

Research Article

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On Periodic solutions for implicit nonlinear Caputo tempered fractional differential problems

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Abstract: The main goal of this article is to study the existence and uniqueness of periodic solutions for the implicit problem with nonlinear fractional differential equation involving the Caputo tempered fractional derivative. The proofs are based upon the coincidence degree theory of Mawhin. To show the efficiency of the stated result, two illustrative examples will be demonstrated.

Keywords: coincidence degree theory, existence, uniqueness, tempered fractional operators, implicit problems

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1 Introduction

Fractional calculus extends beyond traditional differentiation and integration by incorporating noninteger orders, a notion that has captured both theoretical curiosity and practical significance across a wide array of research domains. Its versatility has propelled it to a central position in the field. Recent periods have observed a significant upsurge in dedicated research toward fractional calculus, exploring different outcomes across distinct scenarios and manifestations of fractional differential equations and inclusions. For a more comprehensive understanding of the real-world applications of fractional calculus, readers are directed to the works by Herrmann [1] and Samko et al. [2]. The contributions of Benchohra et al. [3,4] have brought attention to the existence, uniqueness, and stability of various problem classes, each subject to distinct conditions. They introduced an extension of the well-known Hilfer fractional derivative, seamlessly merging the Riemann-Liouville and Caputo fractional derivatives.

The utilization of the coincidence degree theory, introduced by Mawhin [5,6], has found widespread application in the examination of diverse categories of nonlinear differential equations. This method becomes notably advantageous, particularly when conventional techniques like the fixed point principle are inapplicable. In references [7–14], the employment of the coincidence degree theory has yielded results related to fractional-order nonlinear differential equations. These achievements would have remained unattainable through alternative approaches such as the fixed point principle.

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In recent times, the field of tempered fractional calculus has emerged as a significant category of fractional calculus operators. This classification possesses the ability to generalize diverse forms of fractional calculus while also featuring analytical kernels. This establishes it as an extension of fractional calculus capable of characterizing the transition between normal and anomalous diffusion. The foundational definitions of fractional integration involving weak singular and exponential kernels were initially formulated by Buschman in [15]. Further insights into this subject are available in [16–23]. Despite the limited exploration in existing literature, the Caputo tempered fractional derivative harbors substantial potential to make noteworthy contributions to this domain. Our objective in investigating this derivative is to enhance our comprehension of its attributes and potential applications within this distinct mathematical framework, thereby propelling the advancement of fractional calculus.

In the study by Benzenati *et al.* [7], by using the coincidence degree theory of Mawhin, the authors studied the nonlinear pantograph fractional equations with Ψ -Hilfer fractional derivative:

$$\begin{cases} {}^H\mathcal{D}_a^{0,\beta;\Psi} w(\delta) = \kappa(\delta, w(\delta), w(\varepsilon\delta)), & \delta \in (0, \kappa], \\ \mathcal{I}_0^{1-\nu,\Psi} w(0) = \mathcal{I}_0^{1-\nu,\Psi} w(\kappa), \end{cases}$$

where ${}^H\mathcal{D}_0^{0,\beta;\Psi}$ denotes the Ψ -Hilfer fractional derivative of order $0 < \varrho \leq 1$, $0 < \varepsilon < 1$, and type $\beta \in [0, 1]$. $\mathcal{I}_0^{1-\nu,\Psi}$ is the Ψ -Riemann-Liouville fractional integral of order $1 - \nu$, ($\nu = \varrho + \beta - \varrho\beta$). Moreover, $\kappa : (0, \kappa] \times \mathfrak{R}^2 \rightarrow \mathfrak{R}$ is a given continuous function.

Krim *et al.* [18] investigated the following class of Caputo tempered fractional differential equation with finite delay:

$$\begin{cases} \left({}^C\mathcal{D}_\delta^{\kappa,\varepsilon} w \right)(\delta) = \kappa(\delta, w_\delta, \mathcal{D}_0^\kappa w(\delta)); & \delta \in \Theta = [0, \varpi], \\ w(\delta) = \wp(\delta), & \delta \in [-\kappa, 0], \\ j_1 w(0) + j_2 w(\varpi) = j_3, \end{cases}$$

where $0 < \kappa < 1$; $\varepsilon \geq 0$; ${}^C\mathcal{D}_\delta^{\kappa,\varepsilon}$ is the Caputo tempered fractional derivative; $\kappa : \Theta \times C([-\kappa, 0], \mathbb{R}) \times \mathbb{R}$ is a continuous function; $\wp \in C([-\kappa, \varpi], \mathbb{R})$; $0 < \varpi < +\infty$; j_1, j_2 , and j_3 are real constants; and $\kappa > 0$ is the time delay. The results are based on the fixed point theorems of Banach, Schauder, and Schaefer. Observe that this problem encompasses initial, terminal, and anti-periodic problems; however, the employed approach does not yield solutions for the periodic problem.

In this article, we study the existence and uniqueness of periodic solutions for the implicit problem with nonlinear fractional differential equation involving the Caputo tempered fractional derivative:

$$({}^C\mathcal{D}_\delta^{\alpha,\lambda} w)(\delta) = f(\delta, w(\delta), ({}^C\mathcal{D}_\delta^{\alpha,\lambda} w)(\delta)); \quad \delta \in \mathfrak{J} = [0, \kappa], \quad (1)$$

$$w(0) = w(\kappa) = 0, \quad (2)$$

where $0 < \alpha < 1$, $\lambda \geq 0$, ${}^C\mathcal{D}_\delta^{\alpha,\lambda}$ is the Caputo tempered fractional derivative, and $f : \mathfrak{J} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous function.

It is crucial to emphasize that despite its relatively limited coverage in current literature, the Caputo tempered fractional derivative holds the promise of yielding noteworthy advancements within this specific domain. Our research aims to extensively explore the attributes and potential practical applications inherent to the Caputo tempered fractional derivative. This undertaking not only seeks to enhance our comprehension of this unique mathematical concept but also to drive the advancement of fractional calculus as a whole. Furthermore, our study is innovative in that it addresses a specific class of problems: coupled systems involving the Caputo tempered fractional derivative and periodic conditions. These particular problems have yet to be investigated in existing literature. As a result, our contribution naturally extends the progression of this dynamic field, introducing new dimensions and possibilities.

The structure of this article is as follows: Section 2 presents certain notations and preliminaries about the tempered fractional derivatives used throughout this manuscript; in Section 3, we present existence and uniqueness result for the problem (1)–(2) that are based upon the coincidence degree theory of Mawhin; and in the last section, illustrative examples are provided in support of the obtained results.

2 Preliminaries

First, we give the definitions and notations that we will use throughout this article. We denote by $C(\mathfrak{J}, \mathbb{R})$ the Banach space of all continuous functions from $\mathfrak{J}$ into $\mathbb{R}$ with the following norm:

$$\|f\|_{\infty} = \sup_{\delta \in \mathfrak{J}} \{|f(\delta)|\}.$$

As usual, $AC(\mathfrak{J})$ denotes the space of absolutely continuous functions from $\mathfrak{J}$ into $\mathbb{R}$. For any $n \in \mathbb{N}^*$, we denote by $AC^n(\mathfrak{J})$ the space defined as follows:

$$AC^n(\mathfrak{J}) = \left\{ \mathfrak{w} : \mathfrak{J} \rightarrow \mathbb{R} : \frac{d^n}{dt^n} \mathfrak{w}(\delta) \in AC(\mathfrak{J}) \right\}.$$

Consider the space $X_b^p(0, \mathfrak{u})$ ($b \in \mathbb{R}$, $1 \leq p \leq \infty$) of those real-valued Lebesgue measurable functions $\mathfrak{w}$ on $[0, \mathfrak{u}]$ for which $\|\mathfrak{w}\|_{X_b^p} < \infty$, where the norm is defined as follows:

$$\|\mathfrak{w}\|_{X_b^p} = \left(\int_0^{\mathfrak{u}} |\delta^b \mathfrak{w}(\delta)|^p \frac{d\delta}{\delta} \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty, b \in \mathbb{R}).$$

