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Research Article

Fayyaz Ahmad, Kifayat Ullah, Junaid Ahmad, Hasanen A. Hammad*, and Reny George

A novel iterative process for numerical reckoning of fixed points via generalized nonlinear mappings with qualitative study

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Abstract: This manuscript is devoted to constructing a novel iterative scheme and reckoning of fixed points for generalized contraction mappings in hyperbolic spaces. Also, we establish Δ and strong convergence results by the considered iteration under the class of mappings satisfying condition (E). Moreover, some qualitative results of the suggested iteration, like weak w^2-stability and data dependence results, are discussed. Furthermore, to test the efficiency and effectiveness of the proposed iteration, practical experiments are given. To support the theoretical results, illustrative examples are presented. Finally, our results improve and generalize several classical results in the literature of fixed point iterations.

Keywords: iteration process, fixed point, convergence result, data dependence, stability, hyperbolic space

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1 Introduction and preliminaries

Fixed points are considered as the pillar of nonlinear analysis in general and semantic analysis in particular, as they are the bright side that made the turnout for these branches very large by many writers, due to their ease and flexibility. It is common that a self-map J defined on a nonempty subset A of a metric space H is called a Banach-contraction mapping (or a contraction mapping) if for every two elements $a, b \in A$, there exists at least one real number $\lambda \in [0, 1)$ such that

$$\varrho(Ja, Jb) \leq \lambda\varrho(a, b). \tag{1.1}$$

Clearly, if the inequality (1.1) holds for $\lambda = 1$, then J is called a nonexpansive mapping. Once a point a^* is available in the domain A with the property $Ja^* = a^*$, then the point is called a fixed point for J and the relation $Ja = a$ is called a fixed point problem. We denote the set of all fixed points of the mapping J by F_J.

* **Corresponding author: Hasanen A. Hammad,** Department of Mathematics, College of Sciences, Qassim University, Buraydah 52571, Saudi Arabia; Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt, e-mail: hassanein_hamad@science.sohag.edu.eg
Fayyaz Ahmad: Department of Mathematical Sciences, University of Lakki Marwat, Lakki Marwat 28420, Khyber Pakhtunkhwa, Pakistan, e-mail: fayyazahmadmath@gmail.com
Kifayat Ullah: Department of Mathematical Sciences, University of Lakki Marwat, Lakki Marwat 28420, Khyber Pakhtunkhwa, Pakistan, e-mail: kifayat@ulm.edu.pk
Junaid Ahmad: Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad - 44000, Pakistan, e-mail: junaid.phdma126@iiu.edu.pk
Reny George: Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam bin Abdulaziz University, Alkharj, 11942 Saudi Arabia, e-mail: r.kunnelchacko@psau.edu.sa

Almost all problems in engineering, mathematical physics, chemistry, and other subjects can be written as fixed-point problems.

In 1922, Banach [1] carried out a basic fixed point theorem for contractions called the Banach contraction principle (BCP). The BCP is important because it suggests the existence and uniqueness of a fixed point and, at the same time, suggests an iterative process called the Picard iteration for calculating this fixed point. This principle is widely applied for finding sought solutions of differential, integral and other types of equations in applied sciences [2–5].

On the other hand, it was an open question until 1965 whether a nonexpansive mapping admits a fixed point in the closed subset of a complete metric space and, if it admits a fixed point, whether the Picard iteration converges to this fixed point. Thus, in the same year, Browder [6], Gohde [7], and Kirk [8] proposed a basic fixed point theorem for nonexpansive mappings in a Banach space setting.

Osilike [9] introduced a generalization of Banach-contraction as follows: The mapping J is called a generalized contraction if for every $a, b \in A$, there exist $\lambda \in [0, 1)$ and $L \in [0, \infty)$ such that

$$\varrho(Ja, Jb) \leq \lambda\varrho(a, b) + L\varrho(a, Ja). \tag{1.2}$$

Clearly, the Banach contraction mapping satisfies equation (1.2) with $L = 0$, and the class of generalized contractions contains properly the class of the Banach contraction mapping.

In 2008, Suzuki generalized nonexpansive mappings as follows: A mapping J is said to satisfy the condition (C) if, for all $a, b \in A$, one has

$$\frac{1}{2}\varrho(a, Ja) \leq \varrho(a, b) \Rightarrow \varrho(Ja, Jb) \leq \varrho(a, b).$$

Recently, Garcia-Falset et al. [10] generalized the above notion as follows: The mapping J is said to satisfy the condition (E) if, for all $a, b \in A$, there exists $\mu \geq 1$ such that

$$\varrho(a, Jb) \leq \mu\varrho(Ja, Jb) + \varrho(a, b).$$

The same authors suggested the following properties of mappings, which satisfy condition (E):

Proposition 1.1. *Let $J : A \to A$ be a given mapping. Then,*
(*i*) *if J satisfies condition (C), then J satisfies the condition (E);*
(*ii*) *if J satisfies the condition (E), then for all $a \in A$ and for all $a^* \in F_J$, one has $\varrho(Ja, Ja^*) \leq \varrho(a, a^*)$;*
(*iii*) *if J satisfies the condition (E), then the set F_J is essentially closed in A.*

The idea of a hyperbolic space introduced by Kohlenbach [11] is as follows:

Definition 1.2. Let (H, ϱ) be a metric space, then (H, ϱ, S) is called a hyperbolic space if the function $\mathcal{S} : H \times H \times [0, 1] \to H$ fulfills
(S_1) $\varrho(r, \mathcal{S}(a, b, \alpha)) \leq \alpha\varrho(r, a) + (1 - \alpha)\varrho(r, b)$;
(S_2) $\varrho(\mathcal{S}(a, b, \alpha), \mathcal{S}(a, b, \beta)) \leq |\alpha - \beta|\varrho(a, b)$;
(S_3) $\mathcal{S}(a, b, \alpha) = \mathcal{S}(b, a, 1 - \alpha)$;
(S_4) $\varrho(\mathcal{S}(a, r, \alpha), \mathcal{S}(b, s, \alpha)) \leq \alpha\varrho(a, b) + (1 - \alpha)\varrho(r, s)$

for all $a, b, r, s \in H$ and $0 \leq \alpha, \beta \leq 1$.

A Banach space is an example of a hyperbolic metric space that is linear. Examples of this type of space, that are nonlinear, include Hadamard manifolds and the Hilbert open unit ball. The hyperbolic space H defined above is more general than the hyperbolic spaces in the sense of Takahashi [12], Goebel and Kirk [13], and Reich and Shafrir [14]. Hence, our results will be simultaneously valid in the other hyperbolic space mentioned above. All norm space (and hence all inner product spaces), Hadamarad manifolds, and metric trees are the well-known examples of hyperbolic spaces [15,16]. In this case, for any pair of elements $a, b \in H$ and any scalar $\alpha \in [0, 1]$, the following holds:

$$\varrho(a, \mathcal{S}(a, b, \alpha)) = \alpha\varrho(a, b) \quad \text{and} \quad \varrho(a, \mathcal{S}(a, b, \alpha)) = (1 - \alpha)\varrho(a, b).$$

It follows that

$$\mathcal{S}(a, b, 1) = a \quad \text{and} \quad \mathcal{S}(a, b, 0) = b. \tag{1.3}$$

Definition 1.3. A subset A of a hyperbolic space H is called convex, if and only if $\mathcal{S}(a, b, \alpha) \in A$ for all $a, b \in A$ and $\alpha \in [0, 1]$.

Definition 1.4. [17] Assume that (H, ϱ) is hyperbolic space. We say that H is a uniformly convex if for any $0 < r < \infty$ and for each $0 < \varepsilon \leq 2$, there is a $0 < \theta < 1$ in the way that for any choice of $a, u, v \in H$ with $\varrho(u, a) \leq r$, $\varrho(v, a) \leq r$, and $\rho(u, v) \geq \varepsilon r$, we have

$$\frac{\varrho\left(\mathcal{S}\left(u, v, \frac{1}{2}\right), a\right)}{(1 - \theta)} \leq r.$$

Definition 1.5. [17] A monotone modulus of uniform convexity (MMUC) is a function Y on $(0, \infty) \times (0, 2)$ to the interval $(0, 1]$ that satisfy $\eta = Y(r, \varepsilon)$ for any choice of $0 < r < \infty$ and $0 < \varepsilon < 2$ and Y is monotone (i.e., Y decreases with r and ε).

