DE GRUYTER Demonstratio Mathematica 2024; 57: 20230150

Research Article

Daochang Zhang*, Leiming Ma, Jianping Hu, and Chaochao Sun

# Jordan left derivations in infinite matrix rings

https://doi.org/10.1515/dema-2023-0150
received February 25, 2023; accepted January 19, 2024

**Abstract:** Let $R$ be a unital associative ring. Our motivation is to prove that left derivations in column finite matrix rings over $R$ are equal to zero and demonstrate that a left derivation $d:\mathcal{T}\to\mathcal{T}$ in the infinite upper triangular matrix ring $\mathcal{T}$ is determined by left derivations $d_j$ in $R(j=1,2,\ldots)$ satisfying $d((a_{ij}))=(b_{ij})$ for any $(a_{ij})\in\mathcal{T}$, where

$$b_{ij}=\begin{cases}d_j(a_{11}), & i=1,\\ 0, & i\neq 1.\end{cases}$$

The similar results about Jordan left derivations are also obtained when $R$ is 2-torsion free.

**Keywords:** left derivations, Jordan left derivations, column finite matrix rings, infinite upper triangular matrix rings

**MSC 2020:** 15B33, 16W25, 16S50

## 1 Introduction

Let $R$ be an associative ring, and let $M$ be a left $R$-module. An additive mapping $d:R\to M$ is called a left derivation if $d(xy)=xd(y)+yd(x)$ for all $x,y\in R$, and an additive mapping $d:R\to M$ is called a Jordan left derivation if $d(x^2)=2xd(x)$ for all $x\in R$. A (Jordan) left derivation in a ring $R$ means a (Jordan) left derivation from $R$ into itself.

In 1988, the Singer-Wermer theorem was generalized by removing the boundedness of a derivation (see [1]), which was called as the Singer-Wermer conjecture. In 1990, Brešar and Vukman [2] introduced the concept of left derivations. They based upon the rather weak assumptions to exhibit that the existence of a nonzero Jordan left derivation in a 2-torsion free and 3-torsion free ring $R$ implies that $R$ is commutative. Using this result, they showed that continuous linear left derivations in a Banach algebra A map into its radical, which is a noncommutative extension of the classical Singer-Wermer theorem.

The (Jordan) left derivation has been widely studied in some special rings and algebras (see [3–7]). Concretely, related results have been proved in prime rings (see [8]), semi-prime rings (see [9]), and Γ-rings (see [10,11]); on the other hand, related results have been proved in Banach algebras (see [12–17]). Some results were generalized to different types of semi-rings, which play an important role in theoretical computer science (see [18,19]). In 2010, Li and Zhou [20] proved that Jordan left derivations or left derivable mappings at zero or

*** Corresponding author: Daochang Zhang,** College of Sciences, Northeast Electric Power University, Jilin 132012, China, e-mail: daochangzhang@126.com
**Leiming Ma:** College of Sciences, Northeast Electric Power University, Jilin 132012, China, e-mail: maleiming2022@163.com
**Jianping Hu:** College of Sciences, Northeast Electric Power University, Jilin 132012, China, e-mail: neduhjp307@163.com
**Chaochao Sun:** School of Mathematics and Statistics, Linyi University, Linyi 276005, China, e-mail: sunchaochao@lyu.edu.cn

unit on some algebras are zero under certain conditions. With the concept of generalized left derivations developed, the properties of the generalized (Jordan) left derivations have been discussed (see [21–30]).

Specially, in 2001, Cui and Niu [31] showed that an arbitrary derivation of infinite matrix rings of finite nonzero entries is a sum of two special derivations, and came up with the decomposition formula of a series of important derivations in infinite matrix rings. In 2010, Xu and Zhang [32] dealt with the case of left derivations in matrix rings and upper triangular matrix rings over a unital associative ring. It is natural to consider (Jordan) left derivations in infinite matrix rings over a unital associative ring.

In what follows, suppose $R$ is a ring with 1 unless stated otherwise. The aim of this article is to establish explicit representations of left derivations and Jordan left derivations in column finite matrix rings and infinite upper triangular matrix rings over $R$. Let $\mathcal{S}$ denote the column finite matrix ring over $R$. We validate that left derivations in $\mathcal{S}$ are all zero and that Jordan left derivations are all zero when $R$ is 2-torsion free. Let $\mathcal{T}$ be the infinite upper triangular matrix ring over $R$ and $d:\mathcal{T}\to\mathcal{T}$ be a left derivation. We demonstrate that there exist left derivations $d_j: R\to R$ $(j=1,2,\ldots)$ such that $d((a_{ij}))=(b_{ij})$ for any $(a_{ij})\in\mathcal{T}$, where

$$b_{ij}=\begin{cases} d_j(a_{11}), & i=1,\\ 0, & i\neq 1.\end{cases}$$

A similar result for Jordan left derivations is obtained when $R$ is 2-torsion free.