Definition 2.1. (The Riemann-Liouville-tempered fractional integral [19,23,24]) Suppose that the real function $\mathfrak{w}$ is piecewise continuous on $[0, \mathfrak{u}]$ and $\mathfrak{w} \in X_b^p(0, \mathfrak{u})$, $\lambda > 0$. Then, the Riemann-Liouville-tempered fractional integral of order α is defined as follows:

$${}_0\mathcal{I}_{\delta}^{\alpha, \lambda} \mathfrak{w}(\delta) = e^{-\lambda \delta} {}_0\mathcal{I}_{\delta}^{\alpha} (e^{\lambda \delta} \mathfrak{w}(\delta)) = \frac{1}{\Gamma(\alpha)} \int_0^{\delta} \frac{e^{-\lambda(\delta-s)} \mathfrak{w}(s)}{(\delta-s)^{1-\alpha}} ds, \quad (3)$$

where ${}_0\mathcal{I}_{\delta}^{\alpha}$ denotes the Riemann-Liouville fractional integral, defined as follows:

$${}_0\mathcal{I}_{\delta}^{\alpha} \mathfrak{w}(\delta) = \frac{1}{\Gamma(\alpha)} \int_0^{\delta} \frac{\mathfrak{w}(s)}{(\delta-s)^{1-\alpha}} ds. \quad (4)$$

Obviously, the tempered fractional integral (3) reduces to the Riemann-Liouville fractional integral (4) if $\lambda = 0$.

Definition 2.2. (The Riemann-Liouville-tempered fractional derivative [19,24]) For $n-1 < \alpha < n$; $n \in \mathbb{N}^+$, $\lambda \geq 0$. The Riemann-Liouville-tempered fractional derivative is defined as follows:

$${}_0\mathcal{D}_{\delta}^{\alpha, \lambda} \mathfrak{w}(\delta) = e^{-\lambda \delta} {}_0\mathcal{D}_{\delta}^{\alpha} (e^{\lambda \delta} \mathfrak{w}(\delta)) = \frac{e^{-\lambda \delta}}{\Gamma(n-\alpha)} \frac{d^n}{d\delta^n} \int_0^{\delta} \frac{e^{\lambda s} \mathfrak{w}(s)}{(\delta-s)^{\alpha-n+1}} ds,$$

where ${}_0\mathcal{D}_{\delta}^{\alpha} (e^{\lambda \delta} \mathfrak{w}(\delta))$ denotes the Riemann-Liouville fractional derivative, given by

$${}_0\mathcal{D}_{\delta}^{\alpha} (e^{\lambda \delta} \mathfrak{w}(\delta)) = \frac{d^n}{d\delta^n} ({}_0\mathcal{I}_{\delta}^{n-\alpha} (e^{\lambda \delta} \mathfrak{w}(\delta))) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{d\delta^n} \int_0^{\delta} \frac{(e^{\lambda s} \mathfrak{w}(s))}{(\delta-s)^{\alpha-n+1}} ds.$$

Definition 2.3. (The Caputo tempered fractional derivative [19,23]) For $n-1 < \alpha < n$; $n \in \mathbb{N}^+$, $\lambda \geq 0$. The Caputo tempered fractional derivative is defined as follows:

$${}_0^C\mathcal{D}_{\delta}^{\alpha, \lambda} \mathfrak{w}(\delta) = e^{-\lambda \delta} {}_0^C\mathcal{D}_{\delta}^{\alpha} (e^{\lambda \delta} \mathfrak{w}(\delta)) = \frac{e^{-\lambda \delta}}{\Gamma(n-\alpha)} \int_0^{\delta} \frac{1}{(\delta-s)^{\alpha-n+1}} \frac{d^n(e^{\lambda s} \mathfrak{w}(s))}{ds^n} ds,$$

where ${}_0^C\mathcal{D}_{\delta}^{\alpha, \lambda} (e^{\lambda \delta} \mathfrak{w}(\delta))$ denotes the Caputo fractional derivative, given by

$${}_0^C \mathcal{D}_\delta^\alpha (e^{\lambda \delta} \mathfrak{w}(\delta)) = \frac{1}{\Gamma(n - \alpha)} \int_0^\delta \frac{1}{(\delta - s)^{\alpha - n + 1}} \frac{d^n(e^{\lambda s} \mathfrak{w}(s))}{ds^n} ds.$$

Lemma 2.4. [19] For a constant C ,

$${}_0 \mathcal{D}_\delta^{\alpha, \lambda} C = C e^{-\lambda \delta} {}_0 \mathcal{D}_\delta^\alpha e^{\lambda \delta}, \quad {}_0^C \mathcal{D}_\delta^{\alpha, \lambda} C = C e^{-\lambda \delta} {}_0^C \mathcal{D}_\delta^\alpha e^{\lambda \delta}.$$

Obviously, ${}_0 \mathcal{D}_\delta^{\alpha, \lambda}(C) \neq {}_0^C \mathcal{D}_\delta^{\alpha, \lambda}(C)$ and ${}_0^C \mathcal{D}_\delta^{\alpha, \lambda}(C)$ is no longer equal to zero, being different from ${}_0^C \mathcal{D}_\delta^\alpha(C) \equiv 0$.

Lemma 2.5. [19,23] Let $\mathfrak{w}(\delta) \in AC^n[0, \mathfrak{N}]$ and $n - 1 < \alpha < n$. Then, the Caputo tempered fractional derivative and the Riemann-Liouville-tempered fractional integral have the composite properties

$${}_0 \mathcal{I}_\delta^{\alpha, \lambda} [{}_0^C \mathcal{D}_\delta^{\alpha, \lambda} \mathfrak{w}(\delta)] = \mathfrak{w}(\delta) - \sum_{k=0}^{n-1} e^{-\lambda \delta} \frac{(\delta - 0)^k}{k!} \left[\frac{d^k(e^{\lambda \delta} \mathfrak{w}(\delta))}{d\delta^k} \right]_{\delta=0},$$

and

$${}_0^C \mathcal{D}_\delta^{\alpha, \lambda} [{}_0 \mathcal{I}_\delta^{\alpha, \lambda} \mathfrak{w}(\delta)] = \mathfrak{w}(\delta) \quad \text{for } \alpha \in (0, 1).$$

Theorem 2.6. [25] Let $\mathfrak{w}, \mathfrak{v} \in AC^n(\mathfrak{J}, \mathfrak{R})$, $n - 1 < \alpha \leq n$, ($n \in \mathbb{N}$), $\lambda \in [0, +\infty)$, and $\Psi \in C^n(\mathfrak{J}, \mathfrak{R})$, be a nondecreasing function such that $\Psi' \neq 0$ on $\mathfrak{J}$. Then, we have

$${}_0^C \mathcal{D}_{\Psi(\delta)}^{\alpha, \lambda} \mathfrak{w}(\delta) = {}_0^C \mathcal{D}_{\Psi(\delta)}^{\alpha, \lambda} \mathfrak{v}(\delta) \Leftrightarrow \mathfrak{w}(\delta) = \mathfrak{v}(\delta) + e^{-\lambda \Psi(\delta)} \sum_{k=0}^{n-1} c_k (\psi(\delta) - \psi(0))^k, \quad \delta \in \mathfrak{J},$$

where

$$c_k = \frac{1}{k!} \left[\left(\frac{1}{\psi'(\delta)} \frac{d}{d\delta} \right)^k (e^{\lambda \Psi(\delta)} [\mathfrak{w}(\delta) - \mathfrak{v}(\delta)]) \right]_{\delta=0}.$$

Remark 2.7. If we pose $\omega = \mathfrak{w} - \mathfrak{v} \in C^1(\mathfrak{J}, \mathfrak{R})$, $\Psi(\delta) = \delta$, and $0 < \alpha \leq 1$, then, we have

$${}_0^C \mathcal{D}_\delta^{\alpha, \lambda} \omega(\delta) = 0 \Leftrightarrow \omega(\delta) = e^{-\lambda \delta} \omega(0), \quad \delta \in \mathfrak{J}.$$

We will present definitions and the coincidence degree theory that are essential in proofs of our results, see [5,6].