Recently, Deshmukh et al. [18] proposed two new iterative schemes called JF and Thakur iterations for finding fixed points of a general class of nonexpansive mappings. Now, we assume that $\alpha_r, \beta_r, \gamma_r \subset (0, 1)$, then by using hypothesis $(\mathcal{S}_3)$ and equation (1.3), Mann [19], Khan and Picard-Mann [20], Deshmukh et al. [18], Ullah and Arshad [21], Ali and Ali [22], and Deshmukh et al. [18] iterative processes are, respectively, defined as follows (see Ahmad et al. [23], Sahin [24], and others for details):

$$\begin{cases} a_1 \in A, \\ a_{r+1} = \mathcal{S}(Ja_r, a_r, \alpha_r), \quad r \geq 1, \end{cases} \tag{1.4}$$

$$\begin{cases} a_1 \in A, \\ b_r = \mathcal{S}(Ja_r, a_r, \alpha_r), \\ a_{r+1} = Jb_r, r \geq 1. \end{cases} \tag{1.5}$$

$$\begin{cases} a_1 \in A, \\ c_r = \mathcal{S}(Ja_r, a_r, \gamma_r), \\ b_r = \mathcal{S}(Jc_r, c_r, \beta_r), \\ a_{r+1} = \mathcal{S}(Jb_r, Jc_r, \alpha_r), r \geq 1. \end{cases} \tag{1.6}$$

$$\begin{cases} a_1 \in A, \\ c_k = \mathcal{S}(Ja_r, a_r, \alpha_r), \\ b_r = Jc_r, \\ a_{r+1} = Jb_r, r \geq 1. \end{cases} \tag{1.7}$$

$$\begin{cases} a_1 \in A, \\ d_r = \mathcal{S}(Ja_r, a_r, \alpha_r), \\ c_r = Jd_r, \\ b_r = Jc_r, \\ a_{r+1} = Jb_r, r \geq 1. \end{cases} \tag{1.8}$$

$$\begin{cases} a_1 \in A, \\ e_r = \mathcal{S}(Ja_r, a_r, \alpha_r), \\ d_r = Je_r, \\ c_r = Jd_r, \\ b_r = \mathcal{S}(Jc_r, c_r, \beta_r), \\ a_{r+1} = Jb_r, r \geq 1. \end{cases} \tag{1.9}$$

Based on iterations (1.7) and (1.8), we introduce the following iteration:

$$\begin{cases} a_1 \in A, \\ e_r = \mathcal{S}(Ja_r, a_r, \alpha_r), \\ d_r = Je_r, \\ c_r = Jd_r, \\ b_r = Jc_r, \\ a_{r+1} = Jb_r, r \geq 1. \end{cases} \tag{1.10}$$

In addition, we establish strong convergence and Δ results for the class of mappings satisfying the condition (E) under considered iteration. Also, some qualitative results of the suggested iteration are examined, including weak w^2-stability and data dependence results. In addition, real-world tests are conducted to evaluate the effectiveness and efficiency of the suggested iteration, and illustrative examples are provided to corroborate the theoretical results. Ultimately, our results extend and generalize a number of prior results in the literature on fixed-point iterations.

Now, we need some basic definitions and results as follows.

Definition 1.6. Assume that H is a uniformly convex hyperbolic space (UCHS), $\{a_r\}$ is a bounded sequence of H and A is closed convex subset of H. We shall denote the asymptotic radius of $\{a_r\}$ with respect to A by $R(A, \{a_r\})$ and defined by $\inf\{\limsup_{r\to\infty}\varrho(a_r, a) : a \in A\}$. Also, we denote the asymptotic center of $\{a_r\}$ with respect to A by $C(A, \{a_r\})$ and it is defined by $\{a \in A : \limsup_{r\to\infty}\varrho(a_r, a) = R(A, a_r)\}$.

Definition 1.7. [25] Let A be a nonempty subset of H. We say that a mapping $J : A \to A$ satisfies condition (I) if there exists a nondecreasing mapping $g : A \to A$ such that $g(c) = 0$ iff $c = 0$, $g(c) > 0$ for each $c > 0$ and $\varrho(a, Ja) \geq g(\varrho(a, F_J))$.

Lemma 1.8. [26] *Let A be a closed convex subset of a UCHS H with an MMUC and $\{a_r\}$ be any bounded sequence in H. Then, the set $C(A, \{a_r\})$ contains a singleton point.*

Definition 1.9. Let H be a UCHS. A sequence $\{a_r\} \subset H$ is said to be Δ-convergent to a point $a^* \in H$, if $\{u_r\}$ is any subsequence of $\{a_r\}$ and a^* is a unique asymptotic center of $\{u_r\}$.

Lemma 1.10. [27] *Assume that H is a UCHS with an MMUC. If $0 < i \leq \sigma_r \leq j < 1$ such that $\{a_r\}$ and $\{b_r\}$ in H satisfy the conditions* $\limsup_{r\to\infty}\varrho(a_r, a^*) \leq \xi$, $\limsup_{r\to\infty}\varrho(b_r, a^*) \leq \xi$, *and* $\lim_{r\to\infty}\varrho(\mathcal{S}(b_r, a_r, \sigma_r), a^*) = \xi$, *where $\xi \geq 0$ is a real number. Then,* $\lim_{r\to\infty}\varrho(a_r, b_r) = 0$.

2 Convergence results

This section is devoted to approximating fixed points of mappings with condition (E). From here to the end of the article, we shall consider H is a complete UCHS and has an MMUC. We begin this section with the following lemma:

Lemma 2.1. *Let A be a closed convex subset of H and J be a self-mapping on A. If J satisfies condition (E) and $F_J \neq \emptyset$, then the sequence $\{a_r\}$ produced by equation* (1.10) *is essentially Δ-convergent to an element of the set F_J.*

Proof. We split the proof into the following parts:

Part 1. Assume that $a^* \in F_J$, we will prove that

$$\lim_{r\to\infty} \varrho(a_r, a^*) \tag{2.1}$$

exists. Since $a^* \in F_J$, then by Proposition 1.1(ii), we have

$$\begin{aligned} \varrho(e_r, a^*) &= \varrho(\mathcal{S}(Ja_r, a_r, \alpha_r), a^*) \\ &\le \alpha_r \varrho(Ja_r, a^*) + (1 - \alpha_r)\varrho(a_r, a^*) \\ &\le \alpha_r \varrho(a_r, a^*) + (1 - \alpha_r)\varrho(a_r, a^*) \\ &= \varrho(a_r, a^*). \end{aligned} \tag{2.2}$$

Using equation (2.2), we obtain

$$\begin{aligned} \varrho(a_{r+1}, a^*) &= \varrho(Jb_r, a^*) \le \varrho(b_r, a^*) \\ &= \varrho(Jc_r, a^*) \le \varrho(c_r, a^*) \\ &= \varrho(Jd_r, a^*) \le \varrho(d_r, a^*) \\ &= \varrho(Je_r, a^*) \le \varrho(e_r, a^*) \\ &\le \varrho(a_r, a^*). \end{aligned} \tag{2.3}$$

Hence,

$$\varrho(a_{r+1}, a^*) \le \varrho(a_r, a^*).$$

It follows that the sequence $\{\varrho(a_r, a^*)\}$ is bounded and nonincreasing, which implies that equation (2.1) is true.

Part 2. We show that

$$\lim_{r\to\infty} \varrho(a_r, Ja_r) = 0. \tag{2.4}$$

Now, for any $a^* \in F_J$ of F_J, put

$$\lim_{r\to\infty} \varrho(a_r, a^*) = \xi. \tag{2.5}$$

Using equation (2.1) and Proposition 1.1(ii), one has

$$\varrho(Ja_r, a^*) \le \varrho(a_r, a^*).$$

By equation (2.5), we have

$$\limsup_{r\to\infty} \varrho(Ja_r, a^*) \le \xi. \tag{2.6}$$

From equation (2.2), we obtain

$$\varrho(e_r, a^*) \le \varrho(a_r, a^*),$$

Using equation (2.5), we can obtain

$$\limsup_{r\to\infty} \varrho(e_r, a^*) \le \xi. \tag{2.7}$$

Similarly, from equation (2.3), one can write

$$\varrho(a_{r+1}, a^*) \le \varrho(e_r, a^*),$$

From equation (2.5), one has

$$\xi \le \liminf_{r\to\infty} \varrho(e_r, a^*). \tag{2.8}$$

From equations (2.7) and (2.8), one obtains

$$\xi = \lim_{r\to\infty} \varrho(e_r, a^*). \tag{2.9}$$

Since $e_r = \mathcal{S}(Ja_r, a_r, a_r)$, so applying it in equation (2.9), we conclude that

$$\lim_{r\to\infty} \varrho(\mathcal{S}(Ja_r, a_r, a_r), a^*) = \xi. \tag{2.10}$$

Considering equations (2.5), (2.6), and (2.10) with the Lemma 1.10, we have equation (2.4).

