Research Article

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Results on solutions of several systems of the product type complex partial differential difference equations

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Abstract: This article is devoted to exploring the solutions of several systems of the first-order partial differential difference equations (PDDEs) with product type

$$\begin{cases} u(z+c)[\alpha_1 u(z) + \beta_1 u_{z_1} + \gamma_1 u_{z_2} + \alpha_2 v(z) + \beta_2 v_{z_1} + \gamma_2 v_{z_2}] = 1, \\ v(z+c)[\alpha_1 v(z) + \beta_1 v_{z_1} + \gamma_1 v_{z_2} + \alpha_2 u(z) + \beta_2 u_{z_1} + \gamma_2 u_{z_2}] = 1, \end{cases}$$

where $c = (c_1, c_2) \in \mathbb{C}^2, \alpha_j, \beta_j, \gamma_j \in \mathbb{C},\ j = 1, 2$. Our theorems about the forms of the transcendental solutions for these systems of PDDEs are some improvements and generalization of the previous results given by Xu, Cao and Liu. Moreover, we give some examples to explain that the forms of solutions of our theorems are precise to some extent.

Keywords: partial differential difference equation, second order, Nevanlinna theory

MSC 2020: 30D35, 35M30, 39A45

1 Introduction

Khavinson [20] in 1995 described the solutions of the eikonal (eiconal) equation in $\mathbb{C}^2$

$$(u_{z_1})^2 + (u_{z_2})^2 = 1 \tag{1.1}$$

and proved the classical result that any entire solution of (1.1) must be linear of the form $u = c_1 z_1 + c_2 z_2 + c_0$, where $c_1^2 + c_2^2 = 1$ (can also be found in [1]). Of course, equation (1.1) can be seen as a typical partial differential equation (PDE), which are widely applied in various disciplines, including mathematics, physics, etc. [2–6]. Later, Saleeby [7] proved the same conclusion by using the different method, and this result is also included in one corollary [8, Corollary 2.2]. In the past several decades, many mathematics scholars had paid considerable attention on the solutions of the eikonal equation and its variants, and obtained a great number of interest and important results [9–18].

Theorem A. [19, Corollary 2.3] *Let $P(z_1, z_2)$ and $Q(z_1, z_2)$ be arbitrary polynomials in $\mathbb{C}^2$. Then u is an entire solution of the equation*

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$$(Pu_{z_1})^2 + (Qu_{z_2})^2 = 1 \tag{1.2}$$

if and only if $u = c_1z_1 + c_2z_2 + c_3$ *is a linear function, where* c_j *are constants, and exactly one of the following holds*:

(*i*) $c_1 = 0$ *and* Q *is a constant satisfying that* $(c_2Q)^2 = 1$;

(*ii*) $c_2 = 0$ *and* P *is a constant satisfying that* $(c_1P)^2 = 1$;

(*iii*) $c_1c_2 \neq 0$ *and* P, Q *are both constant satisfying that* $(c_1P)^2 + (c_2Q)^2 = 1$.

Khavinson [20] and Li [8] mentioned that equation (1.1) can be reduced to $u_{z_1}u_{z_2} = 1$ by taking the linear transformation $x = z_1 + iz_2$ and $y = z_1 - iz_2$. Differentiating this new equation with respect to z_1 and z_2 yields that $u_{z_1z_1}u_{z_2} = -u_{z_1}u_{z_2z_1}$ and $u_{z_1z_2}u_{z_2} = -u_{z_1}u_{z_2z_2}$, and this leads to $u_{z_1z_1}u_{z_2z_2} - u_{z_1z_2}^2 = 0$. This equation can be seen as a degenerated Monge-Ampère equation, which has the linear function solutions. For the non-degenerated Monge-Ampère equation

$$A(u_{z_1z_1}u_{z_2z_2} - u_{z_1z_2}^2) + Bu_{z_1z_1} + Cu_{z_1z_2} + Du_{z_2z_2} + E = 0,$$

where A, B, C, D, and E are functions depending only on z_1, z_2, u, u_{z_1}, and u_{z_2}. In general, it is very difficult to find solutions of a non-degenerate Monge-Ampère equation. There is a great number references focusing on the study of this class equation.

Motivated by the idea of this remarks of Khavinson [20] and Li [8], Lü [21] focused on entire solutions of a variation of the eikonal equation with product type.

Theorem B. [21, Theorem 1] *Let* g *be a polynomial in* $\mathbb{C}^2$, *and let* m *be a non-negative integer. Then* u *is an entire solution of the PDE* $u_xu_y = x^me^g$ *in* $\mathbb{C}^2$ *if and only if the following assertions hold*:

(1) $u = \phi_1(x) + \phi_2(x)$, *where* $\phi'(x) = x^me^{\alpha(x)}$ *and* $\phi'(y) = e^{\beta(y)}$ *satisfying* $\alpha(x) + \beta(y) = g(x, y)$;

(2) $u = F(y + Ax^{m+1})$, *where* A *is a non-zero constant and* $(m + 1)AF'^2(y + Ax^{m+1}) = e^g$;

(3) $u = (x^{k+1}/(k + 1))e^{ay+b} + C$, *where* $(a/(k + 1))e^{2(ay+b)} = e^g$, $m = 2k + 1$, *and* $a(\neq 0)$, b, *and* C *are constants.*

In 2022, Chen and Han [22] further investigated the entire solutions for a series of product type nonlinear PDEs, and obtained

Theorem C. [22, Theorem 1.1] *Let* $p(z, w) \neq 0$ *be a polynomial in* $\mathbb{C}^2$, *and let* $l \geq 0$ *and* $m, n \geq 1$ *be integers.* $u(z, w)$ *in* $\mathbb{C}^2$ *is an entire solution to the nonlinear first-order PDE*

$$(u^lu_z)^m(u^lu_w)^n = p(z, w) \tag{1.3}$$

if and only if one of the following situations occurs.

(1) $l = 0$, $p(z, w) = q^m(z)r^n(w)$ *for some nonzero polynomials* $q(z), r(w)$ *in* $\mathbb{C}$, *and* $u(z, w) = c_1\int q(z)\mathrm{d}z + c_2\int r(w)\mathrm{d}w + c_0$ *for some constants* c_0, c_1, c_2 *satisfying* $c_1^mc_2^n = 1$; *in particular, when* $p(z, w) = K$ *for a constant* $K(\neq 0)$, *then* $u(z, w)$ *is affine.*

(2) $l \geq 0$ *and* $u(z, w) = \{(l + 1)(c_1\int q(z, w)\mathrm{d}z + c_2\int r(z, w)\mathrm{d}w - c_1\iint q_w(z, w)\mathrm{d}z\mathrm{d}w)\}^{\frac{1}{l+1}}$ *for some constants* c_1, c_2 *with* $c_1^mc_2^n = 1$, *where* $q(z, w), r(z, w)$ *are nonzero polynomials in* $\mathbb{C}^2$ *such that* $c_1q_w(z, w) = c_2r_z(z, w) \neq 0$ *and* $p(z, w) = q^m(z, w)r^n(z, w)$.