Some symbols will be declared throughout this article. An $\mathbb{N}\times\mathbb{N}$ matrix $M$ over $R$ is called column finite if the number of nonzero entries is finite in every column. Let $\mathcal{S}$ be the ring consisting of column finite matrices, and let $\mathcal{A}$ be the ring consisting of matrices with only a finite number of nonzero entries. Denote by $e_{ij}$ the $\mathbb{N}\times\mathbb{N}$ matrix whose entries are all 0 except the $(i,j)$-entry, which is equal to 1. We define $\mathcal{T}=\{(a_{ij})\in\mathcal{S}\,|\,a_{ij}=0 \quad \text{for all} \quad i,j\in\mathbb{N} \ \text{with} \ i>j\}$, and $\mathcal{TF}=\{M\in\mathcal{T}\cap\mathcal{A}\}$. The differences among $\mathcal{A},\mathcal{S},\mathcal{T}$ and $\mathcal{TF}$ can be seen clearly as follows: $\mathcal{A}\subseteq\mathcal{S}, \mathcal{T}\subseteq\mathcal{S}$, and $\mathcal{TF}\subseteq\mathcal{S}$. Specifically, the form of any element in $\mathcal{A}$ is as $\begin{bmatrix} * & 0\\ 0 & 0\end{bmatrix}$, the form of any element in $\mathcal{T}$ is as $T=\begin{bmatrix} a_{11} & *\\ 0 & \ddots\end{bmatrix}$, and the form of any element in $\mathcal{TF}$ is as $\begin{bmatrix} T & 0\\ 0 & 0\end{bmatrix}$.

## 2 Left derivation

We need some key lemmas for proving the results of this article.

**Lemma 2.1.** *Let d be a left derivation in R (not necessarily with 1). Then,*

$$[x,z]d(y)+[y,z]d(x)=0, \quad \text{for all } x,y,z\in R.$$

**Proof.** From $d((xy)z)=d(x(yz))$ for all $x,y,z\in R$, we have that $zd(xy)+xyd(z)=xd(yz)+yzd(x)$, implying

$$z(xd(y)+yd(x))+xyd(z)=x(yd(z)+zd(y))+yzd(x).$$

Hence, $[x,z]d(y)+[y,z]d(x)=0$, for all $x,y,z\in R$. □

**Lemma 2.2.** *Let d be a Jordan left derivation in R (not necessarily with 1). Then,*

$$d(xy)+d(yx)=2xd(y)+2yd(x), \quad \text{for all } x,y\in R.$$

**Proof.** For all $x,y\in R$, from $d((x+y)^2)=2(x+y)d(x+y)=2(x+y)(d(x)+d(y))$ and $d((x+y)^2)=d((x+y)(x+y))=d(x^2+y^2+xy+yx)$, we obtain

$$d(xy)+d(yx)=2xd(y)+2yd(x),$$

as desired. □

**Lemma 2.3.** *Let $d$ be a Jordan left derivation in $R$ (not necessarily with 1). Then, $d(e) = 0$ for any idempotent $e \in R$.*

**Proof.** Since $d(e) = d(e^2) = 2ed(e)$, we have $ed(e) = 2ed(e) = 0$, and then $d(e) = 0$. □

First, we discuss left derivations.

**Lemma 2.4.** *Let $d : \mathcal{A} \to \mathcal{S}$ be a left derivation. Then, $d = 0$.*

**Proof.** For any $i, j \in \mathbb{N}$ with $i \neq j$ and $a \in R$, $e_{ii}$ and $e_{ii} + ae_{ij}$ are all idempotents. By Lemma 2.3, we obtain $d(e_{ii}) = 0$ and $d(e_{ii} + ae_{ij}) = 0$, whence $d(ae_{ij}) = 0$. Note that

$$d(ae_{ii}) = d(ae_{ij}e_{ji}) = ae_{ij}d(e_{ji}) + e_{ji}d(ae_{ij}) = 0, \quad \text{for any } i \neq j.$$

It follows that $d(ae_{ij}) = 0$ for all $i, j \in \mathbb{N}$. For any $A = (a_{ij}) \in \mathcal{A}$, there exists $n \in \mathbb{N}$ such that $A = \sum_{i,j=1}^{n} a_{ij}e_{ij}$. Thus,

$$d(A) = d\left(\sum_{i,j=1}^{n} a_{ij}e_{ij}\right) = \sum_{i,j=1}^{n} d(a_{ij}e_{ij}) = 0.$$

Hence, $d = 0$. □

We note that the first main result is shown as follows.

