementioned operators play a vital role in data manipulation and transformation. These operators are specifically designed to carry out a range of operations on different data types, including arithmetic calculations, logical comparisons, and assignment of values. Programmers rely on these operators to develop intricate algorithms and incorporate specific functionalities into their programs. It is imperative to utilize these operators correctly and efficiently to write code that is both functional and efficient. Additionally, type operators are indispensable for ensuring the proper handling and manipulation of data in programming.

The terminologies that we introduce in our work are specifically linked to the analysis of q.

Each integer $l \geq 0$, $q > 0$, $[l]_q$ is accompanied by a set of corresponding definitions [17]:

$$[l]_q := \begin{cases} \dfrac{1-q^k}{1-q}, & q \in \mathbb{R}^* \backslash \{1\} \\ k, & q = 1. \end{cases} \tag{1}$$

For $l \in \mathbb{N}$ and $[0]_q = 0$, the definition of the q-factorial is as follows:

$$[l]_q! := \begin{cases} [1]_q [2]_q \cdots [l]_q, & \forall l \in \mathbb{N}^*, \\ 1, & l = 0, \end{cases} \tag{2}$$

and in their respective order

$$(1+z)_q^l := \begin{cases} \prod_{i=0}^{n-1} (1+q^i z), & \forall l \in \mathbb{N}^*, \\ 1, & l = 0. \end{cases} \tag{3}$$

The q-binomial coefficient is defined for integers $0 \leq l \leq m$ as follows:

$$\begin{pmatrix} n \\ l \end{pmatrix}_q = \frac{[m]_q!}{[l]_q![m-l]_q!}. \tag{4}$$

Let us denote the q-derivative of $D_q f_q(z)$ of f_q for a fixed value of $q > 1$ as follows:

$$D_q f_q(z) \coloneqq \begin{cases} \dfrac{f_q(qz) - f_q(z)}{(q-1)z}, & z \neq 0, \\ f_q'(0), & z = 0. \end{cases} \tag{5}$$

Also, $D_q^0 f_q \coloneqq f$ and $D_q^m f_q \coloneqq D_q(D_q^{m-1} f_q)$, $\forall m \in \mathbb{N}$.

We have presented two different forms for the q-exponential function. Jackson defined the q-exponential function $e_q(z)$ using $|q| < 1$ and $|z| < \frac{1}{1-q}$:

$$e_q(z) \coloneqq (1 - z + qz)_q^{-\infty}. \tag{6}$$

Assuming $e_q(z)$ is an entire function for $|q| > 1$,

$$e_q(z) = \prod_{m=0}^{\infty} \left(1 + \frac{1}{q^m} - \frac{z}{q^{m+1}}\right). \tag{7}$$

To acquire an alternative exponential function, we need to invert the base in equation (6), for values of q such that $|q|$ is less than 1:

$$E_q(z) = (1 - z + qz)_q^{\infty}. \tag{8}$$

Upon requesting it, we are promptly provided with

$$E_q(z) = \prod_{m=0}^{\infty} (1 + zq^m - zq^{m+1}), \quad |q| \in (0,1), \tag{9}$$

by (7).

Our study focuses on the scrutiny of an analytical function that lies within a disk centered at the origin, denoted as O, and has a specific radius of R.

Consider a disk $D_R = \{z \in \mathbb{C} || z| < R\}$ in the complex plane $\mathbb{C}$.

Let $H(D_R)$ represent the set of analytic functions on D_R, where f_q maps to $f : [R, \infty) \cup \bar{D}_{\mathbb{R}} \to \mathbb{C}$ continuously in $(R, \infty) \cup \bar{D}_R$.

In our writing regarding $f_q \in H(D_R)$, it is permissible to utilize the expression $f_q(z) = \sum_{m=0}^{\infty} c_m z^m$ for any value of z within the domain D_R.

Assume that there exists real-valued continuous functions f_q and f_q defined on the interval $[0, \infty)$, $q \in (0,1)$, $p, l \in \mathbb{N}$, $m \in \mathbb{N}^*$

$$B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(x) = \frac{[m-1]_q}{[m]_q} \sum_{l=0}^{\infty} b_{m,p}^l(x) \int_0^{\infty} s_{m,p}^l(t) f_q\left(\frac{[m]_q t + \alpha_q}{[m]_q + \beta_q}\right) d_q t, \tag{10}$$

where

$$b_{m,p}^l(x) = \begin{pmatrix} m+l \\ l \end{pmatrix}_q q^{l^2} \frac{x^l}{(1+x)^{m+j+1}} \tag{11}$$

and

$$s_{m,p}^l(x) = \frac{([m+1]_q t)^l}{[l]_q!} e_q^{-[m+1]_q t}. \tag{12}$$

2 Additional outcomes

Lemma 1. *The recurrence formula is defined as*

$$T^{(\alpha_q,\beta_q)}_{q,m,p,l+1}(z) = \frac{z(1+z)}{[m+1]_q} D_q T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z) + \frac{[m+1]_q z + l + 1}{[m+1]_q} T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z), \tag{13}$$

where $T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z) \coloneqq B^{(\alpha_q,\beta_q)}_{m,p,q}(e_l)(z)$ *and* $\mathbb{N}^0$ *represents the collection of all non-negative integers. It can be observed that the said recurrence formula applies to all integers within* $m, p, l \in \mathbb{N}^0$, $0 \le \alpha_q \le \beta_q$, *and* $z \in \mathbb{C}$.

Proof. By denoting $e_l(z) = z^l$ and writing,

$$\begin{aligned} z(1+z)D_q T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z) &= \frac{[m-1]_q}{[m]_q} \sum_{l=1}^{\infty} z(z+1) D_q b^l_{m,p}(z) \int_0^{\infty} s^l_{m,p}(t) t^l \mathrm{d}t, \\ &= \frac{[m-1]_q}{[m]_q} \sum_{l=1}^{\infty} (l - [m+1]_q z) b^l_{m,p}(z) \int_0^{\infty} s^l_{m,p}(t) t^l \mathrm{d}t, \\ &= \frac{[m-1]_q}{[m]_q} \sum_{l=1}^{\infty} b^l_{m,p}(z) \int_0^{\infty} [(l - [m+1]_q t) + ([m+1]_q t - [m+1]_q z)] s^l_{m,p}(t) t^l \mathrm{d}t, \\ &= \frac{[m-1]_q}{[m]_q} \sum_{l=1}^{\infty} b^l_{m,p}(z) \int_0^{\infty} D_q s^l_{m,p}(t) t^{l+1} dt + [m+1]_q T^{(\alpha_q,\beta_q)}_{q,m,p,l+1}(z) - [m+1]_q z T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z), \end{aligned}$$

$$\begin{aligned} z(1+z)D_q T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z) &= -(l+1) T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z) - [m+1]_q z T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z) + [m+1]_q T^{(\alpha_q,\beta_q)}_{q,m,p,l+1}(z), \\ [m+1]_q T^{(\alpha_q,\beta_q)}_{q,m,p,l+1}(z) &= z(1+z) D_q T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z) + (l + 1 + [m+1]_q z) T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z), \\ T^{(\alpha_q,\beta_q)}_{q,m,p,l+1}(z) &= \frac{z(1+z)}{[m+1]_q} D_q T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z) + \frac{[m+1]_q z + l + 1}{[m+1]_q} T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z). \end{aligned}$$

□

We were able to obtain the desired result for $z \in \mathbb{C}$.