Definition 2.8. We consider the normed spaces $\mathcal{X}$ and $\mathcal{Y}$. A Fredholm operator of index zero is a linear operator $\mathcal{L} : \text{Dom}(\mathcal{L}) \subset \mathcal{X} \rightarrow \mathcal{Y}$ such that

- (a) $\dim \ker \mathcal{L} = \text{codim } \text{Im} \mathcal{L} < +\infty$.
- (b) $\text{Im} \mathcal{L}$ is a closed subset of $\mathcal{Y}$.

By Definition 2.8, there exist continuous projectors $Q : \mathcal{Y} \rightarrow \mathcal{Y}$ and $\mathcal{P} : \mathcal{X} \rightarrow \mathcal{X}$ satisfying

$$\text{Im} \mathcal{L} = \ker Q, \quad \ker \mathcal{L} = \text{Im} \mathcal{P}, \quad \mathcal{Y} = \text{Im} Q \oplus \text{Im} \mathcal{L}, \quad \mathcal{X} = \ker \mathcal{P} \oplus \ker \mathcal{L}.$$

Thus, the restriction of $\mathcal{L}$ to $\text{Dom} \mathcal{L} \cap \ker \mathcal{P}$, denoted by $\mathcal{L}_{\mathcal{P}}$, is an isomorphism onto its image.

Definition 2.9. Let $\Omega \subseteq \mathcal{X}$ be a bounded subset and $\mathcal{L}$ be a Fredholm operator of index zero with $\text{Dom} \mathcal{L} \cap \Omega \neq \emptyset$. Then, the operator $\mathcal{N} : \overline{\Omega} \rightarrow \mathcal{Y}$ is called to be $\mathcal{L}$ -compact in $\overline{\Omega}$ if

- (a) the mapping $Q\mathcal{N} : \overline{\Omega} \rightarrow \mathcal{Y}$ is continuous and $Q\mathcal{N}(\overline{\Omega}) \subseteq \mathcal{Y}$ is bounded.
- (b) the mapping $(\mathcal{L}_{\mathcal{P}})^{-1}(id - Q)\mathcal{N} : \overline{\Omega} \rightarrow \mathcal{X}$ is completely continuous.

Lemma 2.10. [26] Let X and $\mathcal{Y}$ be Banach spaces, and, $\Omega \subset X$ a bounded open set and symmetric with $0 \in \Omega$. Suppose that $\mathcal{L} : \text{Dom} \mathcal{L} \subset X \rightarrow \mathcal{Y}$ is a Fredholm operator of index zero with $\text{Dom} \mathcal{L} \cap \overline{\Omega} \neq \emptyset$, and $N : X \rightarrow \mathcal{Y}$ is a $\mathcal{L}$ -compact operator on $\overline{\Omega}$. Assume, moreover, that

$$\mathcal{L}x - Nx \neq -\zeta(\mathcal{L}x + N(-x))$$

for any $x \in \text{Dom} \mathcal{L} \cap \partial\Omega$ and $\zeta \in (0, 1]$, where $\partial\Omega$ is the boundary of Ω with respect to X . If these conditions are verified, then there exists at least one solution to the equation $\mathcal{L}x = Nx$ on $\text{Dom} \mathcal{L} \cap \overline{\Omega}$.

3 Main results

Let the spaces

$$\mathcal{X} = \{\mathfrak{w} \in C(\mathfrak{J}, \mathfrak{R}) : \mathfrak{w}(\delta) = {}_0\mathcal{I}_\delta^{a,\lambda} \mathfrak{v}(\delta) : \mathfrak{v} \in C(\mathfrak{J}, \mathfrak{R})\},$$

and

$$\mathcal{Y} = C(\mathfrak{J}, \mathfrak{R}),$$

be endowed with the norms

$$\|\mathfrak{w}\|_{\mathcal{X}} = \|\mathfrak{w}\|_{\mathcal{Y}} = \|\mathfrak{w}\|_{\infty} = \sup_{\delta \in \mathfrak{J}} |\mathfrak{w}(\delta)|.$$

We give now the definition of the operator $\mathcal{L} : \text{Dom} \mathcal{L} \subseteq \mathcal{X} \rightarrow \mathcal{Y}$

$$\mathcal{L}\mathfrak{w} = {}_0^C\mathcal{D}_\delta^{a,\lambda} \mathfrak{w}, \quad (5)$$

where

$$\text{Dom} \mathcal{L} = \{\mathfrak{w} \in \mathcal{X} : {}_0^C\mathcal{D}_\delta^{a,\lambda} \mathfrak{w} \in \mathcal{Y} : \mathfrak{w}(0) = \mathfrak{w}(\mathfrak{u}) = 0\}.$$

Lemma 3.1. Using the definition of $\mathcal{L}$ given in equation (5), then

$$\ker \mathcal{L} = \{\mathfrak{w} \in \mathcal{X} : \mathfrak{w}(\delta) = 0, \delta \in \mathfrak{J}\},$$

and

$$\text{Im} \mathcal{L} = \left\{ \mathfrak{v} \in \mathcal{Y} : \int_0^{\mathfrak{u}} (\mathfrak{u} - s)^{a-1} e^{-\lambda(\mathfrak{u}-s)} \mathfrak{v}(s) ds = 0 \right\}.$$

Proof. By Remark 2.7, we have for all $\mathfrak{w} \in \text{Dom} \mathcal{L} \subset \mathcal{X}$ the equation $\mathcal{L}\mathfrak{w} = {}_0^C\mathcal{D}_\delta^{a,\lambda} \mathfrak{w} = 0$ in $\mathfrak{J}$ has a solution of the form

$$\mathfrak{w}(\delta) = e^{-\lambda\delta} \mathfrak{w}(0) = 0, \delta \in \mathfrak{J},$$

then

$$\ker \mathcal{L} = \{\mathfrak{w} \in \mathcal{X} : \mathfrak{w}(\delta) = 0, \delta \in \mathfrak{J}\}.$$

For $\mathfrak{v} \in \text{Im} \mathcal{L}$, there exists $\mathfrak{w} \in \text{Dom} \mathcal{L}$ such that $\mathfrak{v} = \mathcal{L}\mathfrak{w} \in \mathcal{Y}$. Using Lemma 2.5, we obtain for every $\delta \in \mathfrak{J}$:

$$\begin{aligned} \mathfrak{w}(\delta) &= e^{-\lambda\delta} \mathfrak{w}(0) + {}_0\mathcal{I}_\delta^{a,\lambda} \mathfrak{v}(\delta) \\ &= e^{-\lambda\delta} \mathfrak{w}(0) + \frac{1}{\Gamma(a)} \int_0^\delta (\delta - s)^{a-1} e^{-\lambda(\delta-s)} \mathfrak{v}(s) ds. \end{aligned}$$