In recent years, the topic of solutions of systems of complex functional equations has attracted consideration attention [23–25,27]. In 2023, Xu et al. [26] gave some description of entire solutions of the product type PDEs systems and obtained:

Theorem D. [26, Theorem 2.1] *Let* $D \coloneqq ad - bc \neq 0$ *and* (f, g) *be a pair of transcendental entire solutions with finite-order for system*

$$\begin{cases}(af_{z_1}+bf_{z_2})(cg_{z_1}+dg_{z_2})=1,\\(ag_{z_1}+bg_{z_2})(cf_{z_1}+df_{z_2})=1.\end{cases}\tag{1.4}$$

Then (f,g) is one of the following forms

(*i*)

$$(f(z),g(z))=\left(\frac{1}{a}F_1(z_1),\frac{1}{c}G_1(z_1)\right);$$

(*ii*)

$$(f(z),g(z))=\left(\frac{1}{b}F_2(z_2),\frac{1}{d}G_2(z_2)\right);$$

(*iii*)

$$(f(z),g(z))=\left(\frac{aA^{-1}-c}{D}F_3\left(z_2-\frac{b-dA}{a-cA}z_1\right),\frac{cA-a}{D}G_3\left(z_2-\frac{b-dA}{a-cA}z_1\right)\right),$$

where $A\in\mathbb{C}-\{0\}$, $\varphi_j(t)$, $j=1,2,3$ are nonconstant polynomial in $\mathbb{C}$ and

$$F_j'(t)=e^{\varphi_j(t)},\quad G_j'(t)=e^{-\varphi_j(t)},\ j=1,2,3.$$

By employing the difference Nevanlinna theory of several complex variables [28–30], Xu and Cao [31,32] discussed the transcendental solutions of several partial differential difference equations (PDDEs). In general, *an equation is called as a PDDE, if this equation includes the partial derivatives, shifts, and differences of f*, which can be called as PDDE for short. They obtained the following:

Theorem E. [31, Theorem 1.2] *Let $c=(c_1,c_2)\in\mathbb{C}^2$. Then any transcendental entire solution with finite-order of the PDDE*

$$(f_{z_1})^2+f(z_1+c_1,z_2+c_2)^2=1\tag{1.5}$$

has the form of $f(z_1,z_2)=\sin(Az_1+B)$, where A is a constant on $\mathbb{C}$ satisfying $Ae^{iAc_1}=1$ and B is a constant on $\mathbb{C}$; in the special case whenever $c_1=0$, we have $f(z_1,z_2)=\sin(z_1+B)$.

In the same year, Xu et al. [25] studied the finite-order transcendental entire solutions when equation (1.1) turn to the system of Fermat-type PDDEs and obtained the following:

Theorem F. [25, Theorem 1.3] *Let $c=(c_1,c_2)\in\mathbb{C}^2$. Then any pair of transcendental entire solutions with finite-order for the system of Fermat-type PDDEs*

$$\begin{cases}(f_{z_1})^2+g(z_1+c_1,z_2+c_2)^2=1,\\(g_{z_1})^2+f(z_1+c_1,z_2+c_2)^2=1\end{cases}\tag{1.6}$$

have the following forms

$$(f,g)=\left(\frac{e^{L(z)+B_1}+e^{-(L(z)+B_1)}}{2},\frac{A_{21}e^{L(z)+B_1}+A_{22}e^{-(L(z)+B_1)}}{2}\right),$$

where $L(z)=a_1z_1+a_2z_2$, B_1 is a constant in $\mathbb{C}$, and a_1,c,A_{21},A_{22} satisfy one of the following cases:

(*i*) $A_{21}=-i$, $A_{22}=i$, *and* $a_1=i$, $L(c)=(2k+\frac{1}{2})\pi i$, *or* $a_1=-i$, $L(c)=(2k-\frac{1}{2})\pi i$;

(*ii*) $A_{21}=i$, $A_{22}=-i$, *and* $a_1=i$, $L(c)=(2k-\frac{1}{2})\pi i$, *or* $a_1=-i$, $L(c)=(2k+\frac{1}{2})\pi i$;

(*iii*) $A_{21}=1$, $A_{22}=1$, *and* $a_1=i$, $L(c)=2k\pi i$, *or* $a_1=-i$, $L(c)=(2k+1)\pi i$;

(*iv*) $A_{21}=-1$, $A_{22}=-1$, *and* $a_1=i$, $L(c)=(2k+1)\pi i$, *or* $a_1=-i$, $L(c)=2k\pi i$.

In 2021, Xu et al. [33] further investigated the solutions of several systems of the PDDEs in $\mathbb{C}^2$ and obtained the following:

Theorem G. [33, Theorem 1.3] *Let $c = (c_1, c_2) \in \mathbb{C}^2$ and $c_1 \neq c_2$. Then any pair of transcendental entire solution (f_1, f_2) with finite-order for the system of the PDEs*

$$\begin{cases} f_1(z)^2 + \left[f_2(z+c) + \dfrac{\partial f_1}{\partial z_1} + \dfrac{\partial f_1}{\partial z_2}\right]^2 = 1, \\ f_2(z)^2 + \left[f_1(z+c) + \dfrac{\partial f_2}{\partial z_1} + \dfrac{\partial f_2}{\partial z_2}\right]^2 = 1 \end{cases} \tag{1.7}$$

must be of the form

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+b} + e^{-L(z)-b}}{2}, \frac{A_1 e^{L(z)+b} + A_2 e^{-L(z)-b}}{2}\right),$$

where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$, and $\alpha_1, \alpha_2, b, A_1, A_2 \in \mathbb{C}$ satisfy one of the following cases:

(i) *if $\alpha_1 + \alpha_2 = 0$, then $\alpha_1 c_1 + \alpha_2 c_2 = 2k\pi i + \frac{\pi}{2}i$, $A_1 = A_2 = -1$ or $\alpha_1 c_1 + \alpha_2 c_2 = 2k\pi i + \frac{3\pi}{2}i$, $A_1 = A_2 = 1$ or $\alpha_1 c_1 + \alpha_2 c_2 = 2k\pi i$, $A_1 = -i, A_2 = i$ or $\alpha_1 c_1 + \alpha_2 c_2 = (2k+1)\pi i$, $A_1 = i, A_2 = -i$, $k \in \mathbb{Z}$;*

(ii) *if $\alpha_1 + \alpha_2 = -2i$, then $\alpha_1 c_1 + \alpha_2 c_2 = 2k\pi i + \frac{\pi}{2}i$, $A_1 = A_2 = 1$ or $\alpha_1 c_1 + \alpha_2 c_2 = 2k\pi i + \frac{3\pi}{2}i$, $A_1 = A_2 = -1$, $k \in \mathbb{Z}$.*

The aforementioned results suggest the following question:

Question 1.1. What will happen about the solutions if the system is of the product type PDDEs with more general forms?