Lemma 2. *Denoting* $B^{(0,0)}_{m,p,q}(e_j)(z)$ *as* $B_{m,p,q}(e_j)(z)$, *for all* $m, p, l \in \mathbb{N}^0$, *we have established a recursive relation for the images of monomials* e_l *under* $B^{(\alpha_q,\beta_q)}_{m,p,q}$ *in terms of* $B_{m,p,q}(e_j)$, $j = 0, 1, \cdots, l$:

$$T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z) = \sum_{j=0}^{l} \binom{l}{j}_q \frac{[m+1]_q^j \alpha_q^{l-j}}{([m+1]_q + \beta_q)^l} \cdot B_{m,p,q}(e_j)(z). \tag{14}$$

Proof. When l equals zero, equation (15) remains true. For the specific value

$$T^{(\alpha_q,\beta_q)}_{q,m,p,t}(z) = \sum_{j=0}^{t} \binom{t}{j}_q \frac{[m+1]_q^j \alpha_q^{t-j}}{([m+1]_q + \beta_q)^t} \cdot B_{m,p,q}(e_j)(z), \tag{15}$$

let it be true that l equals t.

Using equation (14),

$$T^{(\alpha_q,\beta_q)}_{m,p,t+1}(z) = \frac{z(1+z)}{[m+1]_q} \sum_{j=0}^{t} \binom{t}{j}_q \frac{[m+1]_q^j \alpha_q^{t-j}}{([m+1]_q + \beta_q)^t} D_q B_{m,p,q}(e_j)(z) \tag{16}$$

$$+ \frac{[m+1]_q z + l + 1}{[m+1]_q} \sum_{j=0}^{t} \binom{t}{j}_q \frac{[m+1]_q^j \alpha_q^{t-j}}{([m+1]_q + \beta_q)^t} B_{m,p,q}(e_j)(z), \tag{17}$$

$$= \sum_{j=0}^{t} \binom{t}{j}_q \frac{[m+1]_q^{j+1} \alpha_q^{t-j}}{([m+1]_q + \beta_q)^{t+1}} \tag{18}$$

$$\times \left[\frac{z(1+z)}{[m+1]_q} D_q B_{m,p,q}(e_j)(z) + \frac{[m+1]_q z + l + 1}{[m+1]_q} B_{m,p,q}(e_j)(z) \right], \tag{19}$$

which has been obtained.

By deriving a recurrence relation for the complex q-Beta-Baskakov-Szasz-Stancu operator, it becomes evident that

$$\begin{aligned} T_{m,p,t+1}^{(\alpha_q,\beta_q)}(z) &= \sum_{j=1}^{t} \binom{t}{j-1}_q \frac{[m+1]_q^{j} \alpha_q^{t-j+1}}{([m+1]_q + \beta_q)^{t+1}} B_{m,p,q}(e_j)(z) \\ &+ \sum_{j=0}^{t} \binom{t}{j}_q \frac{[m+1]_q^{j} \alpha_q^{t-j+1}}{([m+1]_q + \beta_q)^{t+1}} \cdot B_{m,p,q}(e_j)(z), \\ &= \sum_{j=0}^{t+1} \binom{t+1}{j}_q \frac{[m+1]_q^{j} \alpha_q^{t-j+1}}{([m+1]_q + \beta_q)^{t+1}} B_{m,p,q}(e_j)(z), \end{aligned} \tag{20}$$

providing proof for the lemma. □

3 Approximation of complex q-Beta-Baskakov-Szasz-Stancu operators

We proceed to provide numerical approximations in the subsequent section, accompanied by the following theorem.

Theorem 1. *Let R be a real number greater than 1, and let f_q be a function defined on $[R,\infty) \cup \bar{D}_R$ that is continuous and bounded in $[0,\infty)$ and analytic in the disk D_R. In particular, we can write f_q as a power series $\sum_{l=0}^{\infty} c_l z^l$ that converges for all z in D_R and for all non-negative integers l. For any $0 \le \alpha_q \le \beta_q$ and $1 < r < R$ assuming arbitrary but constant values, we present an estimate that holds true for all $|z| \le r$ and $q > 1, m \in \mathbb{N}$,*

$$|B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(z) - f_q(z)| \le \frac{[m+1]_q(\beta_q + 1) + \beta_q}{[m+1]_q([m+1]_q + \beta_q)}, \tag{21}$$

where

$$C_r = 2(r+2) \sum_{l=1}^{\infty} |c_l|(l+2)! r^{l-1} < \infty.$$

Proof. Through the use of Lemma 2, we are able to obtain

$$\begin{aligned} &T_{q,m,p,l}^{(\alpha_q,\beta_q)}(e_l)(z) - e_l(z) \\ &= \sum_{j=0}^{l-1} \binom{l}{j}_q \frac{[m+1]_q^{j} \alpha_q^{l-j}}{([m+1]_q + \beta_q)^{l}} \cdot (B_{m,p,q}(e_j)(z) - e_j(z)) \\ &+ \sum_{j=0}^{l-1} \binom{l}{j}_q \frac{[m+1]_q^{j} \alpha_q^{l-j}}{([m+1]_q + \beta_q)^{l}} \cdot (e_j)(z) + \frac{[m+1]_q^{l}}{([m+1]_q + \beta_q)^{l}} B_{m,p,q}(e_l)(z) - e_l(z). \end{aligned}$$

The utilization of Lemma 1 presents a potential opportunity for the acquisition of

$$T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z) - z^l = \frac{z(1+z)}{[m+1]_q} D_q(T^{(\alpha_q,\beta_q)}_{q,m,p,l-1}(z) - z^{l-1}) + \frac{[m+1]_q z + l}{[m+1]_q}(T^{(\alpha_q,\beta_q)}_{q,m,p,l-1}(z) - z^{l-1}) + \frac{(2l-1)+(l-1)r}{[m+1]_q} z^{l-1}, \forall z \in \mathbb{C},\ l,m,p \in \mathbb{N}. \tag{22}$$