Part 3. We prove the Δ convergence of the sequence $\{a_r\}$ toward the fixed point of J. As the sequence $\{a_r\}$ is bounded in A, so according to Lemma 1.8, $\{a_r\}$ admits a unique asymptotic center $C(A, \{a_r\}) = \{a_0\}$. Suppose that $\{g_r\}$ is any subsequence of the sequence $\{a_r\}$ such that $C(A, \{g_r\}) = \{g_0\}$. Then, by equation (2.4), we have

$$\lim_{r\to\infty} \varrho(g_r, Jg_r) = 0. \tag{2.11}$$

Now, we claim that the element g_0 is belonging to F_J. Since J satisfies condition (E), we obtain

$$\varrho(g_r, Jg_0) \leq \mu\varrho(g_r, Jg_r) + \varrho(g_r, g_0) \quad \text{for some } \mu \geq 1.$$

Taking $\limsup_{r\to\infty}$ in equation (2.11) and in the above inequality, one has

$$\begin{aligned} R(\{g_r\}, Jg_0) &= \limsup_{r\to\infty} \varrho(g_r, Jg_0) \\ &\leq \limsup_{r\to\infty} \varrho(g_r, g_0) \\ &= R(\{g_r\}, g_0). \end{aligned}$$

Since the asymptotic center contains only one point, we have $g_0 = Jg_0$, i.e., g_0 is an element of the set F_J. Putting $g_0 \neq a_0$. So according to equation (2.1), $\lim_{r\to\infty} \varrho(a_r, g_0)$ exists. But in our case, the asymptotic center contains only one point. Thus,

$$\begin{aligned} \limsup_{r\to\infty} \varrho(g_r, g_0) &< \limsup_{r\to\infty} \varrho(g_r, a_0) \\ &\leq \limsup_{r\to\infty} \varrho(a_r, a_0) \\ &< \limsup_{r\to\infty} \varrho(a_r, g_0) \\ &= \limsup_{r\to\infty} \varrho(g_r, g_0). \end{aligned}$$

Subsequently, we proved that $\limsup_{r\to\infty} \varrho(g_r, g_0) < \limsup_{r\to\infty} \varrho(g_r, g_0)$, which is a clear contradiction. This implies that g_0 is the unique asymptotic center for any sub-sequence of $\{a_r\}$. This proves the Δ convergence of the sequence $\{a_r\}$ in F_J. □

Theorem 2.2. *Suppose that A is a compact convex subset of H and J is a self-mapping on A. If J satisfies condition* (E) *and* $F_J \neq \varnothing$, *then the sequence* $\{a_r\}$ *generated by equation* (1.10) *is essentially strongly convergent to an element of the set* F_J.

Proof. Since A is convex, then $\{a_r\}$ is completely continuous in A. Applying compactness property of A, we can find a subsequence $\{a_{r_l}\}$ of $\{a_r\}$ that satisfies $\lim_{l\to\infty} \varrho(a_{r_l}, a^*) = 0$, where a^* is some point of A. As J satisfies condition (E), then, we have

$$\varrho(a_{r_l}, Ja^*) \leq \mu\varrho(a_{r_l}, Ja_{r_l}) + \varrho(a_{r_l}, a^*) \tag{2.12}$$

for each $\mu \geq 1$. In view of equation (2.4), we obtain $\lim_{l\to\infty} \varrho(a_{r_l}, Ja_{r_l}) = 0$.

Using $\lim_{l\to\infty} \varrho(a_{r_l}, Ja_{r_l}) = 0$ and keeping $\lim_{l\to\infty} \varrho(a_{r_l}, y_0) = 0$ in our mind and applying the limit on both sides of equation (2.12), we obtain $\lim_{l\to\infty} \varrho(a_{r_l}, Ja^*) = 0$.

By uniqueness of the limit of a convergent sequence in a metric space, we can write $Ja^* = a^*$, which implies that $a^* \in F_J$.

Hence, according to equation (2.1), we conclude that $\lim_{r\to\infty}\varrho(a_r, a^*)$ exists. Subsequently, a^* is the point of F_J and the sequence $\{a_r\}$ is strongly convergent to a^*. This completes the proof. □

We now include an example of condition (E) on a hyperbolic space that is not linear.

Example 2.3. Consider $H = \{(a_1, a_2) \in \mathbb{R}^2 : a_1, a_2 > 0\}$. Set a metric ϱ on H as $\varrho(a, b) = |a_1 - b_1| + |a_1a_2 - b_1b_2|$ for each $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in H. Then, ϱ form a metric function on H. Now, for any $\beta \in [0, 1]$, we set a function $\mathcal{S}$ as follows:

$$\mathcal{S}(a, b, \beta) = \left((1 - \beta)a_1 + \beta b_1, \frac{(1 - \beta)a_1a_2 + \beta b_1b_2}{(1 - \beta)a_1 + \beta b_1}\right).$$

Then, (H, ϱ, β) is hyperbolic metric space. Assume that $A = [0.5, 3] \times [0.5, 3]$, which is a compact and closed subset of H, and set a mapping J on A as: $J(a_1, a_2) = (1, 1)$ if $(a_1, a_2) \neq (0.5, 0.5)$ and $J(a_1, a_2) = (1.5, 1.5)$ if $(a_1, a_2) = (0.5, 0.5)$. Then J is not Suzuki mapping as $\frac{1}{2}\varrho(a, Ja) \leq \varrho(a, b)$ but $\varrho(Ja, Jb) > \varrho(a, b)$ for $a = (0.5, 0.5)$ and $b = (1.1, 1.1)$. However, J satisfies the condition (E) with $\mu = 7$. Hence, all the requirements for Theorem 2.2 are satisfied. Hence, the sequence $\{a_r\}$ generated by equation (1.10) converges to the fixed point (1, 1). Note that, we cannot apply directly classical results of the literature to this example because this example is not a linear space and is only a hyperbolic space.

Theorem 2.4. *Assume that A is a closed convex subset of H and J is a self-mapping on A. If J satisfies condition* (*E*) *and* $F_J \neq \emptyset$. *Then, the sequence* $\{a_r\}$ *obtained by equation* (1.10) *is essentially strongly convergent to an element of the set* F_J, *provided that the condition* $\liminf_{r\to\infty} d(a_r, F_J) = 0$ *holds.*

Proof. For an arbitrary point $a^* \in F_J$, we proved that $\lim_{r\to\infty}\varrho(a_r, a^*)$ exists. From our assumption, one has

$$\lim_{r\to\infty}\varrho(a_r, F_J) = 0.$$

Based on Proposition 1.1(iii), we have that F_J is closed. The remaining proof follows immediately by [25, Theorem 2]. □

A strong convergence under the condition (I) is as follows:

Theorem 2.5. *Assume that A is a closed convex subset of H and J is a self-mapping on A. If J satisfies condition* (*E*) *and* $F_J \neq \emptyset$, *then the sequence* $\{a_r\}$ *obtained by equation* (1.10) *is essentially strongly convergent to an element of the set,* F_J *provided that the mapping J satisfies condition* (*I*).

Proof. According to equation (2.4), we can write

$$\liminf_{r\to\infty}\varrho(a_r, Ja_r) = 0. \tag{2.13}$$

But J is enriched with the condition (I), then

$$\varrho(a_r, Ja_r) \geq g(\varrho(a_r, F_J)). \tag{2.14}$$

Using equations (2.13) and (2.14), one has

$$\liminf_{r\to\infty} g(\varrho(a_r, F_J)) = 0.$$

But $g(0) = 0$, so eventually, we obtain

$$\liminf_{r\to\infty}\varrho(a_r, F_J) = 0.$$

Therefore, all requirements of Theorem 2.4 are satisfied. Thus, $\{a_r\}$ converges strongly to a point in F_J. This finishes the proof. □

Theorem 2.6. *Assume that A is a closed subset of H and J is a self-mapping on A. If J satisfies* (1.2) *with $F_J = \{a^*\}$, then the sequence $\{a_r\}$ generated by equation* (1.10) *is essentially strongly convergent to an element of the set F_J if $\sum_{r=1}^{\infty}\alpha_r = \infty$.*

Proof. Using ($\mathcal{S}_1$) and equation (1.10), one has

$$\begin{aligned}\varrho(e_r, a^*) &= \varrho(\mathcal{S}(Ja_r, a_r, \alpha_r), a^*) \le \alpha_r\varrho(Ja_r, a^*) + (1 - \alpha_r)(a_r, a^*)\\ &\le (\alpha_r(\lambda\varrho(a_r, a^*) + L\varrho(a^*, Ja^*))) + (1 - \alpha_r)\varrho(a_r, a^*)\\ &= (\alpha_r\lambda + (1 - \alpha_r))\varrho(a_r, a^*)\\ &= (1 - \alpha_r(1 - \lambda))\varrho(a_r, a^*).\end{aligned}$$