Since $\mathfrak{w} \in \text{Dom} \mathcal{L}$, we have $\mathfrak{w}(0) = \mathfrak{w}(\mathfrak{u}) = 0$. Thus,

$$\int_0^{\kappa} (\kappa - s)^{\alpha-1} e^{-\lambda(\kappa-s)} v(s) ds = 0.$$

Furthermore, if $v \in \mathcal{Y}$ and satisfies

$$\int_0^{\kappa} (\kappa - s)^{\alpha-1} e^{-\lambda(\kappa-s)} v(s) ds = 0,$$

then for any $w(\delta) = {}_0\mathcal{I}_{\delta}^{\alpha,\lambda} v(\delta)$, using Lemma 2.5, we obtain $v(\delta) = {}_0\mathcal{D}_{\delta}^{\alpha,\lambda} w(\delta)$. Therefore,

$$w(\kappa) = w(0) = 0,$$

which implies that $w \in \text{Dom } \mathcal{L}$. So $v \in \mathcal{I} \text{mg } \mathcal{L}$ and

$$\mathcal{I} \text{mg } \mathcal{L} = \left\{ v \in \mathcal{Y} : \int_0^{\kappa} (\kappa - s)^{\alpha-1} e^{-\lambda(\kappa-s)} v(s) ds = 0 \right\},$$

which completes the proof. $\square$

Lemma 3.2. Let $\mathcal{L}$ be defined by equation (5). Then, $\mathcal{L}$ is a Fredholm operator of index zero, and the linear continuous projector operators $Q : \mathcal{Y} \rightarrow \mathcal{Y}$ and $\mathcal{P} : \mathcal{X} \rightarrow \mathcal{X}$ can be written as follows:

$$Q(v) = \frac{1}{\varpi(\kappa)} \int_0^{\kappa} (\kappa - s)^{\alpha-1} e^{-\lambda(\kappa-s)} v(s) ds,$$

and

$$\mathcal{P}(w) = 0,$$

where

$$\varpi(\kappa) = \int_0^{\kappa} (\kappa - s)^{\alpha-1} e^{-\lambda(\kappa-s)} ds.$$

Furthermore, the operator $\mathcal{L}_{\mathcal{P}}^{-1} : \mathcal{I} \text{mg } \mathcal{L} \rightarrow \mathcal{X} \cap \ker \mathcal{P}$ can be written as follows:

$$\mathcal{L}_{\mathcal{P}}^{-1}(v)(\delta) = {}_0\mathcal{I}_{\delta}^{\alpha,\lambda} v(\delta), \quad \delta \in \mathcal{J}.$$

Proof. Obviously, for each $v \in \mathcal{Y}$, $Q^2 v = Qv$ and $v = Q(v) + (v - Q(v))$, where $(v - Q(v)) \in \ker Q = \mathcal{I} \text{mg } \mathcal{L}$.

Using the fact that $\mathcal{I} \text{mg } \mathcal{L} = \ker Q$ and $Q^2 = Q$, then $\mathcal{I} \text{mg } \mathcal{L} \cap \mathcal{I} \text{mg } Q = 0$. So,

$$\mathcal{Y} = \mathcal{I} \text{mg } \mathcal{L} \oplus \mathcal{I} \text{mg } Q.$$

Similarly, we obtain that $\mathcal{I} \text{mg } \mathcal{P} = \ker \mathcal{L}$ and $\mathcal{P}^2 = \mathcal{P}$. It follows that for each $w \in \mathcal{X}$, if $w = (w - \mathcal{P}(w)) + \mathcal{P}(w)$, then $\mathcal{X} = \ker \mathcal{P} + \ker \mathcal{L}$. Clearly, we have $\ker \mathcal{P} \cap \ker \mathcal{L} = 0$. So

$$\mathcal{X} = \ker \mathcal{P} \oplus \ker \mathcal{L}.$$

Using Rank-nullity theorem, we obtain

$$\text{codim } \mathcal{I} \text{mg } \mathcal{L} = \dim \mathcal{Y} - \dim \mathcal{I} \text{mg } \mathcal{L} = [\dim \ker Q + \dim \mathcal{I} \text{mg } Q] - \dim \mathcal{I} \text{mg } \mathcal{L},$$

and since $\mathcal{I} \text{mg } \mathcal{L} = \ker Q$, then

$$\text{codim } \mathcal{I} \text{mg } \mathcal{L} = \dim \mathcal{I} \text{mg } Q. \quad (6)$$

Using also Rank-nullity theorem, we obtain

$$\dim \ker \mathcal{L} = \dim \mathcal{X} - \dim \mathcal{I} \text{mg } \mathcal{L} = \text{codim } \mathcal{I} \text{mg } \mathcal{L},$$

which implies that

$$\dim \ker \mathcal{L} = \operatorname{codim} \operatorname{Im} \mathcal{L}. \quad (7)$$

By equations (6) and (7), we have

$$\dim \ker \mathcal{L} = \operatorname{codim} \operatorname{Im} \mathcal{L} = \dim \operatorname{Im} \mathcal{Q},$$

and since $\dim \operatorname{Im} \mathcal{Q} < \infty$,

$$\dim \ker \mathcal{L} = \operatorname{codim} \operatorname{Im} \mathcal{L} < \infty.$$

Since $\operatorname{Im} \mathcal{L}$ is a closed subset of $\mathcal{Y}$, then $\mathcal{L}$ is a Fredholm operator of index zero.

Now, we will show that the inverse of $\mathcal{L}|_{\operatorname{Dom} \mathcal{L} \cap \ker \mathcal{P}}$ is $\mathcal{L}_{\mathcal{P}}^{-1}$. Effectively, for $v \in \operatorname{Im} \mathcal{L}$, by Lemma 2.5, we have

$$\mathcal{L} \mathcal{L}_{\mathcal{P}}^{-1}(v) = {}^C_0 \mathcal{D}_{\delta}^{a, \lambda} ({}_0 I_{\delta}^{a, \lambda} v) = v. \quad (8)$$

Furthermore, for $w \in \operatorname{Dom} \mathcal{L} \cap \ker \mathcal{P}$, we obtain

$$\mathcal{L}_{\mathcal{P}}^{-1}(\mathcal{L}(w(\delta))) = {}_0 I_{\delta}^{a, \lambda} ({}_0^C \mathcal{D}_{\delta}^{a, \lambda} w(\delta)) = w(\delta) - e^{-\lambda \delta} w(0), \quad \delta \in \mathfrak{J}.$$

Using the fact that $w \in \operatorname{Dom} \mathcal{L} \cap \ker \mathcal{P}$, then

$$w(0) = 0.$$

Thus,

$$\mathcal{L}_{\mathcal{P}}^{-1} \mathcal{L}(w) = w. \quad (9)$$

Using equations (8) and (9) together, we obtain $\mathcal{L}_{\mathcal{P}}^{-1} = (\mathcal{L}|_{\operatorname{Dom} \mathcal{L} \cap \ker \mathcal{P}})^{-1}$, which completes the demonstration.

Let us consider the following hypothesis:

(H₁) There exist positive constants γ and η with

$$|f(\delta, w, v) - f(\delta, \bar{w}, \bar{v})| \leq \gamma |w - \bar{w}| + \eta |v - \bar{v}|,$$

for every $\delta \in \mathfrak{J}$ and $w, \bar{w}, v, \bar{v} \in \mathfrak{X}$.

Define $\mathcal{N} : \mathcal{X} \rightarrow \mathcal{Y}$ by

$$\mathcal{N}w(\delta) = f(\delta, w(\delta), ({}_0^C \mathcal{D}_{\delta}^{a, \lambda} w)(\delta)), \quad \delta \in \mathfrak{J}.$$

Then, problems (1) and (2) are equivalent to the problem $\mathcal{L}w = \mathcal{N}w$. □

Lemma 3.3. Suppose that (H₁) is satisfied, then for any bounded open set $\Omega \subset \mathcal{X}$, the operator $\mathcal{N}$ is $\mathcal{L}$ -compact.

Proof. We consider for $M > 0$ the bounded open set $\Omega = \{w \in \mathcal{X} : \|w\|_{\mathcal{X}} < M\}$. We split the proof into three steps:

Step 1: $\mathcal{Q}\mathcal{N}$ is continuous.