We denote the norm $\|\cdot\|$ in $C(\bar{D}_R)$ for $1 \le r \le R$. Here, $\bar{D}_R$ is defined as the set of complex numbers z where $|z| \le r$. Within the closed unit disk, we establish an inequality that states $|D_q P_l(z)| \le \frac{l}{r}\|P_l\|_r$ for all $|z| \le r$. This is applicable to $P_l(z)$, which is a polynomial of a degree less than or equal to l. Using the recurrence relation mentioned earlier, we are able to derive the following result:

$$|T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z) - z^l| \le \frac{r(1+r)}{[m+1]_q}\left(\frac{l-1}{r}\right)\|T^{(\alpha_q,\beta_q)}_{m,p,l-1}\|_r + \frac{[m+1]_q r + l}{[m+1]_q}|T^{(\alpha_q,\beta_q)}_{m,p,l-1}(z) - z^{l-1}| + \frac{l(r+2)}{[m+1]_q} r^{l-1}.$$

The suggestion made using the notation $\iota = r + 2$ is that

$$\begin{aligned}|T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z) - z^k| &\le \left(r + \frac{(2+r)k}{[m+1]_q}\right)\|T^{(\alpha_q,\beta_q)}_{q,m,p,k-1}\|_r + \frac{k}{[m+1]_q}(2+r)r^{k-1}\\ &= \left(r + \frac{\iota k}{[m+1]_q}\right)\|T^{(\alpha_q,\beta_q)}_{q,m,p,k-1}\|_r + \frac{k}{[m+1]_q}\iota r^{k-1}.\end{aligned}$$

The process of deriving outcomes involves utilizing induction in regards to k, and taking into account the condition of $m + 1 \ge \iota$:

$$|T^{(\alpha_q,\beta_q)}_{q,m,p,l}(z) - z^l| \le \frac{\iota(l+2)!}{[m+1]_q} r^{l-1}, \forall l \ge 1. \tag{23}$$

The aforementioned recurrence transforms into

$$\begin{aligned}|T^{(\alpha_q,\beta_q)}_{q,m,p,l+1}(z) - z^{l+1}| &\le \left(r + \frac{\iota(l+1)}{[m+1]_q}\right)\frac{(l+2)!}{[m+1]_q}\iota r^{l-1} + \frac{\iota(l+1)}{[m+1]_q} r^l\\ &\le \left(r + \frac{\iota(l+1)}{[m+1]_q}\right)(l+2)! + r(l+1).\end{aligned}$$

Given that the value of m added to 1 is greater than or equal to ι, we can conclude that:

$$|T^{(\alpha_q,\beta_q)}_{q,m,p,l+1}(z) - z^{l+1}| \le (r + l + 1)(l+2)! + r(l+1) \le (l+3)!r.$$

For any value of l greater than or equal to one and a fixed value of r greater than or equal to one, the validity of this is evident.

In the course of our work, we employed the equation denoted as (23):

$$\begin{aligned}\|T^{(\alpha_q,\beta_q)}_{q,m,p,l}(e_l) - e_l\|_r &\le \sum_{j=0}^{l-1}\binom{l}{j}_q \frac{[m+1]_q^j \alpha_q^{l-j}}{([m+1]_q + \beta_q)^l}\cdot\|B_{m,p,q}(e_j) - e_j\|_r + \sum_{j=0}^{l-1}\binom{l}{j}_q \frac{[m+1]_q^j \alpha_q^{l-j}}{([m+1]_q + \beta_q)^l}\cdot(r^j)(z)\\ &\quad + \frac{[m+1]_q^l}{([m+1]_q + \beta_q)^l}\|B_{m,p,q}(e_l) - e_l\|_r + \left(1 - \frac{[m+1]_q^l}{([m+1]_q + \beta_q)^l}\right)\cdot r^l\\ &\le \frac{([m+1]_q + \alpha_q)^l}{([m+1]_q + \beta_q)^l}\cdot\frac{\iota(l+2)!}{[m+1]_q}\cdot r^{l-1} + r^l\cdot\left[\frac{([m+1]_q + \alpha_q)^l}{([m+1]_q + \beta_q)^l} - \frac{[m+1]_q^l}{([m+1]_q + \beta_q)^l}\right]\\ &\quad + \frac{[m+1]_q^l}{([m+1]_q + \beta_q)^l}\cdot\frac{\iota(l+2)!}{[m+1]_q}\cdot r^{l-1} + \left(1 - \frac{[m+1]_q^l}{([m+1]_q + \beta_q)^l}\right)\cdot r^l\\ &\le \frac{2\iota(l+2)!}{[m+1]_q}\cdot r^{l-1} + 2r^l\cdot\frac{l\beta_q}{[m+1]_q + \beta_q}\\ &\le \frac{([m+1]_q)(\beta_q + 1) + \beta_q}{([m+1]_q)([m+1]_q + \beta_q)}\cdot 2(l+2)!\cdot\iota\cdot r^{l-1}.\end{aligned} \tag{24}$$

Incorporating the inequality $1 - \prod_{j=1}^{l} x_j \le \sum_{j=1}^{l}(1 - x_j)$ is a common practice in mathematical discourse. This equation is applicable when considering variables x_j within the range of 0 to 1, where j is any integer between 1 and l.

Now, we write $B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(z) = \sum_{l=1}^{\infty} c_k T_{q,m,p,l}^{(\alpha_q,\beta_q)}(z)$, which implies

$$\begin{aligned} |B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(z) - f_q(z)| &\le \sum_{l=1}^{\infty} |c_l| \cdot |T_{q,m,p,l}^{(\alpha_q,\beta_q)}(z) - z^l| \\ &\le \sum_{l=1}^{\infty} |c_l| \cdot 2\iota \frac{([m+1]_q)(\beta_q + 1) + \beta_q}{([m+1]_q)([m+1]_q + \beta_q)} \cdot (k+2)! \cdot r^{l-1} \\ &= 2(r+2) \cdot \frac{([m+1]_q)(\beta_q + 1) + \beta_q}{([m+1]_q)([m+1]_q + \beta_q)} \sum_{l=1}^{\infty} |c_l| \cdot (l+2)! \cdot r^{l-1}. \end{aligned} \tag{25}$$

We denote that, for every value of r from one to R, and for every value of z smaller than or equal to r, the following notation holds

$$C_r = 2(r+2) \sum_{l=1}^{\infty} |c_l|(l+2)! r^{l-1} < \infty.$$

As a result, the demonstration has reached its conclusion. □

Lemma 3. *We can establish that the given inequality holds true for all values of $|z|$ less than or equal to r, where r is greater than or equal to 1, q is greater than 1, and m, p, and l are all natural numbers, provided that m is greater than or equal to $r + 1$. The absolute value* $|T_{q,m,p,l}^{(\alpha_q,\beta_q)}(e_l)(z)| \le 4r^l(l+2)!$.