2 Results and examples

Inspired by Question 1.1 and the aforementioned results, we mainly discuss the entire solutions of several systems of the product type PDDEs in $\mathbb{C}^2$. As far as we know, *the systems we are concerned with have not been studied earlier*. In this article, we first assume that the readers are familiar with the Nevanlinna theory and difference Nevanlinna theory with several complex variables (can refer to Korhonen and Cao [28,29,34]). Here and below, we denote $z + w = (z_1 + w_1, z_2 + w_2)$ and $az = (az_1, az_2)$ for any $z = (z_1, z_2)$, $w = (w_1, w_2)$ and $a \in \mathbb{C}$.

Theorem 2.1. *Let $c = (c_1, c_2) \in \mathbb{C}^2$, $c_1, c_2 \in \mathbb{C}$ and $D := \beta_1\gamma_2 - \beta_2\gamma_1$, and assume that (u, v) is a pair of finite-order transcendental entire solutions of system*:

$$\begin{cases} u(z+c)[\alpha_1 u(z) + \beta_1 u_{z_1} + \gamma_1 u_{z_2} + \alpha_2 v(z) + \beta_2 v_{z_1} + \gamma_2 v_{z_2}] = 1, \\ v(z+c)[\alpha_1 v(z) + \beta_1 v_{z_1} + \gamma_1 v_{z_2} + \alpha_2 u(z) + \beta_2 u_{z_1} + \gamma_2 u_{z_2}] = 1. \end{cases} \tag{2.1}$$

(i) *If $\alpha_1 \neq 0$, then* (2.1) *has no any pair of transcendental entire solutions with finite-order*;

(ii) *If $\alpha_1 = 0$ and $D \neq 0$, then (u, v) must be the form of*

$$(u, v) = \left(\frac{1}{\alpha_2 + \beta_2 A_1 + \gamma_2 A_2} e^{A_1 z_1 + A_2 z_2 - B_2}, \frac{1}{\alpha_2 - \beta_2 A_1 - \gamma_2 A_2} e^{-A_1 z_1 + A_2 z_2 - B_1}\right),$$

where $A_1, A_2, B_1, B_2 \in \mathbb{C}$ satisfy $\beta_1 A_1 + \gamma_1 A_2 = 0$ and

$$e^{2(A_1 c_1 + A_2 c_2)} = \frac{\alpha_2 + \beta_2 A_1 + \gamma_2 A_2}{\alpha_2 - \beta_2 A_1 - \gamma_2 A_2}, \quad e^{2(B_1 + B_2)} = \frac{1}{\alpha_2^2 - (\beta_2 A_1 + \gamma_2 A_2)^2}. \tag{2.2}$$

The following examples show the existence of transcendental entire solutions of system (2.1) for every case in Theorem 2.1.

Example 2.1. Let

$$(u, v) = \left(\frac{2}{3}e^{\frac{1}{2}z_2} + \frac{1}{2}\log 3, 2e^{-\frac{1}{2}z_2-\log 2}\right) = \left(\frac{2}{\sqrt{3}}e^{\frac{1}{2}z_2}, e^{-\frac{1}{2}z_2}\right).$$

Thus, (u, v) is a pair of transcendental entire solutions of system (2.1) for the case $\alpha_1 = 0, \beta_1 = 1, \gamma_1 = 0, \alpha_2 = 1, \beta_2 = 0, \gamma_2 = 1, c_2 = \log 3, c_1 \in \mathbb{C}$, and $\rho(u, v) = 1$.

Example 2.2. Let

$$(u, v) = \left(\frac{5}{9}e^{\frac{2}{5}(z_1+z_2)+\log 3}, 5e^{-\frac{2}{5}(z_1+z_2)-\log 5}\right) = \left(\frac{5}{3}e^{\frac{2}{5}(z_1+z_2)}, e^{-\frac{2}{5}(z_1+z_2)}\right).$$

Thus, (u, v) is a pair of transcendental entire solutions of system (2.1) for the case $\alpha_1 = 0, \beta_1 = 1, \gamma_1 = -1, \alpha_2 = 1, \beta_2 = 1, \gamma_2 = 1, c_2 = \frac{3}{2}\log 3, c_1 = \log 3$, and $\rho(u, v) = 1$.

An example shows that the condition $D \neq 0$ can not be removed.

Example 2.3. Let

$$(u, v) = (e^{z_1-2z_2+2(z_1-2z_2)^3}, e^{-(z_1-2z_2)-2(z_1-2z_2)^3}).$$

Then (u, v) is a pair of finite-order transcendental entire solution of system (2.1) for the case $\alpha_1 = 0, \beta_1 = 2, \gamma_2 = 1, \alpha_2 = -1, \beta_2 = 4, \gamma_2 = 2, c_1 = \frac{1}{2}\pi i$, and $c_2 = -\frac{1}{4}\pi i$. Obviously, the form of this solution cannot be included in Theorem 2.1.

By observing (2.2) in Theorem 2.1, we can obtain the following corollary for $c_1 = c_2 = 0$.

Corollary 2.1. *Let $D := \beta_1\gamma_2 - \beta_2\gamma_1 \neq 0$, then the system*

$$\begin{cases} u(z)[\alpha_1 u(z) + \beta_1 u_{z_1} + \gamma_1 u_{z_2} + \alpha_2 v(z) + \beta_2 v_{z_1} + \gamma_2 v_{z_2}] = 1, \\ v(z)[\alpha_1 v(z) + \beta_1 v_{z_1} + \gamma_1 v_{z_2} + \alpha_2 u(z) + \beta_2 u_{z_1} + \gamma_2 u_{z_2}] = 1 \end{cases} \tag{2.3}$$

has not any pair of nonconstant finite-order transcendental entire solutions.