**Theorem 2.5.** *Let $d : \mathcal{S} \to \mathcal{S}$ be a left derivation. Then, $d = 0$.*

**Proof.** By Lemma 2.1, we obtain $[A, C]d(B) + [B, C]d(A) = 0$, where $A, B, C \in \mathcal{S}$. By Lemma 2.4, we obtain $d(A) = 0$ for all $A \in \mathcal{A}$. Then, $[A, C]d(B) = 0$ for all $A \in \mathcal{A}$ and all $B, C \in \mathcal{S}$. Let $A = e_{11}$ and $C = e_{1i}$ with $i > 1$. Then,

$$0 = [e_{11}, e_{1i}]d(B) = e_{1i}d(B), i > 1, \tag{2.1}$$

which means the $i$th row of $d(B)$ is zero for $i > 1$.

Let $A = e_{12}$, $C = e_{21}$. Then,

$$0 = [e_{12}, e_{21}]d(B) = (e_{11} - e_{22})d(B), \tag{2.2}$$

which means the first row of $d(B)$ is zero.

We conclude from (2.1) and (2.2) that $d = 0$. □

Recall that $\mathcal{T} = \{(a_{ij}) \in \mathcal{S} | a_{ij} = 0, \quad \text{for all } i, j \in \mathbb{N} \quad \text{such that } i > j\}$ and $\mathcal{TF} = \{M \in \mathcal{T} | M \text{ is finite}\}$.

**Lemma 2.6.** *Let $d : \mathcal{TF} \to \mathcal{T}$ be a left derivation. Then, there exist left derivations $d_j : R \to R$ $(j = 1, 2, \ldots)$ such that for any $(a_{ij}) \in \mathcal{TF}$, we have $d((a_{ij})) = (b_{ij})$, where*

$$b_{ij} = \begin{cases} d_j(a_{11}), & i = 1, \\ 0, & i \neq 1. \end{cases}$$

**Proof.** For any $i, j \in \mathbb{N}$ with $i < j$ and $a \in R$, $e_{ii}$ and $e_{ii} + ae_{ij}$ are all idempotents of $\mathcal{TF}$. By Lemma 2.3, we have $d(e_{ii}) = 0$ and $d(e_{ii} + ae_{ij}) = 0$, whence

$$d(ae_{ij}) = 0, \quad a \in R, \ i < j. \tag{2.3}$$

Note that $d(ae_{ii}) = d(e_{ii}(ae_{ii})) = e_{ii}d(ae_{ii}) + ae_{ii}d(e_{ii}) = e_{ii}d(ae_{ii})$ for $a \in R$ and $i \in \mathbb{N}$. We observe that $0 = d(ae_{1i}) = d(e_{1i}(ae_{ii})) = e_{1i}d(ae_{ii})$ for $a \in R$ and $i > 1$. We have

$$d(ae_{ii}) = 0, \quad a \in R, \ i > 1. \tag{2.4}$$

Let $A = (a_{ij}) \in \mathcal{TF}$. Then, $A = \sum_{1 \le i \le j \le n} a_{ij}e_{ij}$ for some positive integer $n$. By (2.3) and (2.4), we have

$$d(A) = \sum_{1 \le i \le j \le n} d(a_{ij}e_{ij}) = d(a_{11}e_{11}).$$

Note that $d(a_{11}e_{11}) = e_{11}d(a_{11}e_{11})$. For $j \geqslant 1$, set $d_j : R \to R$ such that $d_j(a)e_{1j} = d(ae_{11})e_{jj}$ for $a \in R$. So $d(a_{11}e_{11}) = (b_{ij})$, where

$$b_{ij} = \begin{cases} d_j(a_{11}), & i = 1, \\ 0, & i \neq 1. \end{cases}$$

Note that $d(abe_{11}) = d((ae_{11})(be_{11})) = ae_{11}d(be_{11}) + be_{11}d(ae_{11})$, which means $d_j(ab)e_{1j} = d(abe_{11})e_{jj} = ae_{11}d(be_{11})e_{jj} + be_{11}d(ae_{11})e_{jj} = ad_j(s)e_{1j} + bd_j(a)e_{1j} = (ad_j(s) + bd_j(a))e_{1j}$. So $d_j : R \to R$ $(j \geqslant 1)$ are left derivations. □

Furthermore, we obtain another main formula.