Let $(v_n)_{n \in \mathbb{N}}$ be a sequence such that $v_n \rightarrow v$ in $\mathcal{Y}$, then for each $\delta \in \mathfrak{J}$, we have

$$|\mathcal{Q}\mathcal{N}(v_n)(\delta) - \mathcal{Q}\mathcal{N}(v)(\delta)| \leq \frac{1}{\varpi(\kappa)} \int_0^{\kappa} (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} |\mathcal{N}(v_n)(s) - \mathcal{N}(v)(s)| ds.$$

By (H₁), we have

$$\begin{aligned} & |\mathcal{Q}\mathcal{N}(v_n)(\delta) - \mathcal{Q}\mathcal{N}(v)(\delta)| \\ & \leq \frac{\gamma}{\varpi(\kappa)} \int_0^{\kappa} (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} |v_n(s) - v(s)| ds + \frac{\eta}{\varpi(\kappa)} \int_0^{\kappa} (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} |{}_0^C \mathcal{D}_s^{a, \lambda} v_n(s) - {}_0^C \mathcal{D}_s^{a, \lambda} v(s)| ds \\ & \leq \frac{\gamma}{\varpi(\kappa)} \int_0^{\kappa} (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} |v_n(s) - v(s)| ds + \frac{\eta}{\varpi(\kappa)} \int_0^{\kappa} (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} |{}_0^C \mathcal{D}_s^{a, \lambda} v_n(s) - {}_0^C \mathcal{D}_s^{a, \lambda} v(s)| ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\gamma}{\varpi(\kappa)} \int_0^\kappa (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} |v_n(s) - v(s)| ds + \frac{\eta\Gamma(a)}{\varpi(\kappa)} {}_0I_{\delta}^{a,\lambda C} \mathfrak{D}_{\delta}^{a,\lambda} |(v_n - v)(s)|(\kappa) \\
&\leq \frac{\gamma}{\varpi(\kappa)} \int_0^\kappa (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} |v_n(s) - v(s)| ds + \frac{\eta\Gamma(a)}{\varpi(\kappa)} [|v_n(\kappa) - v(\kappa)| - e^{-\lambda\kappa} |v_n(0) - v(0)|] \\
&\leq \left[\frac{\eta\Gamma(a)}{\varpi(\kappa)} + \gamma \right] \|v_n - v\|_{\mathcal{Y}}.
\end{aligned}$$

Thus, for each $\delta \in \mathfrak{J}$, we obtain

$$|QN(v_n)(\delta) - QN(v)(\delta)| \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

and hence

$$\|QN(v_n) - QN(v)\|_{\mathcal{Y}} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

We deduce that QN is continuous.

Step 2: $QN(\overline{\mathcal{Q}})$ is bounded

For $\delta \in \mathfrak{J}$ and $v \in \overline{\mathcal{Q}}$, we have

$$\begin{aligned}
&|QN(v)(\delta)| \\
&\leq \frac{1}{\varpi(\kappa)} \int_0^\kappa (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} |N(v)(s)| ds \\
&\leq \frac{1}{\varpi(\kappa)} \int_0^\kappa (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} |f(s, v(s), {}^C_0\mathfrak{D}_s^{a,\lambda} v(s)) - f(s, 0, 0)| ds + \frac{1}{\varpi(\kappa)} \int_0^\kappa (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} |f(s, 0, 0)| ds \\
&\leq f^* + \frac{\gamma}{\varpi(\kappa)} \int_0^\kappa (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} |v(s)| ds + \frac{\eta}{\varpi(\kappa)} \int_0^\kappa (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} {}^C_0\mathfrak{D}_s^{a,\lambda} |v(s)| ds \\
&\leq f^* + \frac{\gamma}{\varpi(\kappa)} \int_0^\kappa (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} |v(s)| ds + \frac{\eta\Gamma(a)}{\varpi(\kappa)} {}_0I_{\delta}^{a,\lambda C} \mathfrak{D}_{\delta}^{a,\lambda} |v(s)|(b) \\
&\leq f^* + \frac{\gamma}{\varpi(\kappa)} \int_0^\kappa (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} |v(s)| ds + \frac{\eta\Gamma(a)}{\varpi(\kappa)} [|v(\kappa)| - e^{-\lambda\kappa} |v(0)|] \\
&\leq f^* + \left[\frac{\eta\Gamma(a)}{\varpi(\kappa)} + \gamma \right] \mathcal{M},
\end{aligned}$$

where $f^* = \|f(\cdot, 0, 0)\|_{\infty}$.

Thus,

$$\|QN(v)\|_{\mathcal{Y}} \leq f^* + \left[\frac{\eta\Gamma(a)}{\varpi(\kappa)} + \gamma \right] \mathcal{M}.$$

So, $QN(\overline{\mathcal{Q}})$ is a bounded set in $\mathcal{Y}$.

Step 3: $\mathfrak{L}_{\mathcal{P}}^{-1}(id - Q)N : \overline{\mathcal{Q}} \rightarrow X$ is completely continuous.

We will use the Arzelà-Ascoli theorem, so we have to show that $\mathfrak{L}_{\mathcal{P}}^{-1}(id - Q)N(\overline{\mathcal{Q}}) \subset X$ is equicontinuous and bounded. First, for any $\mathfrak{w} \in \overline{\mathcal{Q}}$ and $\delta \in \mathfrak{J}$, we obtain

$$\begin{aligned}
&\mathfrak{L}_{\mathcal{P}}^{-1}(N\mathfrak{w}(\delta) - QN\mathfrak{w}(\delta)) \\
&= {}_0I_{\delta}^{a,\lambda} \left[f(\delta, \mathfrak{w}(\delta), {}^C_0\mathfrak{D}_{\delta}^{a,\lambda} \mathfrak{w}(\delta)) - \frac{1}{\varpi(\kappa)} \int_0^\kappa (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{a,\lambda} \mathfrak{w}(s)) ds \right] \\
&= \frac{1}{\Gamma(a)} \int_0^\delta (\delta - s)^{a-1} e^{-\lambda(\delta-s)} f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{a,\lambda} \mathfrak{w}(s)) ds - \frac{\varpi(\delta)}{\Gamma(a)\varpi(\kappa)} \int_0^\kappa (\kappa - s)^{a-1} e^{-\lambda(\kappa-s)} f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{a,\lambda} \mathfrak{w}(s)) ds.
\end{aligned}$$

For all $\mathfrak{w} \in \overline{\mathcal{Q}}$ and $\delta \in \mathfrak{J}$, we obtain

$$\begin{aligned}
& |\mathfrak{L}_{\mathcal{P}}^{-1}(id - Q)\mathcal{N}\mathfrak{w}(\delta)| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^{\delta} (\delta - s)^{\alpha-1} e^{-\lambda(\delta-s)} |f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s)) - f(s, 0, 0)| ds + \frac{1}{\Gamma(\alpha)} \int_0^{\delta} (\delta - s)^{\alpha-1} e^{-\lambda(\delta-s)} |f(s, 0, 0)| ds \\
& \quad + \frac{\varpi(\delta)}{\Gamma(\alpha)\varpi(\mathfrak{u})} \int_0^{\mathfrak{u}} (\mathfrak{u} - s)^{\alpha-1} e^{-\lambda(\mathfrak{u}-s)} |f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s)) - f(s, 0, 0)| ds \\
& \quad + \frac{\varpi(\delta)}{\Gamma(\alpha)\varpi(\mathfrak{u})} \int_0^{\mathfrak{u}} (\mathfrak{u} - s)^{\alpha-1} e^{-\lambda(\mathfrak{u}-s)} |f(s, 0, 0)| ds, \\
& \leq \frac{2f^*\varpi(\delta)}{\Gamma(\alpha)} + \frac{\gamma}{\Gamma(\alpha)} \int_0^{\delta} (\delta - s)^{\alpha-1} e^{-\lambda(\delta-s)} |\mathfrak{w}(s)| ds + \frac{\eta}{\Gamma(\alpha)} \int_0^{\delta} (\delta - s)^{\alpha-1} e^{-\lambda(\delta-s)} |{}_0^C\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s)| ds \\
& \quad + \frac{\gamma\varpi(\delta)}{\Gamma(\alpha)\varpi(\mathfrak{u})} \int_0^{\mathfrak{u}} (\mathfrak{u} - s)^{\alpha-1} e^{-\lambda(\mathfrak{u}-s)} |\mathfrak{w}(s)| ds + \frac{\eta\varpi(\delta)}{\Gamma(\alpha)\varpi(\mathfrak{u})} \int_0^{\mathfrak{u}} (\mathfrak{u} - s)^{\alpha-1} e^{-\lambda(\mathfrak{u}-s)} |{}_0^C\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s)| ds \\
& \leq \frac{2f^*\mathfrak{u}^{\alpha}}{\Gamma(\alpha+1)} + \left[\frac{2\gamma\mathfrak{u}^{\alpha}}{\Gamma(\alpha+1)} + \eta + \frac{\eta\mathfrak{u}^{\alpha}}{\alpha\varpi(\mathfrak{u})} \right] \mathcal{M}.
\end{aligned}$$