The following example shows that the condition “$D \neq 0$” in Corollary 2.1 can not be removed.

Example 2.4. Let

$$(u, v) = (e^{(z_1-z_2)^2+2(z_1-z_2)^3}, e^{-(z_1-z_2)^2-2(z_1-z_2)^3}).$$

Then (u, v) is a pair of finite-order transcendental entire solutions of system (2.3) for the case $\alpha_1 = 0, \beta_1 = 1, \gamma_2 = 1, \alpha_2 = 1, \beta_2 = 1$, and $\gamma_2 = 1$.

For $D = 0$, we obtain

Theorem 2.2. *Let $c = (c_1, c_2) \in \mathbb{C}^2, c_1, c_2 \in \mathbb{C}$, and assume that (u, v) is a pair of finite-order transcendental entire solutions of system*

$$\begin{cases} u(z + c)(u_{z_1} + v + v_{z_1}) = 1, \\ v(z + c)(v_{z_1} + u + u_{z_1}) = 1. \end{cases} \tag{2.4}$$

(*i*) *If* $c_2 = 0$, *then* (u, v) *is of the form*

$$(u, v) = (e^{\phi(z_2)+B_0}, e^{-\phi(z_2)-B_0}),$$

where $\phi(z_2)$ *is a nonconstant polynomial in* z_2.

(*ii*) *If* $c_2 \neq 0$, *then* (u, v) *is of the form*

$$(u, v) = (e^{A_2z_2-B_2}, e^{-A_2z_2-B_1}),$$

where A_2, B_1, *and* B_2 *are constants and satisfy*

$$e^{2A_2c_2} = 1, \quad e^{2(B_1+B_2)} = 1.$$

Theorem 2.3. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$, $c_1, c_2 \in \mathbb{C}$, *and assume that* (u, v) *is a pair of finite-order transcendental entire solutions of system*

$$\begin{cases} u(z+c)(u_{z_1} + u_{z_2} + v + v_{z_1} + v_{z_2}) = 1, \\ v(z+c)(v_{z_1} + v_{z_2} + u + u_{z_1} + u_{z_2}) = 1. \end{cases} \tag{2.5}$$

(*i*) *If* $c_1 = c_2$, *then* (u, v) *is of the form*

$$(u, v) = (e^{\phi(z_1-z_2)+B_0}, \quad e^{-\phi(z_1-z_2)-B_0}),$$

where $\phi(x)$ *is a nonconstant polynomial in* x.

(*ii*) *If* $c_1 \neq c_2$, *then* (u, v) *is of the form*

$$(u, v) = (e^{A(z_1-z_2)-B_2}, e^{-A(z_1-z_2)-B_1}),$$

where A, B_1, *and* B_2 *are constants and satisfy*

$$e^{2A(c_1-c_2)} = 1, \quad e^{2(B_1+B_2)} = 1.$$

The following examples show that our results about the forms of solutions in Theorem 2.3 are precise.

Example 2.5. Let

$$(u, v) = (e^{(z_1-z_2)^3-2(z_1-z_2)^4}, e^{-(z_1-z_2)^2+2(z_1-z_2)^3}).$$

Then (u, v) is a pair of finite-order transcendental entire solutions of system (2.3) for the case $c_1 = c_2$.

Example 2.6. Let

$$(u, v) = (-ie^{\pi i(z_1-z_2)}, -ie^{-\pi i(z_1-z_2)}).$$

Then (u, v) is a pair of finite-order transcendental entire solutions of system (2.3) for the case $c_1 = 2, c_2 = 1$.

Similar to Theorem 2.1, we have the following:

Theorem 2.4. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$, $c_1, c_2 \in \mathbb{C}$, *and* $D \coloneqq \beta_1\gamma_2 - \beta_2\gamma_1 \neq 0$, *and assume that* (u, v) *is a pair of finite-order transcendental entire solutions of system*

$$\begin{cases} v(z+c)[\alpha_1u(z) + \beta_1u_{z_1} + \gamma_1u_{z_2} + \alpha_2v(z) + \beta_2v_{z_1} + \gamma_2v_{z_2}] = 1, \\ u(z+c)[\alpha_1v(z) + \beta_1v_{z_1} + \gamma_1v_{z_2} + \alpha_2u(z) + \beta_2u_{z_1} + \gamma_2u_{z_2}] = 1. \end{cases} \tag{2.6}$$

(*i*) *If* $\alpha_2 \neq 0$, *then* (2.6) *has no any pair of transcendental entire solutions with finite-order*;

(*ii*) *If* $\alpha_2 = 0$, *then* (u, v) *must be the form of*

$$(u, v) = \left(\frac{1}{\alpha_1 + \beta_1A_1 + \gamma_1A_2}e^{A_1z_1+A_2z_2-B_2}, \frac{1}{\alpha_1 - \beta_1A_1 - \gamma_1A_2}e^{-A_1z_1+A_2z_2-B_1}\right),$$

where $A_1, A_2, B_1, B_2 \in \mathbb{C}$ *satisfy* $\beta_2 A_1 + \gamma_2 A_2 = 0$ *and*

$$e^{2(A_1c_1+A_2c_2)} = \frac{\alpha_1 + \beta_1 A_1 + \gamma_1 A_2}{\alpha_1 - \beta_1 A_1 - \gamma_1 A_2}, \quad e^{2(B_1+B_2)} = \frac{1}{\alpha_1^2 - (\beta_1 A_1 + \gamma_1 A_2)^2}. \tag{2.7}$$

Corollary 2.2. *Let* $D := \beta_1\gamma_2 - \beta_2\gamma_1 \neq 0$, *then the system*

$$\begin{cases} v(z)[\alpha_1 u(z) + \beta_1 u_{z_1} + \gamma_1 u_{z_2} + \alpha_2 v(z) + \beta_2 v_{z_1} + \gamma_2 v_{z_2}] = 1, \\ u(z)[\alpha_1 v(z) + \beta_1 v_{z_1} + \gamma_1 v_{z_2} + \alpha_2 u(z) + \beta_2 u_{z_1} + \gamma_2 u_{z_2}] = 1 \end{cases}$$

has not any pair of nonconstant finite-order transcendental entire solutions.

3 Some lemmas

The following lemmas play the key role in proving our results.