**Theorem 2.7.** *Let $d : \mathcal{T} \to \mathcal{T}$ be a left derivation. Then, there exist left derivations $d_j : R \to R$ $(j = 1, 2, \ldots)$ such that for any $(a_{ij}) \in \mathcal{T}$, we have $d((a_{ij})) = (b_{ij})$, where*

$$b_{ij} = \begin{cases} d_j(a_{11}), & i = 1, \\ 0, & i \neq 1. \end{cases}$$

**Proof.** By Lemma 2.1, we obtain $[A, C]d(B) + [B, C]d(A) = 0$ for all $A, B, C \in \mathcal{T}$. Let $A = e_{11}$, $C = e_{1i}$, where $i > 1$. Then, $[e_{11}, e_{1i}]d(B) = e_{1i}d(B) = 0$ for all $B \in \mathcal{T}$ and all $i > 1$. Particularly, for $B \in \mathcal{T}$ such that $Be_{11} = 0$, consider

$$d(B) = d(e_{11}B + (I - e_{11})B).$$

Since

$$d((I - e_{11})B) = d((I - e_{11})(I - e_{11})B) = (I - e_{11})d((I - e_{11})B) = 0$$

and since

$$d(e_{11}B) = d(e_{11}B(I - e_{11})) = (I - e_{11})d(e_{11}B) = 0,$$

it follows that $d(B) = 0$. Let $(a_{ij}) \in \mathcal{T}$. By Lemma 2.6, we have that $d$ determines mappings $d_j : R \to R$ such that $d((a_{ij})) = (b_{ij})$, where

$$b_{ij} = \begin{cases} d_j(a_{11}), & i = 1, \\ 0, & i \neq 1. \end{cases}$$

It is easy to check that $d_j : R \to R$ $(j = 1, 2, \ldots)$ are left derivations. □

## 3 Jordan left derivations

We now turn to discuss Jordan left derivations.

**Lemma 3.1.** *Let $d : \mathcal{A} \to \mathcal{S}$ be a Jordan left derivation, and let $R$ be 2-torsion free. Then, $d = 0$.*

**Proof.** For any $i, j \in \mathbb{N}$ with $i \neq j$ and $a \in R$, $e_{ii}$ and $e_{ii} + ae_{ij}$ are all idempotents. By Lemma 2.3, we give $d(e_{ii}) = 0$ and $d(e_{ii} + ae_{ij}) = 0$, whence $d(ae_{ij}) = 0$. Since $(e_{ij} + ae_{ji})^2 = (e_{ij} + ae_{ji})(e_{ij} + ae_{ji}) = ae_{ii} + ae_{jj}$, it follows that

$$2(e_{ij} + ae_{ji})d(e_{ij} + ae_{ji}) = d((e_{ij} + ae_{ji})^2) = d(ae_{ii} + ae_{jj}) = 0, \tag{3.1}$$

where $i \neq j$. Since $d(e_{ii}) = 0$, we derive

$$d(((a-1)e_{ii} + I)^2) = d((ae_{ii})^2) = 2ae_{ii}d(ae_{ii}) = 2((a-1)e_{ii} + I)d(ae_{ii}),$$

*i.e.*, $2(I - e_{ii})d(ae_{ii}) = 0$. By (3.1) we gain $2(I - e_{jj})d(ae_{jj}) = 2(I - e_{ii})d(ae_{jj}) = 0$ for all $i, j \in \mathbb{N}$, *i.e.*, $2d(ae_{ii}) = 0$. Note that $R$ is 2-torsion free implies $d(ae_{ii}) = 0$. It follows that $d(ae_{ij}) = 0$ for all $i, j \in \mathbb{N}$. For any $A = (a_{ij}) \in \mathcal{A}$, there exists $n \in \mathbb{N}$ such that $A = \sum_{i,j=1}^{n} a_{ij}e_{ij}$. Thus,

$$d(A) = d\left(\sum_{i,j=1}^{n} a_{ij}e_{ij}\right) = \sum_{i,j=1}^{n} d(a_{ij}e_{ij}) = 0.$$

Hence, $d = 0$. □

The following main conclusion is about a Jordan left derivation.