Therefore,

$$\|\mathfrak{L}_{\mathcal{P}}^{-1}(id - Q)\mathcal{N}\mathfrak{w}\|_{\mathcal{X}} \leq \frac{2f^*\mathfrak{u}^{\alpha}}{\Gamma(\alpha+1)} + \left[\frac{2\gamma\mathfrak{u}^{\alpha}}{\Gamma(\alpha+1)} + \eta + \frac{\eta\mathfrak{u}^{\alpha}}{\alpha\varpi(\mathfrak{u})} \right] \mathcal{M}.$$

This means that $\mathfrak{L}_{\mathcal{P}}^{-1}(id - Q)\mathcal{N}(\overline{\mathcal{Q}})$ is uniformly bounded in $\mathcal{X}$.

It remains to show that $\mathfrak{L}_{\mathcal{P}}^{-1}(id - Q)\mathcal{N}(\overline{\mathcal{Q}})$ is equicontinuous.

For $0 < \delta_1 < \delta_2 \leq \mathfrak{u}$, $\mathfrak{w} \in \overline{\mathcal{Q}}$, we have

$$\begin{aligned}
& |\mathfrak{L}_{\mathcal{P}}^{-1}(id - Q)\mathcal{N}\mathfrak{w}(\delta_2) - \mathfrak{L}_{\mathcal{P}}^{-1}(id - Q)\mathcal{N}\mathfrak{w}(\delta_1)| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^{\delta_1} |(\delta_2 - s)^{\alpha-1} e^{-\lambda(\delta_2-s)} - (\delta_1 - s)^{\alpha-1} e^{-\lambda(\delta_1-s)}| |f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s))| ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_{\delta_1}^{\delta_2} (\delta_2 - s)^{\alpha-1} e^{-\lambda(\delta_2-s)} |f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s))| ds \\
& \quad + \frac{|\varpi(\delta_2) - \varpi(\delta_1)|}{\Gamma(\alpha)\varpi(\mathfrak{u})} \int_0^{\mathfrak{u}} (\mathfrak{u} - s)^{\alpha-1} e^{-\lambda(\mathfrak{u}-s)} |f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s))| ds \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^{\delta_1} |(\delta_2 - s)^{\alpha-1} e^{-\lambda(\delta_2-s)} - (\delta_1 - s)^{\alpha-1} e^{-\lambda(\delta_1-s)}| \times |f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s)) - f(s, 0, 0)| ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_0^{\delta_1} |(\delta_2 - s)^{\alpha-1} e^{-\lambda(\delta_2-s)} - (\delta_1 - s)^{\alpha-1} e^{-\lambda(\delta_1-s)}| |f(s, 0, 0)| ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_{\delta_1}^{\delta_2} (\delta_2 - s)^{\alpha-1} e^{-\lambda(\delta_2-s)} |f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s)) - f(s, 0, 0)| ds + \frac{1}{\Gamma(\alpha)} \int_{\delta_1}^{\delta_2} (\delta_2 - s)^{\alpha-1} e^{-\lambda(\delta_2-s)} |f(s, 0, 0)| ds \\
& \quad + \frac{|\varpi(\delta_2) - \varpi(\delta_1)|}{\Gamma(\alpha)\varpi(\mathfrak{u})} \int_0^{\mathfrak{u}} (\mathfrak{u} - s)^{\alpha-1} e^{-\lambda(\mathfrak{u}-s)} |f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s)) - f(s, 0, 0)| ds \\
& \quad + \frac{|\varpi(\delta_2) - \varpi(\delta_1)|}{\Gamma(\alpha)\varpi(\mathfrak{u})} \int_0^{\mathfrak{u}} (\mathfrak{u} - s)^{\alpha-1} e^{-\lambda(\mathfrak{u}-s)} |f(s, 0, 0)| ds \\
& \leq \frac{\gamma\mathcal{M} + f^*}{\Gamma(\alpha)} \int_0^{\delta_1} |(\delta_2 - s)^{\alpha-1} e^{-\lambda(\delta_2-s)} - (\delta_1 - s)^{\alpha-1} e^{-\lambda(\delta_1-s)}| ds + \eta\mathcal{M}[(e^{-\lambda\delta_2} - e^{-\lambda\delta_1}) + 2(1 - e^{-\lambda(\delta_2-\delta_1)})] \\
& \quad + \frac{2\gamma\mathcal{M} + f^*}{\Gamma(\alpha)} \int_{\delta_1}^{\delta_2} (\delta_2 - s)^{\alpha-1} e^{-\lambda(\delta_2-s)} ds + \left[\frac{\gamma\mathcal{M} + f^*}{\Gamma(\alpha)} + \frac{\eta\mathcal{M}}{\varpi(\mathfrak{u})} \right] |\varpi(\delta_2) - \varpi(\delta_1)|.
\end{aligned}$$

The operator $\mathfrak{L}_{\mathcal{P}}^{-1}(id - Q)\mathcal{N}(\overline{\mathcal{Q}})$ is equicontinuous in $\mathcal{X}$ because the right-hand side of the above inequality tends to zero as $\delta_1 \rightarrow \delta_2$, and the limit is independent of $\mathfrak{w}$. The Arzelà-Ascoli theorem implies that $\mathfrak{L}_{\mathcal{P}}^{-1}(id - Q)\mathcal{N}(\overline{\mathcal{Q}})$ is relatively compact in $\mathcal{X}$. As a consequence of steps 1–3, we obtain that $\mathcal{N}$ is $\mathfrak{L}$ -compact in $\overline{\mathcal{Q}}$, which completes the demonstration. $\square$

Lemma 3.4. Assume (H1). If the condition

$$\frac{\gamma \mathfrak{N}^a}{\Gamma(\alpha + 1)} + \eta < \frac{1}{2} \quad (10)$$

is satisfied, then there exists $\mathcal{A} > 0$, which is independent of ζ such that

$$\mathfrak{L}(\mathfrak{w}) - \mathcal{N}(\mathfrak{w}) = -\zeta[\mathfrak{L}(\mathfrak{w}) + \mathcal{N}(-\mathfrak{w})] \Rightarrow \|\mathfrak{w}\|_{\mathcal{X}} \leq \mathcal{A}, \quad \zeta \in (0, 1].$$

Proof. Let $\mathfrak{w} \in \mathcal{X}$ satisfies

$$\mathfrak{L}(\mathfrak{w}) - \mathcal{N}(\mathfrak{w}) = -\zeta \mathfrak{L}(\mathfrak{w}) - \zeta \mathcal{N}(-\mathfrak{w}),$$

then

$$\mathfrak{L}(\mathfrak{w}) = \frac{1}{1 + \zeta} \mathcal{N}(\mathfrak{w}) - \frac{\zeta}{1 + \zeta} \mathcal{N}(-\mathfrak{w}).$$

So, from the expression of $\mathfrak{L}$ and $\mathcal{N}$, we obtain for any $\delta \in \mathfrak{J}$,

$$\mathfrak{L}\mathfrak{w}(\delta) = {}^C_0\mathfrak{D}_{\delta}^{\alpha, \lambda} \mathfrak{w}(\delta) = \frac{1}{1 + \zeta} f(\delta, \mathfrak{w}(\delta), {}^C_0\mathfrak{D}_{\delta}^{\alpha, \lambda} \mathfrak{w}(\delta)) - \frac{\zeta}{1 + \zeta} f(\delta, -\mathfrak{w}(\delta), -{}^C_0\mathfrak{D}_{\delta}^{\alpha, \lambda} \mathfrak{w}(\delta)).$$