Lemma 3.1. [35,36] *For an entire function* F *on* $\mathbb{C}^n$, $F(0) \neq 0$ *and put* $\rho(n_F) = \rho < \infty$. *Then there exist a canonical function* f_F *and a function* $g_F \in \mathbb{C}^n$ *such that* $F(z) = f_F(z)e^{g_F(z)}$. *For the special case* $n = 1$, f_F *is the canonical product of Weierstrass.*

Remark 3.1. Here, let $\rho(n_F)$ be the order of the counting function of zeros of F.

Lemma 3.2. [37] *If* g *and* h *are entire functions on the complex plane* $\mathbb{C}$ *and* $g(h)$ *is an entire function of finite-order, then there are only two possible cases: either*
(*a*) *the internal function* h *is a polynomial and the external function* g *is of finite-order; or else*
(*b*) *the internal function* h *is not a polynomial but a function of finite-order, and the external function* g *is of zero order.*

Lemma 3.3. *Let* $g(u) = g(x, y)$ *be a polynomial in* $\mathbb{C}^2$, *and* $u_0 = (x_0, y_0)$, $x_0, y_0 \in \mathbb{C}$ *be two constants. If* $g(u + u_0) - g(u) = g(x + x_0, y + y_0) - g(x, y)$ *is a constant, then* $g(u)$ *can be represented as the form of*

$$g(x, y) = L(u) + H(s),$$

where $L(u) = \alpha x + \beta y$, α, β *are constants, and* $H(s)$ *is a polynomial in* s *in* $\mathbb{C}$, $s := y_0 x - x_0 y$.

Proof. From the assumption of this lemma, we can write $g(x, y)$ as the form

$$g(u) = g(x, y) = \sum_{j=0}^{n} Q_j(y)x^j = Q_n(y)x^n + Q_{n-1}(y)x^{n-1} + \cdots + Q_1(y)x + Q_0(y), \tag{3.1}$$

where $Q_j(y)$, $j = 0, 1, \ldots, n$ are polynomials in y. Since $g(u + u_0) - g(u) = g(x + x_0, y + y_0) - g(x, y)$ is a constant, let

$$\eta = g(u + u_0) - g(u) = g(x + x_0, y + y_0) - g(x, y). \tag{3.2}$$

Next, three cases will be considered.
Case 1. $x_0 \neq 0, y_0 = 0$. Thus, we have from (3.2) that

$$\begin{aligned} \eta &= g(x + x_0, y) - g(x, y) \\ &= \sum_{j=0}^{n} Q_j(y)[(x + x_0)^j - x^j] \\ &= \sum_{j=1}^{n} Q_j(y)[C_j^1 x_0 x^{j-1} + \cdots + C_j^j (x_0)^j], \end{aligned} \tag{3.3}$$

where $C_j^i = \frac{j(j-1)\cdots(j-i+1)}{i!}$. If $n = 0$, then $g(u) = Q_0(y)$. Obviously, $g(u) = H(s)$.

If $n \geq 1$, noting that $x_0 \neq 0$, we have from (3.3) that

$$Q_n(y) \equiv Q_{n-1}(y) \equiv \cdots \equiv Q_2(y) \equiv 0 \tag{3.4}$$

and

$$Q_1(y) = \frac{\eta}{x_0}(\text{Const.}). \tag{3.5}$$

Thus, we conclude from (3.4) and (3.5) that

$$\begin{aligned} g(x,y) &= Q_1(y)x + Q_0(y) \\ &= \frac{\eta}{x_0}x + \gamma_m y^m + \gamma_{m-1}y^{m-1} + \cdots + \gamma_1 y + \gamma_0 \\ &= \alpha x + \beta y + H(s), \end{aligned} \tag{3.6}$$

where $\alpha = \frac{\eta}{x_0}$, $\beta = 0$, $d_j = \frac{\gamma_j}{(-x_0)^j}$, $j = 0, 1, \ldots, m$ and

$$H(s) = H(-x_0 y) = d_m s^m + d_{m-1}s^{m-1} + \cdots + d_1 s + d_0.$$

Case 2. $y_0 \neq 0, x_0 = 0$. Here, we can rewrite $g(u)$ as the following form

$$g(u) = g(x,y) = \sum_{j=0}^{m} Q_m(x)y^m.$$

By using the same argument as in Case 1, we can prove that $g(x,y)$ is of the form $\alpha x + \beta y + H(s)$.

Case 3. $x_0 \neq 0, y_0 \neq 0$. We have

$$\begin{aligned} \eta = g(u+u_0) - g(u) &= \sum_{j=0}^{n}[Q_j(y+y_0)(x+x_0)^j - Q_j(y)x^j] \\ &= Q_n(y+y_0)(x+x_0)^n - Q_n(y)x^n + Q_{n-1}(y+y_0)(x+x_0)^{n-1} - Q_{n-1}(y)x^{n-1} + \cdots + Q_1(y+y_0)(x+x_0) \\ &\quad - Q_1(y)x + Q_0(y+y_0) - Q_0(y). \end{aligned} \tag{3.7}$$

If $n \leq 1$, by observing the coefficients of x, y in both sides of (3.7), we have

$$Q_1(y+y_0) - Q_1(y) \equiv 0, \tag{3.8}$$

$$x_0 Q_1(y+y_0) + Q_0(y+y_0) - Q_0(y) = \eta. \tag{3.9}$$

Equation (3.8) implies that $Q_1(y)$ is a constant, let $Q_1(y) = \alpha$. Then it follows from (3.9) that

$$Q_0(y+y_0) - Q_0(y) = \eta - \alpha x_0.$$

Since $Q_0(y)$ is a polynomial in y, it yields that $Q_0(y)$ is a polynomial in y with the degree ≤1, that is, $Q_0(y) = \beta y + b_0$, where $\beta = \frac{\eta - \alpha x_0}{y_0}$. Hence, we have that $g(u) = \alpha x + \beta y + H(s)$, where $H(s) = b_0$.