Proof. We start with the following inequality for the norm in the disk of radius r:

$$\|T_{q,m,p,l}^{(\alpha_q,\beta_q)}(e_l)\|_r \le \|T_{q,m,p,l}^{(\alpha_q,\beta_q)}(e_l) - e_l\|_r + \|e_l\|_r.$$

This is just the triangle inequality for the norm.

Next, we employ the relationship indicated in equation (23)

$$\|T_{q,m,p,l}^{(\alpha_q,\beta_q)}(e_l)\|_r \le \frac{(r+2)(l+2)!}{[m+1]_q} \cdot r^{l-1} + r^l = \frac{(r+2)(l+2)!}{r([m+1]_q)} r^l + r^{\cdot}$$

Now, we can simplify the aforementioned expression:

$$\|T_{q,m,p,l}^{(\alpha_q,\beta_q)}(e_l)\|_r \le r^l \left(1 + \frac{2}{r}\right)(l+2)! + r^l \le 3r^l(l+2)! + r^l \le 4r^l(l+2)!,$$

for all $r \ge 1$, $m, p, l \in \mathbb{N}$ and $m \ge r + 1$. □

Theorem 2. *Assuming that the Theorem 1 conditions are met, where $0 \le \alpha_q \le \beta_q$, $1 < r < R$, and $q > 1$, the Voronovskaja-type finding applies to all $|z| \le r$ and $m, p \in \mathbb{N}$. The inequality expression*

$$\begin{aligned} &\left| B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(z) - f_q(z) - \frac{\alpha_q + 1 - \beta_q(z-1)}{[m+1]_q} f_q'(z) - \frac{z(z+2)}{2[m+1]_q} f_q''(z) \right| \\ &\le \frac{M_r(f_q)}{([m+1]_q)^2} + \frac{N_r(f_q)}{[m+1]_q} + \frac{P_r(f_q)}{([m+1]_q + \beta_q)^2} \end{aligned} \tag{26}$$

holds true. The next formulas illustrate a complex mathematical equation with various contributing factors

$$M_r(f_q) = 2(r+1)^{l+1} \sum_{l=2}^{\infty} |c_l| r(l-1)^2 - \frac{l}{4}(l-1) + \frac{(r+2)}{2}(l+3)! < \infty$$

and

$$N_r(f_q) = \sum_{l=m+2}^{\infty} |c_l| \left(\frac{r}{2} + 1\right)(l+2)! r^{l-1} < \infty,$$

$$P_r(f_q) = \sum_{l=0}^{\infty} |c_l| [(l-1)(2\alpha_q^2 + \beta_q^2 - \alpha_q\beta_q + 3\beta_q) + (\beta_q + 1)(\alpha_q + 1) + (r+1)(\alpha_q + \beta_q)](l+2)! l r^l < \infty.$$

Proof. Considering all $z \in D_R$, we examine

$$\begin{aligned} &B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(z) - f_q(z) - \frac{\alpha_q + 1 - \beta_q(z-1)}{[m+1]_q} f_q'(z) - \frac{z(z+2)}{2[m+1]_q} f_q''(z) \\ &= B_{m,p,q}(f_q)(z) - f_q(z) - \frac{1}{[m+1]_q} f_q'(z) - \frac{z(z+2)}{2[m+1]_q} f_q''(z) + B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(z) - B_{m,p,q}(f_q)(z) \\ &- \frac{\alpha_q - \beta_q(z-1)}{[m+1]_q} f_q'(z). \end{aligned}$$

Using the expression $f_q(z) = \sum_{l=0}^{\infty} c_l z^l$ in the second term, we are able to obtain our desired result:

$$\begin{aligned} &\left| B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(z) - f_q(z) - \frac{\alpha_q + 1 - \beta_q(z-1)}{[m+1]_q} f_q'(z) - \frac{(z+2)z}{2[m+1]_q} f_q''(z) \right| \\ &\leq \left| B_{m,p,q}(f_q)(z) - f_q(z) - \frac{1}{[m+1]_q} f_q'(z) - \frac{(z+2)z}{2[m+1]_q} f_q''(z) \right| \\ &+ \left| \sum_{l=0}^{\infty} |c_l| \left(B_{m,p,q}^{(\alpha_q,\beta_q)}(e_l)(z) - B_{m,p,q}(e_l)(z) - \frac{\alpha_q - \beta_q(z-1)}{[m+1]_q} l z^{l-1} \right) \right|. \end{aligned}$$

The equation involves several absolute values of the difference between certain functions and their approximations, as well as a sum of absolute values of coefficients and differences between certain modified and unmodified functions, along with a polynomial expression with coefficients that depend on the function being approximated.

We have developed a method to estimate the first sum by expressing $B_{m,p,q}(f_q)(z)$ as a summation of $|c_l|$ multiplied by $B_{m,p,q}(e_l)(z)$, with l ranging from 0 to infinity.

Thus,

$$\begin{aligned} &\left| B_{m,p,q}(f_q)(z) - f_q(z) - \frac{1}{[m+1]_q} f_q'(z) - \frac{z(z+2)}{2[m+1]_q} f_q''(z) \right| \\ &\leq \sum_{l=0}^{\infty} |c_l| \left| B_{m,p,q}(e_l)(z) - e_l(z) - \frac{1}{[m+1]_q} l z^{l-1} - \frac{(z+2)z}{2[m+1]_q} l(l-1) z^{l-2} \right|, \\ &\left| B_{m,p,q}(f_q)(z) - f_q(z) - \frac{1}{[m+1]_q} f_q'(z) - \frac{(z+2)z}{2[m+1]_q} f_q''(z) \right| \\ &\leq \sum_{l=0}^{\infty} |c_l| \left| B_{m,p,q}(e_l)(z) - e_l(z) - \frac{[l(l-1)z + 2l(2-l)] z^{l-1}}{2[m+1]_q} \right|. \end{aligned}$$

For any natural numbers m, p, and l, along with any complex number z, we possess Lemma 1.

$$B_{m,p,q}(e_{l+1})(z) = \frac{(1+z)z}{[m+1]_q} D_q B_{m,p,q}(e_l)(z) + \frac{[m+1]_q z + l + 1}{[m+1]_q} B_{m,p,q}(e_l)(z).$$