By Lemma 2.5, we obtain

$$\mathfrak{w}(\delta) = e^{-\lambda \delta} \mathfrak{w}(0) + \frac{1}{\zeta + 1} [{}_0\mathcal{I}_{\delta}^{\alpha, \lambda} (f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s)))(\delta) - \zeta {}_0\mathcal{I}_{\delta}^{\alpha, \lambda} (f(s, -\mathfrak{w}(s), -{}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s)))(\delta)].$$

Thus, for every $\delta \in \mathfrak{J}$, we obtain

$$\begin{aligned} |\mathfrak{w}(\delta)| &\leq |\mathfrak{w}(0)| + \frac{1}{(\zeta + 1)\Gamma(\alpha)} \int_0^{\delta} (\delta - s)^{\alpha-1} e^{-\lambda(\delta-s)} |f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s))| ds \\ &\quad + \frac{\zeta}{(\zeta + 1)\Gamma(\alpha)} \int_0^{\delta} (\delta - s)^{\alpha-1} e^{-\lambda(\delta-s)} |f(s, -\mathfrak{w}(s), -{}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s))| ds \\ &\leq |\mathfrak{w}(0)| + \frac{1}{(\zeta + 1)\Gamma(\alpha)} \int_0^{\delta} (\delta - s)^{\alpha-1} e^{-\lambda(\delta-s)} |f(s, \mathfrak{w}(s), {}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s)) - f(s, 0, 0)| ds \\ &\quad + \frac{1}{(\zeta + 1)\Gamma(\alpha)} \int_0^{\delta} (\delta - s)^{\alpha-1} e^{-\lambda(\delta-s)} |f(s, 0, 0)| ds \\ &\quad + \frac{\zeta}{(\zeta + 1)\Gamma(\alpha)} \int_0^{\delta} (\delta - s)^{\alpha-1} e^{-\lambda(\delta-s)} |f(s, -\mathfrak{w}(s), -{}^C_0\mathfrak{D}_s^{\alpha, \lambda} \mathfrak{w}(s)) - f(s, 0, 0)| ds \\ &\quad + \frac{\zeta}{(\zeta + 1)\Gamma(\alpha)} \int_0^{\delta} (\delta - s)^{\alpha-1} e^{-\lambda(\delta-s)} |f(s, 0, 0)| ds \\ &\leq |\mathfrak{w}(0)| + \frac{2f^* \mathfrak{N}^a}{\Gamma(\alpha + 1)} + 2 \left(\frac{\gamma \mathfrak{N}^a}{\Gamma(\alpha + 1)} + \eta \right) \|\mathfrak{w}\|_{\mathcal{X}}, \end{aligned}$$

thus

$$\|\mathfrak{w}\|_{\mathcal{X}} \leq |\mathfrak{w}(0)| + \frac{2f^* \mathfrak{N}^\alpha}{\Gamma(\alpha+1)} + 2 \left(\frac{\gamma \mathfrak{N}^\alpha}{\Gamma(\alpha+1)} + \eta \right) \|\mathfrak{w}\|_{\mathcal{X}}.$$

We deduce that

$$\|\mathfrak{w}\|_{\mathcal{X}} \leq \frac{|\mathfrak{w}(0)| + \frac{2f^* \mathfrak{N}^\alpha}{\Gamma(\alpha+1)}}{\left[1 - 2 \left(\frac{\gamma \mathfrak{N}^\alpha}{\Gamma(\alpha+1)} + \eta \right) \right]} =: \mathcal{A}.$$

The demonstration is completed. $\square$

Lemma 3.5. *If conditions (H1) and (10) are verified, then there exist a bounded open set $\Omega \subset \mathcal{X}$ with*

$$\mathfrak{L}(\mathfrak{w}) - \mathcal{N}(\mathfrak{w}) \neq -\zeta[\mathfrak{L}(\mathfrak{w}) + \mathcal{N}(-\mathfrak{w})], \quad (11)$$

for any $\mathfrak{w} \in \partial\Omega$ and $\zeta \in (0, 1]$.

Proof. Using Lemma 3.4, there exists a positive constant $\mathcal{A}$ that is independent of ζ such that, if $\mathfrak{w}$ verifies

$$\mathfrak{L}(\mathfrak{w}) - \mathcal{N}(\mathfrak{w}) = -\zeta[\mathfrak{L}(\mathfrak{w}) + \mathcal{N}(-\mathfrak{w})], \quad \zeta \in (0, 1],$$

thus $\|\mathfrak{w}\|_{\mathcal{X}} \leq \mathcal{A}$. So, if

$$\Omega = \{\mathfrak{w} \in \mathcal{X}; \|\mathfrak{w}\|_{\mathcal{X}} < \vartheta\} \quad (12)$$

such that $\vartheta > \mathcal{A}$, we deduce that

$$\mathfrak{L}(\mathfrak{w}) - \mathcal{N}(\mathfrak{w}) \neq -\zeta[\mathfrak{L}(\mathfrak{w}) - \mathcal{N}(-\mathfrak{w})]$$

for all $\mathfrak{w} \in \partial\Omega = \{\mathfrak{w} \in \mathcal{X}; \|\mathfrak{w}\|_{\mathcal{X}} = \vartheta\}$ and $\zeta \in (0, 1]$. $\square$

Theorem 3.6. *Assume (H1) and equation (10), then problems (1) and (2) have a unique solution in $\text{Dom}\mathfrak{L} \cap \overline{\Omega}$.*

Proof. It is clear that the set Ω defined in equation (12) is symmetric, $0 \in \Omega$, and $\mathcal{X} \cap \overline{\Omega} = \overline{\Omega} \neq \emptyset$. In addition, by Lemma 3.5, assume (H1) and equation (10), then

$$\mathfrak{L}(\mathfrak{w}) - \mathcal{N}(\mathfrak{w}) \neq -\zeta[\mathfrak{L}(\mathfrak{w}) - \mathcal{N}(-\mathfrak{w})]$$

for each $\mathfrak{w} \in \mathcal{X} \cap \partial\Omega = \partial\Omega$ and each $\zeta \in (0, 1]$. By Lemma 2.10, problems (1) and (2) have at least one solution in $\text{Dom}\mathfrak{L} \cap \overline{\Omega}$.

Now, we prove the uniqueness result. Suppose that problems (1) and (2) have two different solutions $\mathfrak{w}_1, \mathfrak{w}_2 \in \text{Dom}\mathfrak{L} \cap \overline{\Omega}$. Then, we have for each $\delta \in \mathfrak{J}$,

$${}_0^C \mathfrak{D}_\delta^{\alpha, \lambda} \mathfrak{w}_1(\delta) = f(\delta, \mathfrak{w}_1(\delta), {}_0^C \mathfrak{D}_\delta^{\alpha, \lambda} \mathfrak{w}_1(\delta)),$$

$${}_0^C \mathfrak{D}_\delta^{\alpha, \lambda} \mathfrak{w}_2(\delta) = f(\delta, \mathfrak{w}_2(\delta), {}_0^C \mathfrak{D}_\delta^{\alpha, \lambda} \mathfrak{w}_2(\delta)),$$

and

$$\mathfrak{w}_1(0) = \mathfrak{w}_1(\mathfrak{N}) = 0, \quad \mathfrak{w}_2(0) = \mathfrak{w}_2(\mathfrak{N}) = 0.$$

Let $\mathfrak{U}(\delta) = \mathfrak{w}_1(\delta) - \mathfrak{w}_2(\delta)$ for all $\delta \in \mathfrak{J}$.