If $n \geq 2$, by observing the coefficients of x^n, x^{n-1} on both sides of (3.7), we have

$$Q_n(y+y_0) - Q_n(y) \equiv 0, \tag{3.10}$$

$$Q_n(y+y_0)C_n^1 x_0 + Q_{n-1}(y+y_0) - Q_{n-1}(y) \equiv 0. \tag{3.11}$$

Equation (3.10) implies that $Q_n(y)$ is a constant, let $Q_n(y) = a_n^0$. Thus, it follows from (3.11) that $Q_{n-1}(y)$ is a polynomial in y with degree ≤1, let $Q_{n-1}(y) = a_{n-1}^0 y + a_{n-1}^1$, where a_{n-1}^0, a_{n-1}^1 are two constants satisfying

$$n a_n^0 x_0 = -a_{n-1}^0 y_0,$$

that is,

$$\frac{a_n^0}{a_{n-1}^0} = -\frac{1}{n}\frac{y_0}{x_0}. \tag{3.12}$$

Now, we continue to analyze the coefficient of x^{n-2} on both sides of (3.7) and obtain

$$C_n^2 a_n^0 (y_0)^2 + Q_{n-1}(y + y_0)(n - 1)x_0 + Q_{n-2}(y + y_0) - Q_{n-2}(y) \equiv 0, \tag{3.13}$$

which implies that $Q_{n-2}(y)$ is a polynomial in y with degree ≤ 2, let

$$Q_{n-2}(y) = a_{n-2}^0 y^2 + a_{n-2}^1 y + a_{n-2}^2,$$

where $a_{n-2}^0, a_{n-2}^1, a_{n-2}^2$ are constants. Substituting the aforementioned into (3.13), we have

$$2y_0 a_{n-2}^0 = -a_{n-1}^0 x_0 (n - 1),$$

that is,

$$\frac{a_{n-1}^0}{a_{n-2}^0} = -\frac{2}{n - 1}\frac{y_0}{x_0}. \tag{3.14}$$

Similar to the same argument as in the aforementioned equation, we have that $Q_j(y)$ is a polynomial in y with degree $\leq n - j$ for $j = 1, \ldots, n$. Let

$$Q_j(y) = a_j^0 y^{n-j} + a_j^1 y^{n-j-1} + \cdots + a_j^{n-j-1} y + a_j^{n-j},$$

where $a_j^0, a_j^1, \ldots, a_j^{n-j}$ are constants. Thus, we have

$$\frac{a_{j+1}^0}{a_j^0} = -\frac{C_n^{n-j-1}(x_0)^{n-j-1}(y_0)^{j+1}}{C_n^{n-j}(x_0)^{n-j}(y_0)^j} = -\frac{n - j}{j + 1}\frac{y_0}{x_0}, \quad j = 1, 2 \ldots, n. \tag{3.15}$$

Hence, $g(x, y)$ can be represented as the following form:

$$\begin{aligned} g(x, y) &= Q_n(y)x^n + Q_{n-1}(y)x^{n-1} + \cdots + Q_1(y)x + Q_0(y) \\ &= a_n^0 x^n + (a_{n-1}^0 y + a_{n-1}^1)x^{n-1} + (a_{n-2}^0 y^2 + a_{n-2}^1 y + a_{n-2}^2)x^{n-2} + \cdots + (a_1^0 y^{n-1} + a_1^1 y^{n-2} + \cdots + a_1^{n-1})x + Q_0(y) \\ &= a_n^0 x^n + a_{n-1}^0 y x^{n-1} + a_{n-2}^0 y^2 x^{n-2} + \cdots + a_1^0 y^{n-1} x + a_0^0 y^n - a_0^0 y^n + Q'_{n-1}(y)x^{n-1} + Q'_{n-2}(y)x^{n-2} + \cdots + Q_0(y), \end{aligned}$$

where $Q'_j(y) = Q_j(y) - a_j^0 y^{n-j}$, $j = 1, 2, \ldots, n - 1$, and a_0^0 is a constant satisfying

$$\frac{a_1^0}{a_0^0} = -n\frac{y_0}{x_0}. \tag{3.16}$$

Denote

$$P_n(x, y) = a_n^0 x^n + a_{n-1}^0 y x^{n-1} + a_{n-2}^0 y^2 x^{n-2} + \cdots + a_1^0 y^{n-1} x + a_0^0 y^n,$$

by a simple calculation, we can deduce from (3.12)–(3.15) that

$$P_n(x, y) = a_n^0 x^n + a_{n-1}^0 y x^{n-1} + a_{n-2}^0 y^2 x^{n-2} + \cdots + a_1^0 y^{n-1} x + a_0^0 y^n = b_0(y_0 x - x_0 y)^n, \tag{3.17}$$

where

$$b_0 = \frac{a_{n-j}^0}{C_n^j (-x_0)^j y_0^{n-j}}, \quad j = 0, 1, \ldots, n.$$

Thus, we have

$$g(x, y) = P_n(x, y) + g_1(x, y) = b_0(y_0 x - x_0 y)^n + g_1(x, y), \tag{3.18}$$

where

$$g_1(x, y) = Q'_{n-1}(y)x^{n-1} + Q'_{n-2}(y)x^{n-2} + \cdots + Q_0(y) - a_0^0 y^n.$$

Noting that $P_n(x + x_0, y + y_0) - P_n(x, y) \equiv 0$, we have from (3.2) that

$$\eta = g_1(x + x_0, y + y_0) - g_1(x, y). \tag{3.19}$$

Similar to the aforementioned discussion for $g_1(x, y)$, we can obtain that

$$g(x, y) = P_n(x, y) + P_{n-1}(x, y) + g_2(x, y) = b_0(y_0 x - x_0 y)^n + b_1(y_0 x - x_0 y)^{n-1} + g_2(x, y),$$

where

$$g_2(x, y) = Q''_{n-2}(y)x^{n-2} + Q''_{n-3}(y)x^{n-3} + \dots + Q_0(y) - a_0^0 y^n - a_1^0 y^{n-1}.$$

Repeat the aforementioned discussion several times, we have

$$\begin{aligned} g(x, y) &= P_n(x, y) + P_{n-1}(x, y) + \dots + P_2(x, y) + g_{n-1}(x, y) \\ &= b_0(y_0 x - x_0 y)^n + b_1(y_0 x - x_0 y)^{n-1} + \dots + b_2(y_0 x - x_0 y)^2 + g_{n-1}(x, y), \end{aligned} \tag{3.20}$$

where

$$g_{n-1}(x, y) = a_1^{n-1}x + Q_0(y) - a_0^0 y^n - a_1^0 y^{n-1} - \dots - a_{n-2}^0 y^2 = a_1^{n-1}x + Q_0'(y). \tag{3.21}$$

Noting that $g_{n-1}(x + x_0, y + y_0) - g_{n-1}(x, y)$ is a constant, we have that $Q_0'(y)$ is a polynomial in y with degree ≤ 1. Thus, we can denote that $Q_0'(y) = b_1^{n-1}y + b_0$. Hence, we can deduce that

$$g(x, y) = b_0(y_0 x - x_0 y)^n + b_1(y_0 x - x_0 y)^{n-1} + b_2(y_0 x - x_0 y)^2 + \dots + b_{n-1}(y_0 x - x_0 y) + b_n.$$

Therefore, this completes the proof of Lemma 3.3. □

4 Proofs of Theorems 2.1 and 2.4

Here, we only give the details of the proof of Theorem 2.1 since the proof of Theorem 2.4 is similar to the proof of Theorem 2.1.