This implies that

$$\begin{aligned}\varrho(a_{r+1}, a^*) &= \varrho(Jb_r, a^*) \le \lambda\varrho(b_r, a^*) + L\varrho(a^*, Ja^*)\\ &= \lambda\varrho(b_r, a^*) = \lambda\varrho(Jc_r, a^*)\\ &\le \lambda(\lambda\varrho(c_r, a^*) + L\varrho(c_0, Ja^*))\\ &= (\lambda)^2\varrho(c_r, a^*) = (\lambda)^2\varrho(Jd_r, a^*)\\ &\le (\lambda)^2((\lambda)\varrho(d_r, a^*) + L\varrho(a^*, Ja^*))\\ &= (\lambda)^3\varrho(d_r, a^*) = (\lambda)^3\varrho(Je_r, a^*)\\ &\le (\lambda)^3(\lambda\varrho(e_r, a^*) + L\varrho(a^*, Ja^*))\\ &= (\lambda)^4\varrho(e_r, a^*)\\ &\le (\lambda)^4(1 - \alpha_r(1 - \lambda))\varrho(a_r, a^*).\end{aligned}$$

It follows that

$$\begin{aligned}\varrho(a_{r+1}, a^*) &\le (\lambda)^4(1 - \alpha_r(1 - \lambda))\varrho(a_r, a^*)\\ &\le (\lambda)^4(1 - \alpha_r(1 - \lambda))(\lambda)^4(1 - \alpha_{r-1}(1 - \lambda))\varrho(a_{r-1}, a^*)\\ &\ \ \vdots\\ &\le ((\lambda)^4)^r\Pi_{m=1}^{r}(1 - \alpha_m(1 - \lambda))\varrho(a_1, a^*).\end{aligned}$$

Consequently, we obtain

$$\varrho(a_{r+1}, a^*) \le ((\lambda)^4)^r\Pi_{m=1}^{r}(1 - \alpha_m(1 - \lambda))\varrho(a_1, a^*). \tag{2.15}$$

Using equation (2.15) and the fact $1 - a \le e^{-a}$ for each $a \in [0, 1]$, we have

$$\varrho(a_{r+1}, a^*) \le ((\lambda)^4)^r e^{-(1-\lambda)\sum_{m=1}^{r}\alpha_r}\varrho(a_1, a^*).$$

Since $\lambda \in (0, 1)$, then by the assumption $\sum_{r=1}^{\infty}\alpha_r = \infty$, one can write

$$\lim_{r\to\infty}\varrho(a_{r+1}, a^*) = 0,$$

which implies that

$$\lim_{r\to\infty} a_r = a^* \in F_J.$$

Therefore, the sequence $\{a_r\}$ is strongly convergent to a point in F_J. □

Now, to support the above results, we construct the following examples:

Example 2.7. Let $A = [0, 1]$ and $J : A \to A$ be a mapping described as follows:

$$J = \begin{cases} 0, & \text{if } 0 \le b < \dfrac{1}{55},\\ \dfrac{16b}{17}, & \text{if } \dfrac{1}{55} \le b \le 1.\end{cases}$$

Now, we illustrate the following:
(a) The set F_J is nonempty.
(b) The mapping J is neither Suzuki nonexpansive nor nonexpansive on A;
(c) The mapping J satisfies condition (E).

For any $a^* \in F_J$, we obtain $a^* = 0$, which implies that $F_J = \{0\}$ and proves (a). Next, choose $a = \frac{1}{55}$ and $b = \frac{1}{100}$, then it is easy to see that (b) is true. Ultimately, for (c), take $\mu = 17$, then we realized the following cases:

Case (i): Consider $a, b \in [0, \frac{1}{55})$. It follows that $Ja = Jb = 0$ and

$$\begin{aligned}\varrho(a, Jb) &= |a - Jb| = |a| \leq 17|a| = 17|a - Ja| \\ &\leq 17|a - Ja| + |a - b| \\ &= 17\varrho(a, Ja) + \varrho(a, b) \\ &= \mu\varrho(a, Ja) + \varrho(a, b).\end{aligned}$$

Case (ii): If $a, b \in [\frac{1}{55}, 1]$. It follows that $Ja = \frac{16a}{17}$, $Jb = \frac{16b}{17}$, and

$$\begin{aligned}\varrho(a, Jb) &= |a - Jb| \leq |a - Ja| + |Ja - Jb| \\ &= |a - Ja| + \left|\frac{16a}{17} - \frac{16b}{17}\right| \\ &= |a - Ja| + \frac{16}{17}|a - b| \\ &\leq |a - Ja| + |a - b| \\ &\leq 17|a - Ja| + |a - b| \\ &= 17\varrho(a, Ja) + \varrho(a, b) \\ &= \mu\varrho(a, Ja) + \varrho(a, b).\end{aligned}$$

Case (iii): Consider $a \in [0, \frac{1}{55})$ and $b \in [\frac{1}{55}, 1]$. It follows that $Ja = 0$, $Jb = \frac{16b}{17}$, and

$$\begin{aligned}\varrho(a, Jb) &= |a - Jb| = \left|a - \frac{16b}{17}\right| = \left|\frac{17a - 16b}{17}\right| = \left|\frac{a + 16a - 16b}{17}\right| \\ &\leq \left|\frac{a}{17}\right| + \left|\frac{16a - 16b}{17}\right| \\ &= \frac{1}{17}|a| + \frac{16}{17}|a - b| \\ &= |a| + |a - b| \\ &\leq |a - Ja| + |a - b| \\ &\leq 17|a - Ja| + |a - b| \\ &= 17\varrho(a, Ja) + \varrho(a, b) \\ &= \mu\varrho(a, Ja) + \varrho(a, b).\end{aligned}$$

Case (iv): If $a \in [\frac{1}{55}, 1]$ and $b \in [0, \frac{1}{55}]$. It follows that $Ja = \frac{16a}{17}$, $Jb = 0$, and

$$\begin{aligned}\varrho(a, Jb) &= |a - Jb| = |a| = 17\left|\frac{a}{17}\right| = 17|a - Ja| \\ &\leq 17|a - Ja| + |a - b| \\ &= 17\varrho(a, Ja) + \varrho(a, b) \\ &= \mu\varrho(a, Ja) + \varrho(a, b).\end{aligned}$$

From the above cases, we conclude that for each $a, b \in A$, $\varrho(a, Jb) \leq 17\varrho(a, Ja) + \varrho(a, b)$ and this proves (c). For numerical results, take $\alpha_r = \beta_r = \gamma_r = 0.87$, then we obtain the comparison data shown in Table 1, which were produced by iterations (1.10), (1.9), (1.8), (1.7), (1.6), (1.5), and (1.4).

Table 1: Comparison of various iterative processes using J of Example 2.7

r	New	JF	F	M	Thakur	Picard-Mann	Picard	Mann
1	0.5000000	0.5000000	0.5000000	0.5000000	0.5000000	0.5000000	0.5000000	0.5000000
2	0.3722542	0.375279	0.3955201	0.4202401	0.426625	0.4465051	0.4705882	0.4744117
3	0.2771464	0.281668	0.3128724	0.3532036	0.364018	0.3987337	0.4429065	0.4501330
4	0.2063379	0.211408	0.2474946	0.2968607	0.310599	0.3560734	0.4168532	0.4270968
5	0.1536203	0.158674	0.1957782	0.2495055	0.26501	0.3179772	0.3923324	0.4052395
6	0.1143716	0.119094	0.1548685	0.2097045	0.226127	0.2839569	0.3692540	0.3845007
7	0.0851506	0.089387	0.1225072	0.1762525	0.192943	0.2535765	0.3475332	0.3648233
8	0.0633954	0.0670901	0.0969081	0.1481368	0.164629	0.2264464	0.3270901	0.3461530
9	0.0471984	0.05035	0.0766582	0.1245060	0.140469	0.2022190	0.3078495	0.3284381
10	0.0351396	0.0377943	0.0606397	0.1046449	0.119856	0.1805837	0.2897407	0.3116298
11	0.0261617	0.0283668	0.0479685	0.0879519	0.102267	0.1612631	0.2726971	0.2956817
12	0.0000000	0.021290	0.0379450	0.0739219	0.0872593	0.1440096	0.2566561	0.2805497
13	0.0000000	0.0000000	0.0300160	0.0621299	0.074454	0.1286021	0.2415587	0.2661922
14	0.0000000	0.0000000	0.0237439	0.0522189	0.0635279	0.1148430	0.2273493	0.2525694
15	0.0000000	0.0000000	0.0187823	0.0438890	0.0542052	0.1025560	0.2139759	0.2396438
16	0.0000000	0.0000000	0.0000000	0.0368878	0.0462506	0.0915835	0.2013890	0.2273797
17	0.0000000	0.0000000	0.0000000	0.0310035	0.0394634	0.0817850	0.1895426	0.2157432
18	0.0000000	0.0000000	0.0000000	0.0260578	0.0336721	0.0730349	0.1783930	0.2047022
19	0.0000000	0.0000000	0.0000000	0.0219011	0.0287308	0.0652209	0.1678993	0.1942263
20	0.0000000	0.0000000	0.0000000	0.0184074	0.0245145	0.0582429	0.1580229	0.1842864
21	0.0000000	0.0000000	0.0000000	0.0000000	0.020917	0.1805837	0.1487274	0.1748553
22	0.0000000	0.0000000	0.0000000	0.0000000	0.0178475	0.161263	0.139979	0.165907
23	0.0000000	0.0000000	0.0000000.	0.0000000	0.0000000	0.144011	0.131745	0.157416
.	.	.	.	.	. .	.	.	
.	.	.	.	.	.	.	.	
.	.	.	.	.	.	.	.	
31	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0811151	0.1034030	
.	.	.	.	.	.	.	.	
.	.	.	.	.	.	.	.	
.	.	.	.	.	.	.	.	
57	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0263848	
.	.	.	.	.	.	.	.	
.	.	.	.	.	.	.	.	
.	.	.	.	.	.	.	.	
71	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	