Then,

$$\begin{aligned} \mathfrak{L}\mathfrak{U}(\delta) &= {}_0^C \mathfrak{D}_\delta^{\alpha, \lambda} \mathfrak{U}(\delta) \\ &= {}_0^C \mathfrak{D}_\delta^{\alpha, \lambda} \mathfrak{w}_1(\delta) - {}_0^C \mathfrak{D}_\delta^{\alpha, \lambda} \mathfrak{w}_2(\delta) \\ &= f(\delta, \mathfrak{w}_1(\delta), {}_0^C \mathfrak{D}_\delta^{\alpha, \lambda} \mathfrak{w}_1(\delta)) - f(\delta, \mathfrak{w}_2(\delta), {}_0^C \mathfrak{D}_\delta^{\alpha, \lambda} \mathfrak{w}_2(\delta)). \end{aligned} \quad (13)$$

On the other hand, by Lemma 2.5, we have

$${}_0\mathcal{I}_\delta^{\alpha,\lambda C} {}_0^C\mathcal{D}_\delta^{\alpha,\lambda} \mathfrak{U}(\delta) = \mathfrak{U}(\delta) - e^{-\lambda\delta} \mathfrak{U}(0) = \mathfrak{U}(\delta).$$

By equation (13) and (H1), for all $\delta \in \mathfrak{J}$, we have

$$\begin{aligned} |\mathfrak{U}(\delta)| &= |{}_0\mathcal{I}_\delta^{\alpha,\lambda C} {}_0^C\mathcal{D}_\delta^{\alpha,\lambda} \mathfrak{U}(\delta)| \\ &\leq {}_0\mathcal{I}_\delta^{\alpha,\lambda} [|f(s, \mathfrak{w}_1(s), {}_0^C\mathcal{D}_s^{\alpha,\lambda} \mathfrak{w}_1(s)) - f(s, \mathfrak{w}_2(s), {}_0^C\mathcal{D}_s^{\alpha,\lambda} \mathfrak{w}_2(s))|](\delta) \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^\delta (\delta - s)^{\alpha-1} e^{-\lambda(\delta-s)} |f(s, \mathfrak{w}_1(s), {}_0^C\mathcal{D}_s^{\alpha,\lambda} \mathfrak{w}_1(s)) - f(s, \mathfrak{w}_2(s), {}_0^C\mathcal{D}_s^{\alpha,\lambda} \mathfrak{w}_2(s))| ds \\ &\leq \frac{\gamma}{\Gamma(\alpha)} \int_0^\delta (\delta - s)^{\alpha-1} e^{-\lambda(\delta-s)} |\mathfrak{U}(s)| ds + \frac{\eta}{\Gamma(\alpha)} \int_0^\delta (\delta - s)^{\alpha-1} e^{-\lambda(\delta-s)} {}_0^C\mathcal{D}_s^{\alpha,\lambda} |\mathfrak{U}(s)| ds \\ &\leq \left[\frac{\gamma \kappa^\alpha}{\Gamma(\alpha + 1)} + \eta \right] \|\mathfrak{U}\|_\mathcal{X}. \end{aligned}$$

Therefore,

$$\|\mathfrak{U}\|_\mathcal{X} \leq \left[\frac{\gamma \kappa^\alpha}{\Gamma(\alpha + 1)} + \eta \right] \|\mathfrak{U}\|_\mathcal{X}.$$

Hence, by equation (10), we conclude that

$$\|\mathfrak{U}\|_\mathcal{X} = 0.$$

As a result, for any $\delta \in \mathfrak{J}$, we obtain

$$\mathfrak{U}(\delta) = 0 \Rightarrow \mathfrak{w}_1(\delta) = \mathfrak{w}_2(\delta).$$

This completes the proof. $\square$

4 Examples

Example 4.1. Consider the following problem for nonlinear fractional differential equations:

$$\begin{aligned} {}_0^C\mathcal{D}_\delta^{\frac{1}{3};2} \mathfrak{w}(\delta) &= f\left(\delta, \mathfrak{w}(\delta), {}_0^C\mathcal{D}_\delta^{\frac{1}{3};2} \mathfrak{w}(\delta)\right), \quad \delta \in \mathfrak{J} = [0, 1], \\ \mathfrak{w}(0) &= \mathfrak{w}(1) = 0, \end{aligned}$$

where

$$f\left(\delta, \mathfrak{w}(\delta), {}_0^C\mathcal{D}_\delta^{\frac{1}{3};2} \mathfrak{w}(\delta)\right) = \frac{\mathfrak{w}(\delta)}{5(1 + \delta)} + \frac{1}{13\sqrt{\pi}} \sin\left({}_0^C\mathcal{D}_\delta^{\frac{1}{3};2} \mathfrak{w}(\delta)\right) + \frac{e^\delta}{3}.$$

Here, $\alpha = \frac{1}{3}$, $\lambda = 2$, and $\kappa = 1$.

It is clear that the function $f \in C([0, 1], \mathfrak{R})$. Let $\mathfrak{w}, \bar{\mathfrak{w}}, \mathfrak{v}, \bar{\mathfrak{v}} \in \mathfrak{R}$, and $\delta \in \mathfrak{J}$, then

$$|f(\delta, \mathfrak{w}, \mathfrak{v}) - f(\delta, \bar{\mathfrak{w}}, \bar{\mathfrak{v}})| \leq \frac{1}{5} |\mathfrak{w} - \bar{\mathfrak{w}}| + \frac{1}{13\sqrt{\pi}} |\mathfrak{v} - \bar{\mathfrak{v}}|,$$

which implies that (H1) is satisfied with $\gamma = \frac{1}{5}$ and $\eta = \frac{1}{13\sqrt{\pi}}$.

Furthermore, by some simple calculations, we see that

$$\frac{\gamma \kappa^\alpha}{\Gamma(\alpha + 1)} + \eta \approx 0.267 < \frac{1}{2}.$$

With the use of Theorem 3.6, our problem has a unique solution.

Example 4.2. Consider the following problem for nonlinear fractional differential equations:

$${}_0^C\mathcal{D}_\delta^{\frac{1}{2};4}\mathfrak{w}(\delta) = g\left(\delta, \mathfrak{w}(\delta), {}_0^C\mathcal{D}_\delta^{\frac{1}{2};4}\mathfrak{w}(\delta)\right), \quad \delta \in \mathfrak{J} = [0, 1],$$

$$\mathfrak{w}(0) = \mathfrak{w}(1) = 0,$$

where

$$g\left(\delta, \mathfrak{w}(\delta), {}_0^C\mathcal{D}_\delta^{\frac{1}{2};4}\mathfrak{w}(\delta)\right) = \ln(\delta + 2) + \frac{e^{-\pi\delta}\left(|\mathfrak{w}(\delta)| + \left|{}_0^C\mathcal{D}_\delta^{\frac{1}{2};4}\mathfrak{w}(\delta)\right|\right)}{110\left(1 + |\mathfrak{w}(\delta)| + \left|{}_0^C\mathcal{D}_\delta^{\frac{1}{2};4}\mathfrak{w}(\delta)\right|\right)}.$$

Here, $\alpha = \frac{1}{2}$, $\lambda = 4$, and $\kappa = 1$.

It is easy to see that $g \in C([0, 1], \mathfrak{R})$. Let $\mathfrak{w}, \bar{\mathfrak{w}}, \mathfrak{v}, \bar{\mathfrak{v}} \in \mathfrak{R}$, and $\delta \in \mathfrak{J}$, then

$$|g(\delta, \mathfrak{w}, \mathfrak{v}) - g(\delta, \bar{\mathfrak{w}}, \bar{\mathfrak{v}})| \leq \frac{1}{110}|\mathfrak{w} - \bar{\mathfrak{w}}| + \frac{1}{110}|\mathfrak{v} - \bar{\mathfrak{v}}|.$$

Hence, the assumption (H1) is satisfied with $\gamma = \eta = \frac{1}{110}$.

By simple calculations, we see that

$$\frac{\gamma\kappa^\alpha}{\Gamma(\alpha + 1)} + \eta \approx 0.019 < \frac{1}{2}.$$

So, by Theorem 3.6, our problem has a unique solution.

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