We currently denote

$$O_{m,l}(z) = B_{m,p,q}(e_l)(z) - e_l(z) - \frac{[l(l-1)z + 2l(2-l)]z^{l-1}}{2[m+1]_q}. \tag{27}$$

The polynomial $O_{m,l}(z)$ has a degree that does not exceed l; after some basic computation and utilizing the aforementioned recurrence relation, we arrive at their conclusion:

$$O_{m,l}(z) = \frac{z(1+z)}{[m+1]_q} D_q O_{m,l-1}(z) + \frac{[m+1]_q z + l}{[m+1]_q} O_{m,l-1}(z) + E_{m,l}(z),$$

where

$$E_{m,l}(z) = \frac{2z^{l-2}}{[m+1]_q^2}(l-1)^2(l-2)(l^2-2l+2) + \frac{z^{l-1}(l-1)^2(l-2)^2(z+1)}{2[m+1]_q^2}.$$

Given $l \geq 2, m, p \in \mathbb{N}$ and $|z| \leq r$, the statement holds true. Using theorem 1, we were able to derive an estimate. The following inequality is true for the polynomial function

$$|B_{m,p,q}(e_l)(z) - e_l(z)| \leq \frac{(r+2)(l+2)!}{[m+1]_q} r^{l-1}.$$

The inequality expression states that

$$|O_{m,l}(z)| \leq \frac{r(r+1)}{[m+1]_q}|D_q O_{m,l-1}(z)| + \left(r + \frac{l}{[m+1]_q}\right)|O_{m,l-1}(z)| + |E_{m,l}(z)|.$$

Due to the fact that $O_{m,l-1}(z)$ is a polynomial with a degree of less than or equal to $l-1$, we are able to obtain

$$|D_q O_{m,l-1}(z)| \leq \frac{l-1}{r}||O_{m,l-1}(z)||_r.$$

From (27),

$$\begin{aligned}
|D_q O_{m,l-1}(z)| &\leq \frac{l-1}{r}\left\{||B_{m,p,q}(e_{l+1}) - e_{l+1}||_r + \left|\left| \frac{[(l-1)(l-2)e_1 + 2(l-1)(3-l)]e_{l-2}}{2[m+1]_q} \right|\right|\right\} \\
&\leq \frac{l-1}{r}\left[\frac{(r+2)(l+2)!}{[m+1]_q} r^{l-2} + \frac{[(l-1)(l-2)r + 2(l-1)(3-l)]r^{l-2}}{2[m+1]_q}\right] \\
&\leq \frac{l-1}{r} \cdot \frac{r^{l-2}}{[m+1]_q} \cdot [(r+2)(l+2)! + (l-1)(l-2)(r+2)] \\
&= \frac{(l-1)r^{l-2}(r+2)[(l+2)! + (l-1)(l-2)]}{r[m+1]_q} \\
&\leq \frac{r^{l-3}(r+2)(l+3)!}{[m+1]_q}.
\end{aligned}$$

The equation can be simplified to

$$\frac{r(1+r)}{[m+1]_q}|D_q O_{m,l-1}(z)| \leq \frac{r^{l-2}(r+1)(r+2)(l+3)!}{[m+1]_q^2}$$

and

$$|O_{m,l}(z)| \leq \frac{r^{l-2}(r+1)(r+2)(l+3)!}{[m+1]_q^2} + \left(r + \frac{l}{[m+1]_q}\right)|O_{m,l-1}(z)| + |E_{m,l}(z)|.$$

Assuming l is greater than or equal to two, and with m, p, and r being natural numbers, it is required that m is greater than $r+1$. We were able to obtain the following result for $l \le m$, $m > r+1$, and $|z| \le r$, by using the inequality $r + \frac{l}{[m+1]_q} \le r+1$.

$$|O_{m,l}(z)| \le \frac{r^{l-2}(r+1)(r+2)(l+3)!}{[m+1]_q^2} + (r+1)|O_{m,l-1}(z)| + |E_{m,l}(z)|,$$

with

$$|E_{m,l}(z)| \le \frac{2r^{l-2}}{[m+1]_q^2}(l-1)^2(l-2)(l^2-2l+2) + \frac{r^{l-1}(l-1)^2(l-2)^2(r+1)}{2[m+1]_q^2}.$$

We were able to derive

$$\begin{aligned}|O_{m,l}(z)| &\le \frac{2(r+1)^l}{[m+1]_q^2}\sum_{j=2}^{l}(j-1)^2(j-2)(j^2-2j+2) + \frac{r(r+1)^{l+1}}{2[m+1]_q^2}\sum_{j=2}^{l}(j-1)^2(j-2)^2 + \frac{(r+1)^{l+1}(r+2)}{[m+1]_q^2}\sum_{j=2}^{l}(j+3)!\\ &\le \frac{2(l-1)^2 r(r+1)^{l+1}}{[m+1]_q}\left\{(l-2)(l^2-2l+2) + \frac{(l-2)^2}{4r}\right\} + \frac{(l-1)(r+2)(r+1)^{l+1}}{[m+1]_q}(l+3)!,\end{aligned}$$

for any $z \in \mathbb{C}$ and $2 \le l \le m+1$, where $|O_{m,0}(z)| = |O_{m,1}(z)| = 0$.

To deduce or conclude something, one needs to have a premise or condition, which is typically represented by

$$\begin{aligned}&\left|B_{m,p,q}(f_q)(z) - f_q(z) - \frac{1}{[m+1]_q}f_q'(z) - \frac{z(z+2)}{2[m+1]_q}f_q''(z)\right|\\ &\le \sum_{l=2}^{n+2}|c_l||O_{m,l}(z)| + \sum_{l=m+3}^{\infty}|c_l||O_{m,l}(z)|\\ &\le \frac{1}{[m+1]_q^2}\sum_{l=2}^{\infty}2|c_l|(l-1)^2 r(r+1)^{l+1}\left\{(l-2)(l^2-2l+2) + \frac{(l-2)^2}{4r}\right\}\\ &\quad + \frac{1}{[m+1]_q^2}\sum_{l=2}^{\infty}(l-1)(r+2)(r+1)^{l+1}(l+3)!\\ &\quad + \sum_{l=m+3}^{\infty}|c_l||B_{m,p,q}(e_l)(z) - e_l(z)| - \frac{[l(l-1)z - 2l(l-2)]z^{l-1}}{2[m+1]_q}\\ &\le \frac{1}{[m+1]_q^2}\sum_{l=2}^{\infty}|c_l|(l-1)^2 r(r+1)^{l+1}\left\{(l-2)(l^2-2l+2) + \frac{(l-2)^2}{4r}\right\}\\ &\quad + \frac{1}{[m+1]_q^2}\sum_{l=2}^{\infty}(l-1)(r+2)(r+1)^{l+1}(l+3)! + \sum_{l=m+3}^{\infty}|c_l|\frac{(r+2)(l+2)!}{[m+1]_q}r^{l-1}\\ &\le \frac{T_r(f_q)}{([m+1]_q)^2} + \frac{N_r(f_q)}{[m+1]_q},\end{aligned} \tag{28}$$

where

$$M_r(f_q) = 2(r+1)^{l+1}\sum_{l=2}^{\infty}|c_l|r(l-1)^2 - \frac{l}{4}(l-1) + \frac{(r+2)}{2}(l+3)! < \infty$$

and

$$N_r(f_q) = \sum_{l=m+2}^{\infty}|c_l|\left(\frac{r}{2}+1\right)(l+2)!r^{l-1} < \infty.$$