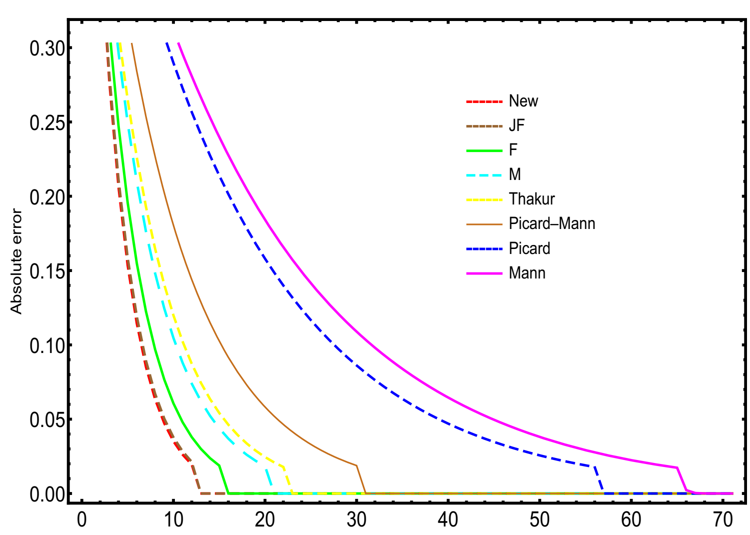

Figure 1: Behaviors of various iterative processes using Example 2.7.

The CPU times in seconds for our new iteration, JF, F, M, Thakur, Picard-Mann, Picard, and Mann iterations shown in Table 1 are, respectively, 0.047, 0.048, 0.048, 0.049, 0.050, 0.050, 0.049, and 0.051. Clearly, our new iteration needs less time for reaching to the fixed point.

Based on Table 1 and Figure 1, it is clear that the suggested iteration (1.10) converges faster than both iterations (1.9), (1.8), (1.7), (1.6), (1.5), and (1.4).

Example 2.8. Let $A = [0, 1]$ and $J : A \to A$ be a mapping defined as follows:

$$J = \begin{cases} \frac{b}{3}, & \text{if} \quad 0 \le b < \frac{1}{2}, \\ \frac{b}{4}, & \text{if} \quad \frac{1}{2} \le b \le 1. \end{cases}$$

Here, we illustrate the following:

(i) The set F_J contains only one point.
(ii) The mapping J is neither Banach-contraction nor nonexpansive on A;
(iii) The mapping J satisfies (1.2).

Now, for any $a^* \in F_J$, $a^* = 0$, which shows that $F_J = \{0\}$, and this proves (i). Since the mapping J is discontinuous, it follows that J cannot form a Banach-contraction or nonexpansive on the set A. This proves (ii). For showing (iii), we will take $\lambda = \frac{1}{3}$ and $L = \frac{3}{4}$ and discuss the cases below:

P_1. For any $a, b \in [0, \frac{1}{2})$, we have

$$\begin{aligned} \varrho(Ja, Jb) = |Ja - Jb| &= \left|\frac{a}{3} - \frac{b}{3}\right| \\ &= \frac{1}{3}|a - b| \le \frac{1}{3}|a - b| + \frac{3}{4}|a - Ja| \\ &= \lambda\varrho(a, b) + L\varrho(a, Ja). \end{aligned}$$

P_2. For any $a, b \in (\frac{1}{2}, 1]$, we obtain

$$\begin{aligned} \varrho(Ja, Jb) = |Ja - Jb| &= \left|\frac{a}{4} - \frac{b}{4}\right| \\ &= \frac{1}{4}|a - b| \le \frac{1}{3}|a - b| \\ &\le \frac{1}{3}|a - b| + \frac{3}{4}|a - Ja| \\ &= \lambda\varrho(a, b) + L\varrho(a, Ja). \end{aligned}$$

P_3. For any $a \in [0, \frac{1}{2})$ and $b \in (\frac{1}{2}, 1]$, we can write

$$\begin{aligned} \varrho(Ja, Jb) = |Ja - Jb| &= \left|\frac{a}{3} - \frac{b}{4}\right| \\ &= \left|\frac{4a - 3b}{12}\right| = \left|\frac{3a - 3b + a}{12}\right| \\ &\le \left|\frac{3a - 3b}{12}\right| + \left|\frac{a}{12}\right| \\ &= \frac{1}{4}|a - b| + \frac{1}{12}|a| \\ &\le \frac{1}{3}|a - b| + \frac{6}{12}|a| \\ &= \frac{1}{3}|a - b| + \frac{3}{4}\left|\frac{2a}{3}\right| \\ &= \frac{1}{3}|a - b| + \frac{3}{4}|a - Ja| \\ &= \lambda\varrho(a, b) + L\varrho(a, Ja). \end{aligned}$$

P_4. For any $a \in (\frac{1}{2}, 1]$ and $b \in [0, \frac{1}{2})$, we obtain that

$$\begin{aligned}
\varrho(Ja, Jb) = |Ja - Jb| &= \left|\frac{a}{4} - \frac{b}{3}\right| \\
&= \left|\frac{3a - 4b}{12}\right| = \left|\frac{4a - 4b - a}{12}\right| \\
&\leq \left|\frac{4a - 4b}{12}\right| + \left|\frac{a}{12}\right| \\
&= \frac{1}{3}|a - b| + \frac{1}{12}|a| \\
&\leq \frac{1}{3}|a - b| + \frac{9}{16}|a| \\
&= \frac{1}{3}|a - b| + \frac{3}{4}\left|\frac{3a}{4}\right| \\
&= \frac{1}{3}|a - b| + \frac{3}{4}|a - Ja| \\
&= \lambda\varrho(a, b) + L\varrho(a, Ja).
\end{aligned}$$

According to the above cases, we conclude that $\varrho(Ja, Jb) \leq \lambda\varrho(a, b) + L\varrho(a, Ja)$. Hence, the required proof of (c) is completed.

For numerical results, choose $\alpha_r = \beta_r = \gamma_r = 0.87$, so we generate some numerical values given in Table 2 from our iteration (1.10) and JF (1.9), F(1.8), M(1.7), Thakur (1.6), Picard-Mann (1.5), and Mann (1.4) iterations. Clearly, once more, our iteration (1.10) gives better approximations in less steps of iterations. Also, the graphical comparison in this case is given in Figure 2.

The CPU times in seconds for our new iteration and JF, F, M, Thakur, Picard-Mann, Picard and Mann iterations in Table 1 are, respectively, 0.047, 0.048, 0.048, and 0.049, 0.050, 0.050, 0.049, and 0.051. Clearly, our new iteration needs less time for reaching to the fixed point.

Table 2: Comparison of various iterative processes using J of Example 2.8

r	New	JF	F	M	Thakur	Picard-Mann	Picard	Mann
1	0.4900000	0.4900000	0.4900000	0.4900000	0.4900000	0.4900000	0.4900000	0.4900000
2	0.0025407	0.0032013	0.0076222	0.0228666	0.0339844	0.0686000	0.1633333	0.2058000
3	0.0000131	0.0000209	0.0001185	0.0010671	0.0023570	0.0096000	0.0544444	0.0864360
4	0.0000000	0.0000002	0.0000011	0.0000497	0.0001634	0.0013445	0.0181481	0.0363031
5	0.0000000	0.0000000	0.0000000	0.0000021	0.0000113	0.0001882	0.0060493	0.0152473
6	0.0000000	0.0000000	0.0000000	0.0000001	0.0000001	0.0000263	0.0020164	0.0064038
7	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000036	0.0006721	0.0026896
8	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000005	0.0002240	0.0011296
9	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000746	0.0004744
10	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000248	0.0001992
11	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000082	0.0000836
12	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000027	0.0000351
13	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000009	0.0000147
14	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000003	0.0000062
15	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000001	0.0000026
16	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000010
17	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000004
18	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000001
19	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
20	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
21	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
22	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

3 Weak w^2 stability result

In 1987, Harder [28] rigorously examined the idea of stability of an fixed point (FP) iteration process in her PhD thesis as follows:

Definition 3.1. [28] Let $J : H \to H$ be a given mapping and $a_{r+1} = h(J, a_r)$ be a fixed point iteration so that $\{a_r\}$ converges to $a \in F_J$. For a chosen sequence $\{q_r\}$ in H, define

$$\varepsilon_r = \varrho(q_{r+1}, h(J, q_r)) \quad \text{for all } r \in \mathbb{N}.$$

Then, the fixed point iteration method is called J-stable if the assertion below holds

$$\lim_{r\to\infty} \varepsilon_r = 0 \quad \text{iff} \quad \lim_{r\to\infty} q_r = a.$$

Several writers have lately examined the idea of stability in Definition 3.1 for various classes of contraction mappings, e.g., see [29–34]. Because the sequence $\{q_r\}$ is arbitrarily chosen, Berinde pointed out in [35] that the concept of stability in Definition 3.1 is not precise. To obtain over this restriction, the same author noted that if $\{q_r\}$ were approximate sequence of $\{a_r\}$, then the definition would make sense. As a result, any iteration process will be weakly stable if it is stable, but the converse is not true in general.