First, assume that (u, v) is a pair of transcendental entire solutions with finite-order of system (2.1). Then we conclude that $u(z + c)$, $v(z + c)$, $\alpha_1 u(z) + \beta_1 u_{z_1} + \gamma_1 u_{z_2} + \alpha_2 v(z) + \beta_2 v_{z_1} + \gamma_2 v_{z_2}$ and $\alpha_1 v(z) + \beta_1 v_{z_1} + \gamma_1 v_{z_2} + \alpha_2 u(z) + \beta_2 u_{z_1} + \gamma_2 u_{z_2}$ have no any zero and pole. Otherwise, we have a contradiction with the assumption of u, v being entire functions. Thus, by Lemmas 3.1 and 3.2, there exist two nonconstant polynomials $p(z), q(z) \in \mathbb{C}^2$ such that

$$u(z + c) = e^{p(z)}, \quad \alpha_1 u(z) + \beta_1 u_{z_1} + \gamma_1 u_{z_2} + \alpha_2 v(z) + \beta_2 v_{z_1} + \gamma_2 v_{z_2} = e^{-p(z)} \tag{4.1}$$

and

$$v(z + c) = e^{q(z)}, \quad \alpha_1 v(z) + \beta_1 v_{z_1} + \gamma_1 v_{z_2} + \alpha_2 u(z) + \beta_2 u_{z_1} + \gamma_2 u_{z_2} = e^{-q(z)}. \tag{4.2}$$

In view of (4.1) and (4.2), it follows

$$(\alpha_1 + \beta_1 p_{z_1} + \gamma_1 p_{z_2})e^{p(z)+p(z+c)} + (\alpha_2 + \beta_2 q_{z_1} + \gamma_2 q_{z_2})e^{q(z)+p(z+c)} \equiv 1 \tag{4.3}$$

and

$$(\alpha_1 + \beta_1 q_{z_1} + \gamma_1 q_{z_2})e^{q(z)+q(z+c)} + (\alpha_2 + \beta_2 p_{z_1} + \gamma_2 p_{z_2})e^{p(z)+q(z+c)} \equiv 1. \tag{4.4}$$

(i) $\alpha_1 \neq 0$. If $\alpha_1 + \beta_1 p_{z_1} + \gamma_1 p_{z_2} = 0$, then it yields from (4.3) that

$$(\alpha_2 + \beta_2 q_{z_1} + \gamma_2 q_{z_2})e^{q(z)+p(z+c)} \equiv 1,$$

which implies that $q(z) + p(z + c)$ is a constant. Set $q(z) + p(z + c) = \eta_1$, $\eta_1 \in \mathbb{C}$. Noting that $\alpha_1 + \beta_1 p_{z_1} + \gamma_1 p_{z_2} = 0$, we have

$$q_{z_1} = -p_{z_1}, \quad q_{z_2} = -p_{z_2}, \quad \alpha_1 + \beta_1 q_{z_1} + \gamma_1 q_{z_2} = 2\alpha_1. \tag{4.5}$$

By substituting (4.5) into (4.4), we have

$$2\alpha_1 e^{q(z)+q(z+c)} + (\alpha_2 - \beta_2 q_{z_1} - \gamma_2 q_{z_2})e^{p(z)+q(z+c)} \equiv 1. \tag{4.6}$$

Obviously, $\alpha_2 - \beta_2 q_{z_1} - \gamma_2 q_{z_2} \not\equiv 0$. Otherwise, it yields from (4.6) that $2\alpha_1 e^{q(z)+q(z+c)} \equiv 1$, which is impossible since $q(z) + q(z+c)$ are nonconstant polynomial. Thus, by applying the second basic theorem for the function $2\alpha_1 e^{q(z)+q(z+c)}$ and combining with $\alpha_1 \neq 0$, we have

$$\begin{aligned} T(r,G) &\le N(r,G) + N\left(r,\frac{1}{G}\right) + N\left(r,\frac{1}{G-1}\right) + S(r,G) \\ &\le N\left(r,\frac{1}{(\alpha_2 - \beta_2 q_{z_1} - \gamma_2 q_{z_2})e^{p(z)+q(z+c)}}\right) + S(r,G) \\ &\le S(r,G) + \log r, \end{aligned}$$

where $G = 2\alpha_1 e^{q(z)+q(z+c)}$, this is impossible.

If $\alpha_1 + \beta_1 p_{z_1} + \gamma_1 p_{z_2} \not\equiv 0$, then $\alpha_2 + \beta_2 q_{z_1} + \gamma_2 q_{z_2} \not\equiv 0$. Otherwise, it yields from (4.3) that $(\alpha_1 + \beta_1 p_{z_1} + \gamma_1 p_{z_2})e^{p(z)+p(z+c)} \equiv 1$, which implies that $p(z) + p(z+c)$ is a constant. This is a contradiction with the assumption of p being a nonconstant polynomial. Similarly, by applying the second basic theorem for the function $(\alpha_1 + \beta_1 p_{z_1} + \gamma_1 p_{z_2})e^{p(z)+p(z+c)}$, we can also obtain a contradiction. Hence, the system (2.1) has no any pair of transcendental entire solution with finite-order for $\alpha_1 \neq 0$.