We proceeded to estimate the second sum by using Lemma 2, which involved rewriting the sum in the following manner:

$$
\begin{aligned}
&B_{m,p,q}^{(\alpha_q,\beta_q)}(e_l)(z) - B_{m,p,q}(e_l)(z) + \frac{\beta_q(z-1)-\alpha_q}{[m+1]_q} l z^{l-1} \\
&= \sum_{j=0}^{l-1} \binom{l}{j}_q \frac{[m+1]_q^j \alpha_q^{l-j}}{([m+1]_q+\beta_q)^l} \cdot B_{m,p,q}(e_j)(z) - \left[1 - \left(\frac{[m+1]_q}{[m+1]_q+\beta_q}\right)^l\right] \cdot B_{m,p,q}(e_l)(z) \\
&\quad + \frac{\beta_q(z-1)-\alpha_q}{[m+1]_q+\beta_q} l z^{l-1} \\
&= \sum_{j=0}^{l-2} \binom{l}{j}_q \frac{[m+1]_q^j \alpha_q^{l-j}}{([m+1]_q+\beta_q)^l} \cdot B_{m,p,q}(e_j)(z) + \frac{l\alpha_q [m+1]_q^{l-1}}{([m+1]_q+\beta_q)^l} B_{m,p,q}(e_{l-1})(z) \\
&\quad - \sum_{j=0}^{l-1} \binom{l}{j}_q \frac{[m+1]_q^j \beta_q^{l-j}}{([m+1]_q+\beta_q)^l} \cdot B_{m,p,q}(e_l)(z) + \frac{\beta_q(z-1)-\alpha_q}{[m+1]_q+\beta_q} l z^{l-1} \\
&= \sum_{j=0}^{l-2} \binom{l}{j}_q \frac{[m+1]_q^j \alpha_q^{l-j}}{([m+1]_q+\beta_q)^l} B_{m,p,q}(e_j)(z) + \frac{l\alpha_q [m+1]_q^{l-1}}{([m+1]_q+\beta_q)^l} (B_{m,p,q}(e_{l-1})(z) - z^{l-1}) \\
&\quad - \sum_{j=0}^{l-2} \binom{l}{j}_q \frac{[m+1]_q^j \beta_q^{l-j}}{([m+1]_q+\beta_q)^l} B_{m,p,q}(e_l)(z) - \frac{l\beta_q [m+1]_q^{l-1}}{([m+1]_q\beta_q)^l} (B_{m,p,q}(e_l)(z) - z^l) \\
&\quad + \frac{l\alpha_q}{[m+1]_q+\beta_q} z^{l-1} \cdot \left(\frac{[m+1]_q^{l-1}}{([m+1]_q+\beta_q)^{l-1}} - 1\right) + \frac{l\beta_q}{[m+1]_q+\beta_q} z^l \left(1 - \frac{[m+1]_q^{l-1}}{([m+1]_q+\beta_q)^{l-1}}\right) \\
&\quad - \frac{\beta_q(\alpha_q+\beta_q+\beta_q z+[m+1]_q)}{[m+1]_q([m+1]_q+\beta_q)}.
\end{aligned}
$$

Using Lemma 3 in conjunction with the subsequent inequality, we can proceed as follows:

$$
1 - \left(\frac{[m+1]_q}{[m+1]_q+\beta_q}\right)^l \leq \sum_{j=1}^{l} \left(1 - \frac{[m+1]_q}{[m+1]_q+\beta_q}\right) = \frac{l\beta_q}{[m+1]_q+\beta_q}. \tag{29}
$$

We obtain

$$
\begin{aligned}
&\left| \sum_{j=0}^{l-2} \binom{l}{j}_q \frac{[m+1]_q^j \alpha_q^{l-j}}{([m+1]_q+\beta_q)^l} B_{m,p,q}(e_j)(z) \right| \\
&\leq \sum_{j=0}^{l-2} \binom{l}{j}_q \frac{[m+1]_q^j \alpha_q^{l-j}}{([m+1]_q+\beta_q)^l} \cdot |B_{m,p,q}(e_j)(z)| \\
&= \sum_{j=0}^{l-2} \frac{l(l-1)}{(l-j)(l-1-j)} \binom{l-2}{j}_q \frac{[m+1]_q^j \alpha_q^{l-j}}{([m+1]_q+\beta_q)^l} \cdot |B_{m,p,q}(e_j)(z)| \\
&\leq \frac{l(l-1)}{2} \cdot \frac{\alpha_q^2}{([m+1]_q+\beta_q)^2} \cdot 4 l! r^{l-2} \sum_{j=0}^{l-2} \binom{l-2}{j}_q \frac{[m+1]_q^j \alpha_q^{l-j-2}}{([m+1]_q+\beta_q)^{l-2}} \\
&\leq \frac{2l(l-1) l! r^{l-2} \alpha_q^2}{([m+1]_q+\beta_q)^2}.
\end{aligned}
$$