Definition 3.2. [35] A sequence $\{q_r\} \subset H$ is called an approximate sequence of $\{a_r\} \subset H$ if, for any $b \geq 1$, there is $\alpha = \alpha(b)$ so that

$$\|a_r - q_r\| \leq \alpha \quad \text{for all } r \geq b.$$

Definition 3.3. [35] Let $\{a_r\}$ be an iterative process defined for $a_0 \in H$, and

$$a_{r+1} = h(J, a_r), \quad r \geq 0, \tag{3.1}$$

where $J : H \to H$ be a given mapping. Suppose that $\{q_r\}$ converges to a fixed point a^* of J, and for any approximate sequence $\{q_r\} \subset H$ of $\{a_r\}$:

$$\lim_{r\to\infty} \varepsilon_r = \lim_{r\to\infty} \varrho(q_{r+1}, h(J, q_r)) = 0 \quad \text{implies} \quad \lim_{r\to\infty} q_r = a^*,$$

then equation (3.1) is called weakly stable with respect to J or weakly J-stable.

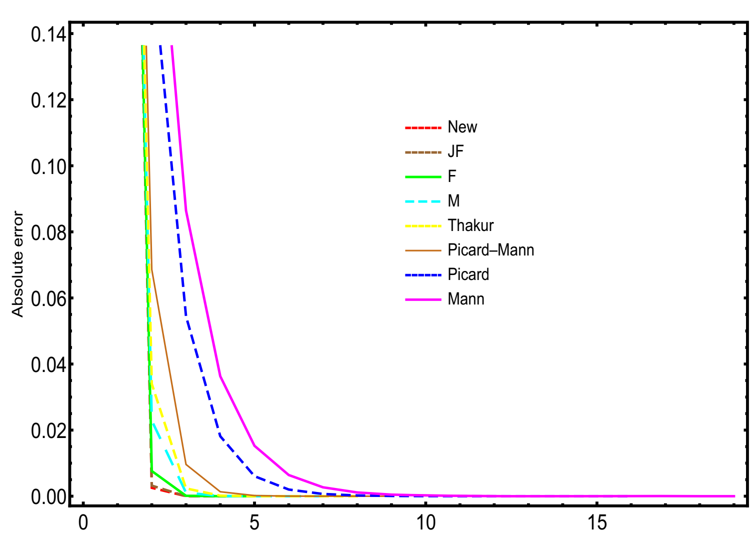

Figure 2: Behaviors of various iterative processes using Example 2.8.

By using the more general concept of the equivalent sequence in place of the approximate sequence in Definition 3.3, Timis [36] studied a new concept of weak stability in 2012 as follows:

Definition 3.4. [37] The sequences $\{a_r\}$ and $\{q_r\}$ are called equivalent if

$$\lim_{r\to\infty} \varrho(q_r, a_r) = 0.$$

Definition 3.5. [36] Assume that $\{a_r\}$ is an iterative procedure defined for $a_0 \in H$ and

$$a_{r+1} = h(J, a_r), r \geq 0, \tag{3.2}$$

where $J : H \to H$ be a self-mapping. Suppose that $\{a_r\}$ converges to a fixed point a^* of J, and for any equivalent sequence $\{q_r\} \subset H$ of $\{a_r\}$:

$$\lim_{r\to\infty} \varepsilon_r = \lim_{r\to\infty} \varrho(q_{r+1}, h(J, q_r)) = 0 \quad \text{implies} \lim_{r\to\infty} q_r = a^*,$$

then equation (3.2) is called weakly w^2-stable with respect to $\mathfrak{I}$.

Any analogous sequence is an approximate sequence, as demonstrated with an example in [36], but the converse is not true.

Here, we provide weak w^2-stability and data dependence theorems for our new iterative process (1.10) in the class of mappings satisfying equation (1.2).

Theorem 3.6. *If all requirements of* Theorem 2.6 *are true. Then, the sequence considered by equation* (1.10) *is essentially weak w^2-stable.*

Proof. Assume that $\{a_r\}$ is a sequence generated by equation (1.10), $\{q_r\}$ is an equivalent sequence of $\{a_r\}$, and

$$\varepsilon_r = \varrho(p_{r+1}, Jq_r).$$

Putting $q_r = Jr_k$, $r_r = Jg_r$, $g_r = Jt_r$, and $t_r = \mathcal{S}(Jp_r, p_r, \alpha_r)$. If $\varepsilon_r \longrightarrow 0$, as $r \to \infty$, then by equations (1.2) and (1.10), and $(\mathcal{S}_4)$, we have

$$\begin{aligned}\varrho(e_r, t_r) &= \varrho(\mathcal{S}(Ja_r, a_r, \alpha_r), \mathcal{S}(Jp_r, p_r, \alpha_r))\\ &\leq \alpha_r \varrho(J_r, Jp_r) + (1 - \alpha_r)\varrho(a_r, p_r)\\ &\leq \alpha_r(\lambda\varrho(a_r, p_r) + L\varrho(a_r, Ja_r)) + (1 - \alpha_r)\varrho(a_r, p_r)\\ &= (1 - \alpha_r(1 - \lambda))\varrho(a_r, p_r) + \alpha_r L\varrho(a_r, Ja_r).\end{aligned}$$

Also,

$$\begin{aligned}\varrho(d_r, g_r) &= \varrho(Je_r, Jt_r)\\ &\leq \lambda\varrho(e_r, t_r) + L\varrho(e_r, Je_r)\\ &= \lambda((1 - \alpha_r(1 - \lambda))\varrho(a_r, p_r) + \alpha_r L\varrho(a_r, Ja_r)) + L\varrho(e_r, Je_r).\end{aligned}$$

Now,

$$\begin{aligned}\varrho(b_r, q_r) &= \varrho(Jc_r, Jr_r)\\ &\leq \lambda\varrho(c_r, r_r) + L\varrho(c_r, Jc_r)\\ &= \lambda\varrho(Jd_r, Jg_r) + L\varrho(c_r, Jc_r)\\ &\leq \lambda(\lambda\varrho(d_r, g_r) + L\varrho(d_r, Jd_r) + L\varrho(c_r, Jc_r)\\ &\leq (\lambda)^2\varrho(d_r, g_r) + \lambda L\varrho(d_r, Jd_r) + L\varrho(c_r, Jc_r)\\ &\leq (\lambda)^2(\lambda((1 - \alpha_r(1 - \lambda))\varrho(a_r, p_r) + \alpha_r L\varrho(a_r, Ja_r))) + L\varrho(e_r, Je_r)) + \lambda L\varrho(d_r, Jd_r) + L\varrho(c_r, Jc_r)\\ &= (\lambda)^3(1 - \alpha_r(1 - \lambda))\varrho(a_r, p_r) + (\lambda)^3\alpha_r L\varrho(a_r, Ja_r) + (\lambda)^2 L\varrho(e_r, Je_r) + \lambda L\varrho(d_r, Jd_r) + L\varrho(c_r, Jc_r).\end{aligned}$$