(ii) $\alpha_1 = 0$. In view of (4.3) and (4.4), we have

$$\beta_1 p_{z_1} + \gamma_1 p_{z_2} = 0, \quad (\alpha_2 + \beta_2 q_{z_1} + \gamma_2 q_{z_2})e^{q(z)+p(z+c)} \equiv 1. \tag{4.7}$$

Noting that the first equation of (4.7), we have $p(z) = \phi(\gamma_1 z_1 - \beta_1 z_2)$, where $\phi(x)$ is a nonconstant polynomial. By observing the second equation of (4.7), we can conclude that $q(z) + p(z+c)$ must be a constant. Thus, it follows that $p_{z_1} = -q_{z_1}$ and $p_{z_2} = -q_{z_2}$. It yields from (4.4) that $\beta_1 q_{z_1} + \gamma_1 q_{z_2} = 0$ and

$$(\alpha_2 + \beta_2 p_{z_1} + \gamma_2 p_{z_2})e^{p(z)+q(z+c)} \equiv 1. \tag{4.8}$$

In view of (4.7) and (4.8), we have $p(z) + q(z+c) = \eta_1$ and $q(z) + p(z+c) = \eta_2$ where η_1 and η_2 are constants. It follows that $p(z+2c) - p(z) = \eta_2 - \eta_1$ and $q(z+2c) - q(z) = \eta_1 - \eta_2$. By Lemma 3.3, we have

$$p(z) = L(z) + B_1 + H(c_2 z_1 - c_1 z_2), \quad q(z) = -L(z) - H(c_2 z_1 - c_1 z_2) + B_2, \tag{4.9}$$

where $H(s)$ is a polynomial in $s \coloneqq c_2 z_1 - c_1 z_2$, $L(z) = A_1 z_1 + A_2 z_2$, $A_1, A_2, B_1, B_2 \in \mathbb{C}$. By substituting $p(z) = \phi(\gamma_1 z_1 - \beta_1 z_2)$ into (4.8), we have

$$[\alpha_2 + (\beta_2\gamma_1 - \gamma_2\beta_1)\phi']e^{\eta_1} \equiv 1.$$

Noting that $D \coloneqq \beta_1\gamma_2 - \gamma_1\beta_2 \neq 0$, we have $\deg_x \phi(x) \le 1$. By combining with (4.9), we have $\deg_s H = n \le 1$. Thus, we can still denote that

$$p(z) = L(z) + B_1, \quad q(z) = -L(z) + B_2. \tag{4.10}$$

By substituting (4.10) into (4.7) and (4.8), we have

$$\begin{cases} (\alpha_2 - \beta_2 A_1 - \gamma_2 A_2)e^{-L(c)+B_1+B_2} \equiv 1, \\ (\alpha_2 + \beta_2 A_1 + \gamma_2 A_2)e^{L(c)+B_1+B_2} \equiv 1. \end{cases} \tag{4.11}$$

In view of (4.7), (4.8), and (4.10), we have $\beta_1 A_1 + \gamma_1 A_2 = 0$ and

$$e^{2L(c)} = \frac{\alpha_2 - \beta_2 A_1 - \gamma_2 A_2}{\alpha_2 + \beta_2 A_1 + \gamma_2 A_2}, \quad e^{2(B_1+B_2)} = \frac{1}{\alpha_2^2 - (\beta_2 A_1 + \gamma_2 A_2)^2}. \tag{4.12}$$

And in view of (4.1), (4.2), (4.11), and (4.12), we can conclude that

$$\begin{cases} f = e^{p(z-c)} = e^{L(z)+B_1-L(c)} = \dfrac{1}{\alpha_2 + \beta_2 A_1 + \gamma_2 A_2} e^{A_1z_1+A_2z_2-B_2}, \\ g = e^{q(z-c)} = e^{-L(z)+L(c)+B_2} = \dfrac{1}{\alpha_2 - \beta_2 A_1 - \gamma_2 A_2} e^{-A_1z_1-A_2z_2-B_1}. \end{cases} \tag{4.13}$$

Therefore, this completes the proof of Theorem 2.1.

5 Proofs of Theorems 2.2 and 2.3

Here, we only give the details of the proof of Theorem 2.3 since the proof of Theorem 2.3 is similar to the proof of Theorem 2.2.

Let (u, v) be a pair of transcendental entire solutions with finite-order of system (2.5). Similar to the arguments as in Theorem 2.1, there exist two nonconstant polynomials $p(z), q(z) \in \mathbb{C}^2$ such that

$$u(z + c) = e^{p(z)}, \quad u_{z_1} + u_{z_2} + v(z) + v_{z_1} + v_{z_2} = e^{-p(z)} \tag{5.1}$$

and

$$v(z + c) = e^{q(z)}, \quad v_{z_1} + v_{z_2} + u(z) + u_{z_1} + u_{z_2} = e^{-q(z)}. \tag{5.2}$$

In view of (5.1) and (5.2), we have

$$(p_{z_1} + p_{z_2})e^{p+p(z+c)} + (1 + q_{z_1} + q_{z_2})e^{q+p(z+c)} \equiv 1 \tag{5.3}$$

and

$$(q_{z_1} + q_{z_2})e^{q+q(z+c)} + (1 + p_{z_1} + p_{z_2})e^{p+q(z+c)} \equiv 1. \tag{5.4}$$

If $p_{z_1} + p_{z_2} \neq 0$, by using the second basic theorem of Nevanlinna, we can obtain a contradiction easily.

If $p_{z_1} + p_{z_2} \equiv 0$, then $p(z) = \phi(z_1 - z_2)$, where $\phi(x)$ is a nonconstant polynomial in x and

$$(1 + q_{z_1} + q_{z_2})e^{q+p(z+c)} \equiv 1, \tag{5.5}$$

which implies that $q + p(z + c)$ is a constant. Thus, it follows that $p_{z_1} = -q_{z_1}$ and $p_{z_2} = -q_{z_2}$. This leads to $q_{z_1} + q_{z_2} = 0$ and

$$e^{q+p(z+c)} \equiv 1, \quad e^{p+q(z+c)} \equiv 1. \tag{5.6}$$

Thus, by Lemma 3.3, we have

$$p(z) = \phi(z_1 - z_2) + B_1, \quad q(z) = -\phi(z_1 - z_2) + B_2, \tag{5.7}$$

where B_1 and B_2 are constants.

If $c_1 = c_2$, then it yields from (5.6) and (5.7) that $e^{B_1+B_2} = 1$. By combining with (5.1) and (5.2), we can deduce

$$u(z) = e^{\phi(z_1-z_2)+B_1}, \quad v(z) = e^{-\phi(z_1-z_2)-B_1}, \tag{5.8}$$

where B_1 is a constant.

If $c_1 \neq c_2$, noting that $p(z + 2c) - p(z)$ and $q(z + 2c) - q(z)$ are constants, it follows that $p(z) = A(z_1 - z_2) + B_1$ and $q(z) = -A(z_1 - z_2) + B_2$, where A, B_1, and B_2 are constants satisfying

$$e^{2A(c_1-c_2)} = 1, \quad e^{2(B_1+B_2)} = 1.$$

Thus, we can deduce from (5.1) and (5.2) that

$$u(z) = e^{A(z_1-z_2)-B_2}, \quad v(z) = e^{-A(z_1-z_2)-B_1}, \tag{5.9}$$

where B_1 and B_2 is a constant.

Therefore, this completes the proof of Theorem 2.3.

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