Now, we have obtained

$$\begin{aligned}&\left|B_{m,p,q}^{(\alpha_q,\beta_q)}(e_l)(z) - B_{m,p,q}(e_l)(z) + \frac{\beta_q(z-1)-\alpha_q}{[m+1]_q+\beta_q} l z^{l-1}\right|\\ &\le \frac{2l(l-1)l!r^{l-2}\alpha_q^2}{([m+1]_q+\beta_q)^2} + \frac{l(r+2)\alpha_q}{([m+1]_q+\beta_q)^2}(l+1)!r^{l-2} + \frac{2l(l-1)\beta_q^2}{([m+1]_q+\beta_q)^2} l! r^{l-2}\\ &\quad + \frac{l(l-1)\alpha_q\beta_q}{([m+1]_q+\beta_q)^2} r^{l-1} + \frac{l\beta_q(r+2)(l+2)!}{([m+1]_q+\beta_q)^2} r^{l-1} + \frac{l(l-1)\beta_q^2}{([m+1]_q+\beta_q)^2} r^l\\ &\quad + \frac{\beta_q(\alpha_q+\beta_q+\beta_q r+[m+1]_q) l r^{l-1}}{([m+1]_q+\beta_q)^2},\end{aligned}$$

which led to

$$\left|B_{m,p,q}^{(\alpha_q,\beta_q)}(e_l)(z) - B_{m,p,q}(e_l)(z) + \frac{\beta_q(z-1)-\alpha_q}{[m+1]_q+\beta_q} l z^{l-1}\right| \le \frac{H_r(f_q)}{([m+1]_q+\beta_q)^2},$$

$$P_r(f_q) = \sum_{l=0}^{\infty} |c_l| [(l-1)(2\alpha_q^2+\beta_q^2-\alpha_q\beta_q+3\beta_q) + (\beta_q+1)(\alpha_q+1) + (r+1)(\alpha_q+\beta_q)](l+2)! l r^l < \infty.$$

Using (28) in conjunction with this, we are able to establish the intended outcome. □

Theorem 3. *Assuming that q is greater than 1 and* $0 \le \alpha_q \le \beta_q$, *and the conditions of the function f are met as described in Theorem* 1, *suppose that* $1 < r < R$ *is a constant. Then, for any m and p that belong to the set of natural numbers, where m is greater than* $r+1$ *and* $|z|$ *is less than or equal to r, we have demonstrated that*

$$\|B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q) - (f_q)\|_r \sim \frac{1}{[m+1]_q}. \tag{30}$$

The equivalence's constants are solely dependent on the values of f_q, α_q, β_q, *and* r, *provided that* f_q *is not a polynomial of degree less than or equal to zero.*

Proof. We state that, for any natural numbers n and p, as well as for any complex number z with an absolute value less than or equal to r, the following holds:

$$\begin{aligned}B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(z) - f_q(z) &= \frac{\alpha_q-\beta_q+1-\beta_q z}{[m+1]_q} f_q'(z) + \frac{z(z+2)}{2[m+1]_q} f_q''(z) + \frac{1}{[m+1]_q^2}[m+1]_q^2\\ &\quad \times \left[B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(z) - f_q(z) + \frac{\beta_q z - \beta_q - \alpha_q - 1}{[m+1]_q} f_q'(z) - \frac{z(z+2)}{2[m+1]_q} f_q''(z)\right].\end{aligned}$$

By making use of the inequality that follows

$$\|F_q + G_q\|_r \ge |\|F_q\|_r - \|G_q\|_r| \ge \|F_q\|_r - \|G_q\|_r,$$

we were able to obtain the following result:

$$\begin{aligned}\|B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q) - f_q\|_r &\ge \frac{1}{[m+1]_q}\cdot \left\|(\alpha_q+1-\beta_q e_1+\beta_q)f_q' + \frac{e_1(e_1+2)}{2} f_q''(z)\right\|_r - \frac{1}{[m+1]_q^2}[m+1]_q^2\cdot\\ &\quad \cdot \left\|B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q) - f + \frac{\beta_q e_1 - \beta_q - \alpha_q - 1}{[m+1]_q} f_q' - \frac{(e_1+2)e_1}{2} f_q''(z)\right\|_r.\end{aligned}$$

We are able to write, based on our hypotheses on f, that f cannot be considered a polynomial with a degree of ≤1 in D_R:

$$\left\|(\alpha_q + 1 - \beta_q e_1 + \beta_q)f' + \frac{e_1(e_1+2)}{2}f_q''(z)\right\|_r > 0.$$

The application of Theorem 2 leads to the conclusion that:

$$[m+1]_q^2\left\|B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q) - f_q + \frac{\beta_q e_1 - (\alpha_q+\beta_q+1)}{[m+1]_q}f_q' - \frac{e_1(e_1+2)}{2}f_q''(z)\right\|_r \le C.$$

An independent constant C with a positive value is present.

We have established that, given $f_q, \alpha_q, \beta_q, q$, and r, there exists an index m_0 such that for all n greater than or equal to m_0, the condition $\frac{1}{[m+1]_q} \to 0$ as $m \to \infty$ is satisfied.

$$\begin{aligned}&\left\|(\alpha_q + 1 - \beta_q(e_1-1))f' + \frac{(e_1+2)e_1}{2}f_q''(z)\right\|_r - [m+1]_q \\ &\cdot \left\|B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q) - f + \frac{\beta_q e_1 - (\alpha_q+\beta_q+1)}{[m+1]_q+\beta_q}f' - \frac{(e_1+2)e_1}{2[m+1]_q}f_q''(z)\right\|_r \\ &\ge \frac{1}{2}\left\|(\alpha_q + 1 - \beta_q(e_1-1))f' + \frac{(e_1+2)e_1}{2}f_q''(z)\right\|_r.\end{aligned}$$

This suggests

$$||B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q) - f_q||_r \ge \frac{1}{2[m+1]_q}\left\|(\alpha_q + 1 - \beta_q(e_1-1))f_q' + \frac{(e_1+2)e_1}{2}f_q''(z)\right\|_r.$$

Whenever the value of the parameter m is greater than or equal to the baseline value of m_0, the following statement applies for all m. On the other hand, if the value of m is between 1 and $m_0 - 1$, inclusive, we obtain the following outcome:

$$||B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q) - f_q||_r \ge \frac{M_{r,n}(f_q)}{[m+1]_q}. \tag{31}$$

If $M_{r,m}(f_q) = [m+1]_q||B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q) - f||_r$ is taken into consideration, it is evident that the value obtained will always be greater than zero. Ultimately, we are able to acquire.

$$||B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q) - f_q||_r \ge \frac{M_r^{\alpha_q,\beta_q}(f_q)}{[m+1]_q}. \tag{32}$$

Using the function $M_r^{\alpha_q,\beta_q}(f_q)$, defined as the minimum of $M_{r,1}(f_q)$ through $M_{r,m_0-1}(f_q)$, and also incorporating $\frac{1}{2}||(\alpha_q - \beta_q e_1 + \beta_q)\cdot f' + \frac{(e_1+2)e_1}{2}\cdot f''||_r$, we, as stated in Theorem 1, are able to attain the desired conclusion.