This implies that

$$\begin{aligned}
\varrho(p_{r+1}, a^*) &\leq \varrho(p_{r+1}, a_{r+1}) + \varrho(a_{r+1}, a^*)\\
&\leq \varrho(p_{r+1}, Jq_r) + \varrho(Jq_r, Jb_r) + \varrho(a_{r+1}, a^*)\\
&\leq \varepsilon_r + \varrho(Jq_r, Jb_r) + \varrho(a_{r+1}, a^*)\\
&\leq \varepsilon_r + \lambda\varrho(q_r, b_r) + L\varrho(b_r, Jb_r) + \varrho(a_{r+1}, a^*)\\
&\leq \varepsilon_r + \lambda[(\lambda)^3(1 - \alpha_r(1 - \lambda))\varrho(a_r, p_r) + (\lambda)^3\alpha_r L\varrho(a_r, Ja_r) + (\lambda)^2 L\varrho(e_r, Je_r) + \lambda L\varrho(d_r, Jd_r) + L\varrho(c_r, Jc_r)]\\
&\quad + L\varrho(b_r, Jb_r) + \varrho(a_{r+1}, a^*)\\
&= \varepsilon_r + (\lambda)^4(1 - \alpha_r(1 - \lambda))\varrho(a_r, p_r) + (\lambda)^4\alpha_r L\varrho(a_r, Ja_r) + (\lambda)^3 L\varrho(e_r, Je_r) + (\lambda)^2 L\varrho(d_r, Jd_r) + \lambda L\varrho(c_r, Jc_r)\\
&\quad + L\varrho(b_r, Jb_r) + \varrho(a_{r+1}, a^*).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\varrho(p_{r+1}, a^*) \leq{}& \varepsilon_r + (\lambda)^4(1 - \alpha_r(1 - \lambda))\varrho(a_r, p_r) + (\lambda)^4\alpha_r L\varrho(a_r, Ja_r) + (\lambda)^3 L\varrho(e_r, Je_r) + (\lambda)^2 L\varrho(d_r, Jd_r)\\
&+ \lambda L\varrho(c_r, Jc_r) + L\varrho(b_r, Jb_r) + \varrho(a_{r+1}, a^*).
\end{aligned} \tag{3.3}$$

Since $\lim_{r\to\infty}\varrho(a_r, Ja_r) = \lim_{r\to\infty}\varrho(b_r, Jb_r) = \lim_{r\to\infty}\varrho(d_r, Jd_r) = \lim_{r\to\infty}\varrho(e_r, Je_r) = 0$ and $\lim_{r\to\infty}\varrho(a_r, a^*) = 0$, then, one has

$$\begin{aligned}
\varrho(a_r, Ja_r) &\leq \varrho(a_r, a^*) + \varrho(Ja_r, a^*)\\
&\leq \varrho(a_r, a^*) + \lambda\varrho(a_r, a^*) + L\varrho(a^*, Ja^*)\\
&= (1 + \lambda)\varrho(a_r, a^*) \longrightarrow 0.
\end{aligned}$$

By the proof of Theorem 2.6, $\varrho(e_r, Ja^*) \leq (1 - \alpha_r(1 - \lambda))\varrho(a_r, a^*)$, we obtain

$$\begin{aligned}
\varrho(e_r, Je_r) &\leq \varrho(e_r, a^*) + \varrho(Je_r, a^*)\\
&\leq \varrho(e_r, a^*) + \lambda\varrho(e_r, a^*) + L\varrho(a^*, Ja^*)\\
&= (1 + \lambda)\varrho(e_r, a^*)\\
&\leq (1 + \lambda)[(1 - \alpha_r(1 - \lambda))]\varrho(a_r, a^*)\\
&\leq (1 + \lambda)\varrho(a_r, a^*) \longrightarrow 0.
\end{aligned}$$

Also,

$$\begin{aligned}
\varrho(d_r, Jd_r) &\leq \varrho(d_r, a^*) + \varrho(Jd_r, a^*)\\
&\leq \varrho(d_r, a^*) + \lambda\varrho(d_r, a^*) + L\varrho(a^*, Ja^*)\\
&= (1 + \lambda)\varrho(d_r, a^*)\\
&\leq (1 + \lambda)[(1 - \alpha_r(1 - \lambda))]\varrho(a_r, a^*)\\
&\leq (1 + \lambda)\varrho(a_r, a^*) \longrightarrow 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\varrho(b_r, Jb_r) &\leq \varrho(b_r, a^*) + \varrho(Jb_r, a^*)\\
&\leq \varrho(b_r, a^*) + \lambda\varrho(b_r a^*) + L\varrho(a^*, Ja^*)\\
&= (1 + \lambda)\varrho(b_r, a^*)\\
&= (1 + \lambda)\varrho(Jc_r, a^*)\\
&\leq (1 + \lambda)(\lambda\varrho(c_r, a^*) + L\varrho(a^*, Ja^*)\\
&= (1 + \lambda)\lambda\varrho(c_r, a^*)\\
&= (1 + \lambda)\lambda\varrho(Jd_r, a^*)\\
&\leq (1 + \lambda)\lambda(\lambda\varrho(d_r, a^*) + L\varrho(a^*, Ja^*)\\
&= (1 + \lambda)(\lambda)^2\varrho(d_r, a^*)\\
&= (1 + \lambda)(\lambda)^2\varrho(Je_r, a^*)\\
&\leq (1 + \lambda)(\lambda)^2(\lambda\varrho(e_r, a^*) + L\varrho(a^*, Ja^*))\\
&= (1 + \lambda)(\lambda)^3\varrho(e_r, a^*)\\
&\leq (1 + \lambda)(\lambda)^3[(1 - \alpha_r(1 - \lambda))]\varrho(a_r, a^*)\\
&\leq (1 + \lambda)(\lambda)^3\varrho(a_r, a^*) \longrightarrow 0.
\end{aligned}$$

Since $\{p_r\}$ and $\{a_r\}$ are equivalent, we have $\lim_{r\to\infty}\varrho(a_r, p_r) = 0$. Using $\lim_{r\to\infty}\varrho(a_r, Ja_r) = \lim_{r\to\infty}\varrho(b_r, Jb_r) = \lim_{r\to\infty}\varrho(d_r, Jd_r) = \lim_{r\to\infty}\varrho(e_r, Je_r) = 0 = \lim_{r\to\infty}\varepsilon_r$. Hence, equation (3.3) can be written as follows:

$$\lim_{r\to\infty}\varrho(p_{r+1}, a^*) = 0.$$

This implies that $\{a_r\}$ is weak w^2-stable with respect to J. □

4 Data dependence result

When we compute the fixed point using an iterative process, we may use an approximate mapping of the given operator. In this case, we may have a certain estimate between the original fixed point and the approximate fixed point. This approach is known as data dependence and is widely studied by authors [38,39]. Now, we carry out a data dependency result for our iterative process.

Definition 4.1. [35] A mapping $\tilde{J}$ is said to be approximate mapping for J if and only if, for any choice of $a, b \in H$ and any $\varepsilon > 0$, we have $\varrho(Ja, \tilde{J}b) \le \varepsilon$.

Lemma 4.2. [40]. *Let $\{a_r\}$, $\{b_r\}$, and $\{c_r\}$ be three sequences of nonnegative real numbers such that $\{b_r\} \in (0, 1)$ satisfying $\sum_{r=1}^{\infty} b_r = \infty$ and $c_r \ge 0$ for any $r \in \mathbb{N}$. If $r_1 \in \mathbb{N}$ such that $r \ge r_1$, and*

$$a_{r+1} \le (1 - b_r)a_r + b_r c_r,$$

then,

$$0 \le \limsup_{r\to\infty} a_r \le \limsup_{r\to\infty} c_r.$$

Now, we introduce our main result in this section.

Theorem 4.3. *Assume that H, A, and J are defined as in* Theorem 2.6. *Let $\tilde{J} : A \to A$ be an approximate mapping of J for some ε. Suppose that the sequences $\{a_r\}$ and $\{\tilde{a}_r\}$ are defined by* (1.10) *and*

$$\begin{cases} \tilde{a}_1 \in A, \\ \tilde{e}_r = \mathcal{S}(\tilde{J}\tilde{a}_r, \tilde{a}_r, \alpha_r), \\ \tilde{d}_r = \tilde{J}\tilde{e}_r, \\ \tilde{c}_r = \tilde{J}\tilde{d}_r, \\ \tilde{b}_r = \tilde{J}\tilde{c}_r, \\ \tilde{a}_{r+1} = \tilde{J}\tilde{b}_r, \ r \ge 1. \end{cases} \tag{4.1}$$

respectively, where $\alpha_r \in (0, 1)$ satisfying the condition $\sum_{r=1}^{\infty}\alpha_r = \infty$. Then, if $a^* = Ja^*$ and $\tilde{a}^* = \tilde{J}\tilde{a}^*$, we have

$$\varrho(a^*, \tilde{a}^*) \le \frac{(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1)\varepsilon}{1 - \lambda^4},$$

where $\lambda \in [0, 1)$.