Theorem 4. *If $q > 1$, $0 \le \alpha_q \le \beta_q$, and the conditions of f in Theorem 1 are met, then suppose $1 < r < r_1 < R$ are the constants. In this scenario, and for all $m, p, u \in \mathbb{N}$, where $m > r + 2$ and $|z| \le r$, we have established the following*:

$$||(B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q))^{(u_q)} - f_q^{(u_q)}||_r \sim \frac{1}{[m+1]_q}. \tag{33}$$

The constants present in the equivalence are dependent solely on $f_q, q, u_q, r_1, \alpha_q, \beta_q$, and r, provided that f_q is not a polynomial of degree less than or equal to $u_q - 1$.

Proof. Let ι be the circle with center O and a radius of r_1, where $1 \le r < r_1 < R$. Because $|z| \le r$ and ι is a part of Γ, it follows that $|\iota - z| \ge r_1 - r$. Applying Cauchy's formulas, we can deduce that for any $|z| \le r$ as well as $nm, p \in \mathbb{N}$ where $m > r + 2$, the following is true:

$$\begin{aligned}(B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(z))^{(u_q)} - f_q^{(u_q)}(z) &\le \frac{u_q!}{2\pi}\left|\int_\iota \frac{B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(\iota) - f_q(\iota)}{(\iota - z)^{u_q+1}}\mathrm{d}\iota\right| \\ &\le \frac{M_{r_1}^{(\alpha_q,\beta_q)}(f_q)}{[m+1]_q}\cdot\frac{u_q!}{2\pi}\cdot\frac{2\pi r_1}{(r_1-r)^{u_q+1}} \\ &= \frac{M_{r_1}^{(\alpha_q,\beta_q)}(f_q)}{[m+1]_q}\cdot\frac{u_q! r_1}{(r_1-r)^{u_q+1}}.\end{aligned} \tag{34}$$

This serves as confirmation for one of the inequalities present in the equivalence.

We obtain a result from Cauchy's formula

$$(B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(z))^{(u_q)} - f_q^{(u_q)}(z) = \frac{u_q!}{2\pi i}\int_\iota \frac{B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(\iota) - f_q(\iota)}{(\iota - z)^{u_q+1}}\mathrm{d}\iota. \tag{35}$$

We have considered all values of ι in the set Γ, along with natural numbers p and m:

$$\begin{aligned}B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(\iota) - f_q(\iota) = {}& \frac{1}{[m+1]_q}\left[(\alpha_q + 1 - \beta_q\iota)f_q'(\iota) + \frac{\iota(\iota+2)}{2}f_q''(\iota)\right] \\ &+ \frac{1}{[m+1]_q}[m+1]_q^2\left(B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(\iota) - f_q(\iota) - \frac{(\alpha_q + 1 - \beta_q\iota)}{[m+1]_q + \beta_q}f_q'(\iota) - \frac{\iota(\iota+2)}{2[m+1]_q}f_q''(\iota)\right).\end{aligned} \tag{36}$$

Using Cauchy's formula, we were able to derive our results:

$$\begin{aligned}&B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(z)^{(u_q)} - f_q^{(u_q)}(z) \\ &= \frac{1}{[m+1]_q}\left[(\alpha_q + 1 - \beta_q\iota)f_q'(z) + \frac{z(z+2)}{2}f_q''(z)\right]^{(u_q)} \\ &+ \frac{1}{[m+1]_q}\left[\frac{u_q!}{2\pi i}\int_\iota\left(\frac{[m+1]_q^2}{(\iota-z)^{u_q+1}}B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(\iota) - f(\iota) + \frac{\beta_q\iota - \alpha_q - 1}{[m+1]_q}f_q'(\iota) - \frac{\iota(\iota+2)}{2[m+1]_q}f_q''(\iota)\right)\mathrm{d}\iota\right].\end{aligned} \tag{37}$$

By transitioning to the norm $||\cdot||_r$, we were able to obtain the following result:

$$\begin{aligned}&||B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)^{(u_q)} - f_q^{(u_q)}||_r \\ &\ge \frac{1}{[m+1]_q}\left|\left|\left[(\alpha_q + 1 - \beta_q e_1)f_q' + \frac{e_1(e_1+2)}{2}f_q''(z)\right]^{(u_q)}\right|\right|_r \\ &- \frac{1}{[m+1]_q}\left|\left|\frac{u_q!}{2\pi i}\int_\iota\frac{[m+1]_q^2}{(\iota-z)^{u_q+1}}(B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(\iota) - f(\iota) + \frac{\beta_q\iota - \alpha_q - 1}{[m+1]_q + \beta_q}f_q'(\iota) - \frac{\iota(\iota+2)}{2[m+1]_q}f_q''(\iota))\mathrm{d}\iota\right|\right|_r.\end{aligned} \tag{38}$$

We have, according to Theorem 2, established that for any natural numbers p and m, the statement holds true:

$$\begin{aligned}&\left|\left|\frac{u_q!}{2\pi i}\int_\iota\frac{[m+1]_q^2}{(\iota-z)^{u_q+1}}(B_{m,p,q}^{(\alpha_q,\beta_q)}(f_q)(\iota) - f(\iota) + \frac{\beta_q\iota - \alpha_q - 1}{[m+1]_q + \beta_q}f_q'(\iota) - \frac{\iota(\iota+2)}{2[m+1]_q}f_q''(\iota))\mathrm{d}\iota\right|\right|_r \\ &\le \frac{u_q!}{2\pi}\cdot\frac{2\pi r_1}{(r_1-r)^{u_q+1}}\sum_{j=1}^{5}M_{r_1,j}^{(\alpha_q,\beta_q)}(f_q).\end{aligned} \tag{39}$$

It is possible to express f in D_R by means other than a polynomial of degree ≤0, given that it does not belong to that category:

$$\left\| \left[(\alpha_q + 1 - \beta_q(e_1 - 1)) f_q' + \frac{(e_1 + 2)e_1}{2[m+1]_q} f_q'' \right]^{(u)} \right\|_r > 0. \tag{40}$$

One can obtain the remaining portion of the proof in a similar manner as demonstrated in the proof for Theorem 3 (see [12]). □

4 Conclusions

The utilization of q-calculus has enabled the creation of q-analogues of numerous traditional operators and has been explored in terms of their approximation properties. In the subsequent article, we introduced a Beta-type generalization that had been modified for complex Baskakov-Szasz-Stancu-type operators and applied it to their q-analogues on compact disks. We also confirmed the convergence of these operators and provided estimates of the errors in the modified Beta-type generalization of complex Baskakov-Szasz-Stancu-type operators and their corresponding q-analogues. The properties of our approximations have been a subject of interest. In response to the limited capabilities of q-Beta operators to only replicate constant functions, we propose a solution to enhance the convergence of approximating functions on compact disks. Despite this limitation, the operators remain versatile due to their ability to handle complex variables and conform to various weight functions. This flexibility makes them a useful tool for various applications.

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