Proof. It follows from equations (1.2), (1.10), and (4.1) that

$$\begin{aligned} \varrho(e_r, \tilde{e}_r) &= \varrho(\mathcal{S}(Ja_r, a_r, \alpha_r), \mathcal{S}(\tilde{J}\tilde{a}_r, \tilde{a}_r, \alpha_r)) \\ &\le \alpha_r\varrho(Ja_r, \tilde{J}\tilde{a}_r) + (1 - \alpha_r)\varrho(a_r, \tilde{a}_r)) \\ &\le \alpha_r\varrho(Ja_r, J\tilde{a}_r) + \alpha_r\varrho(J\tilde{a}_r, \tilde{J}\tilde{a}_r) + (1 - \alpha_r)\varrho(a_r, \tilde{a}_r) \\ &\le \alpha_r\varrho(Ja_r, J\tilde{a}_r) + \alpha_r\varepsilon + (1 - \alpha_r)\varrho(a_r, \tilde{a}_r) \\ &\le \alpha_r(\lambda\varrho(a_r, \tilde{a}_r) + L\varrho(a_r, Ja_r)) + \alpha_r\varepsilon + (1 - \alpha_r)\varrho(a_r, \tilde{a}_r) \\ &= (1 - \alpha_r(1 - \lambda))\varrho(a_r, \tilde{a}_r) + \alpha_r L\varrho(a_r, Ja_r) + \alpha_r\varepsilon. \end{aligned}$$

This implies that

$$
\begin{aligned}
\varrho(a_{r+1}, \tilde{a}_{r+1}) &= \varrho(Jb_r, \tilde{J}\tilde{b}_r)\\
&\le \varrho(Jb_r, J\tilde{b}_r) + \varrho(J\tilde{b}_r, \tilde{J}\tilde{b}_r)\\
&\le \varrho(Jb_r, J\tilde{b}_r) + \varepsilon\\
&\le \lambda\varrho(b_r, \tilde{b}_r) + L\varrho(b_r, J_r) + \varepsilon\\
&\le \lambda[\varrho(Jc_r, \tilde{J}\tilde{c}_r)] + L\varrho(b_r, Jb_r) + \varepsilon\\
&\le \lambda[\varrho(Jc_r, J\tilde{c}_r) + \varrho(J\tilde{c}_r, \tilde{J}\tilde{c}_r)] + L\varrho(b_r, Jb_r) + \varepsilon\\
&\le \lambda[\varrho(Jc_r, J\tilde{c}_r) + \varepsilon] + L\varrho(b_r, Jb_r) + \varepsilon\\
&\le \lambda[\lambda\varrho(c_r, \tilde{c}_r) + L\varrho(c_r, Jc_r) + \varepsilon] + L\varrho(b_r, Jb_r) + \varepsilon\\
&= (\lambda)^2\varrho(c_r, \tilde{c}_r) + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon\\
&\le (\lambda)^2[\varrho(Jd_r, \tilde{J}\tilde{d}_r)] + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon\\
&\le (\lambda)^2[\varrho(Jd_r, J\tilde{d}_r) + \varrho(J\tilde{d}_r, \tilde{J}\tilde{d}_r)] + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon\\
&\le (\lambda)^2[\varrho(Jd_r, J\tilde{d}_r) + \varepsilon] + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon\\
&\le (\lambda)^2[\lambda\varrho(d_r, \tilde{d}_r) + L\varrho(d_r, Jd_r) + \varepsilon] + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon\\
&= (\lambda)^3\varrho(d_r, \tilde{d}_r) + (\lambda)^2L\varrho(d_r, Jd_r) + (\lambda)^2\varepsilon + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon\\
&= (\lambda)^3\varrho(Je_r, \tilde{J}\tilde{e}_r) + (\lambda)^2L\varrho(d_r, Jd_r) + (\lambda)^2\varepsilon + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon\\
&\le (\lambda)^3(\varrho(Je_r, J\tilde{e}_r) + \varrho(J\tilde{e}_r, \tilde{J}\tilde{e}_r)) + (\lambda)^2L\varrho(d_r, Jd_r) + (\lambda)^2\varepsilon + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon\\
&\le (\lambda)^3(\varrho(Je_r, J\tilde{e}_r) + \varepsilon) + (\lambda)^2L\varrho(d_r, Jd_r) + (\lambda)^2\varepsilon + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon\\
&\le (\lambda)^3(\lambda\varrho(e_r, \tilde{e}_r) + L\varrho(e_r, Je_r) + \varepsilon) + (\lambda)^2L\varrho(d_r, Jd_r) + (\lambda)^2\varepsilon + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon\\
&\le (\lambda)^4\varrho(e_r, \tilde{e}_r) + (\lambda)^3L\varrho(e_r, Je_r) + (\lambda)^3\varepsilon + (\lambda)^2L\varrho(d_r, Jd_r) + (\lambda)^2\varepsilon + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon\\
&\le (\lambda)^4[(1-\alpha_r(1-\lambda))\varrho(a_r, \tilde{a}_r) + \alpha_r L\varrho(a_r, Ja_r) + \alpha_r\varepsilon] + (\lambda)^3L\varrho(e_r, Je_r) + (\lambda)^3\varepsilon + (\lambda)^2L\varrho(d_r, Jd_r)\\
&\quad + (\lambda)^2\varepsilon + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon\\
&= (\lambda)^4[(1-\alpha_r(1-\lambda))\varrho(a_r, \tilde{a}_r)] + (\lambda)^4\alpha_r L\varrho(a_r, Ja_r) + (\lambda)^4\alpha_r\varepsilon + (\lambda)^3L\varrho(e_r, Je_r) + (\lambda)^3\varepsilon\\
&\quad + (\lambda)^2L\varrho(d_r, Jd_r) + (\lambda)^2\varepsilon + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\varrho(a_{r+1}, \tilde{a}_{r+1}) \le{} & (\lambda)^4[(1-\alpha_r(1-\lambda))\varrho(a_r, \tilde{a}_r)] + (\lambda)^4\alpha_r L\varrho(a_r, Ja_r) + (\lambda)^4\alpha_r\varepsilon\\
& + (\lambda)^3L\varrho(e_r, Je_r) + (\lambda)^3\varepsilon + (\lambda)^2L\varrho(d_r, Jd_r) + (\lambda)^2\varepsilon\\
& + \lambda L\varrho(c_r, Jc_r) + \lambda\varepsilon + L\varrho(b_r, Jb_r) + \varepsilon.
\end{aligned}
\tag{4.2}
$$

Since $\lambda \in [0, 1)$, we have $(\lambda)^4 \in (0, 1)$. Thus, we can find a constant $f \in (0, 1)$ such that

$$(\lambda)^4 = 1 - f. \tag{4.3}$$

From equation (4.3) and the assumption $\alpha_r \le 1$, we obtain $1 - \alpha_r(1 - \lambda) \le 1$ for all $r \in \mathbb{N}$. Hence, (4.2) can be written as follows:

$$
\begin{aligned}
&\varrho(a_{r+1}, \tilde{a}_{r+1})\\
&\quad \le (1 - f)\varrho(a_r, \tilde{a}_r)\\
&\qquad + f\frac{(\lambda)^4L\varrho(a_r, Ja_r) + (\lambda)^3L\varrho(e_r, Je_r) + (\lambda)^2L\varrho(d_r, Jd_r) + \lambda L\varrho(c_r, Jc_r) + L\varrho(b_r, Jb_r)}{f}.\\
&\qquad + f\frac{\lambda^4\varepsilon + \lambda^3\varepsilon + \lambda^2\varepsilon + \lambda\varepsilon + \varepsilon}{f}.
\end{aligned}
\tag{4.4}
$$

Now, set

$$
\begin{aligned}
A_r &= \varrho(a_{r+1}, \tilde{a}_{r+1}),\\
B_r &= f,\\
C_k &= ((\lambda)^4L\varrho(a_r, Ja_r) + (\lambda)^3L\varrho(e_r, Je_r) + (\lambda)^2L\varrho(d_r, Jd_r) + \lambda L\varrho(c_r, Jc_r)\\
&\quad + L\varrho(b_r, Jb_r) + (\lambda)^4\varepsilon + (\lambda)^3\varepsilon + (\lambda)^2\varepsilon + \lambda\varepsilon + \varepsilon)/(1 - (\lambda)^4).
\end{aligned}
$$

Considering equation (4.4) with Lemma 4.2 and using $\lim_{r\to\infty}\varrho(a_r, Ja_r) = \lim_{r\to\infty}\varrho(c_r, Jc_r) = \lim_{r\to\infty}\varrho(b_r, Jb_r) = \lim_{r\to\infty}\varrho(d_r, Jd_r) = \lim_{r\to\infty}\varrho(e_r, Je_r) = 0$, we have

$$\varrho(a^*, \tilde{a}^*) \le \frac{(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1)\varepsilon}{1 - \lambda^4}.$$

This completes the proof. □

5 Conclusion and future work

A new general fixed-point iterative process is constructed in the hyperbolic space setting for finding fixed points of mappings with condition (E) and generalized contractions. We provided a Δ-convergence theorem for mappings with condition (E). Also, some strong convergence theorems are obtained for our new iterative process in the class of mappings with condition (E). All these results are supported by a numerical example. It is seen that our new iterative process has a leading rate of convergence over the classical iterative processes of the literature. Second, we analyzed our new iteration with a class of generalized contractions and obtained a strong convergence theorem. Eventually, weak w^2-stability and data dependence results are obtained. Our results improve the corresponding results in [24] and [23]. In future, the authors have a plan to extend the main results of this article to the setting of multivalued mappings and to the setting of common fixed points. The authors also made a plan to prove the main results of this article in the setting of modular metric spaces.

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