**Theorem 3.2.** *Let $d : \mathcal{S} \to \mathcal{S}$ be a Jordan left derivation, and let $R$ be 2-torsion free. Then, $d = 0$.*

**Proof.** Using Lemma 2.2, we obtain $d(e_{ii}A + Ae_{ii}) = 2e_{ii}d(A)$ for $A \in \mathcal{S}$ and $i = 1, 2, \ldots$. By Lemma 3.1, we have $d(Ae_{ii}) = 0$. Then,

$$(I - e_{ii})d(e_{ii}A) = 0. \tag{3.2}$$

Let $\alpha = e_{ii}A - e_{ii}Ae_{ii}$, $\beta = Ae_{ii} - e_{ii}Ae_{ii}$, $\gamma = (I - e_{ii})A(I - e_{ii})$. Combining

$$d((I - e_{ii})A + A(I - e_{ii})) = d(\alpha) + d(\beta) + 2d(\gamma)$$

and $d((I - e_{ii})A + A(I - e_{ii})) = 2(I - e_{ii})d(A)$ implies

$$d(\alpha) + d(\beta) + 2d(\gamma) = 2(I - e_{ii})d(A) = 2(I - e_{ii})(d(\alpha) + d(\gamma) + d(\beta) + d(e_{ii}Ae_{ii})).$$

Using Lemma 3.1, we obtain

$$d(\beta) = 0, d(e_{ii}Ae_{ii}) = 0.$$

By (3.2), we obtain

$$2(I - e_{ii})(d(\alpha) + d(\gamma)) = 2(I - e_{ii})d(\gamma).$$

Hence, $d(\alpha) + 2d(\gamma) = 2(I - e_{ii})d(\gamma)$, *i.e.*,

$$d(\alpha) + 2e_{ii}d(\gamma) = 0, \tag{3.3}$$

for all $i = 1, 2, \ldots$. On the one hand,

$$d((I - e_{ii})\gamma + \gamma(I - e_{ii})) = 2d(\gamma).$$

On the other hand,

$$d((I - e_{ii})\gamma + \gamma(I - e_{ii})) = 2(I - e_{ii})d(\gamma).$$

Hence, $2e_{ii}d(\gamma) = 0$. Observe that $R$ is 2-torsion free, which implies

$$e_{ii}d(\gamma) = 0. \tag{3.4}$$

By (3.3), we derive $d(\alpha) = 0$. Since $0 = d(Ae_{ii}) = d(\beta + e_{ii}Ae_{ii}) = d(e_{ii}Ae_{ii})$, which follows that

$$d(e_{ii}A) = d(\alpha + e_{ii}Ae_{ii}) = 0, \tag{3.5}$$

(3.4) and (3.5) imply

$$e_{ii}d(A) = e_{ii}d((I - e_{ii})A + e_{ii}A) = 0,$$

for all $i = 1, 2, \ldots$. Finally, $d(A) = 0$ for all $A \in \mathcal{S}$. □

Recall that $\mathcal{T} = \{(a_{ij}) \in \mathcal{S} | a_{ij} = 0 \text{ for all } i, j \in \mathbb{N} \text{ such that } i > j\}$ and $\mathcal{TF} = \{M \in \mathcal{T} | M \text{ is finite}\}$.

**Lemma 3.3.** *Let $d : \mathcal{TF} \to \mathcal{T}$ be a Jordan left derivation, and let R be 2-torsion free. Then, there exist Jordan left derivations $d_j : R \to R(j = 1, 2, \ldots)$ such that $d((a_{ij})) = (b_{ij})$, where $(a_{ij}) \in \mathcal{TF}$* and

$$b_{ij} = \begin{cases} d_j(a_{11}), & i = 1, \\ 0, & i \neq 1. \end{cases}$$

**Proof.** For any $i, j \in \mathbb{N}$ with $i < j$ and $a \in R$, $e_{ii}$ and $e_{ii} + ae_{ij}$ are all idempotents of $\mathcal{TF}$. By Lemma 2.3, we have $d(e_{ii}) = 0$ and $d(e_{ii} + ae_{ij}) = 0$, whence

$$d(ae_{ij}) = 0, \quad a \in R, \ i < j. \tag{3.6}$$

Since

$$2ae_{ii}d(ae_{ii}) = d((ae_{ii})^2) = d(((a-1)e_{ii} + I)^2) = 2((a-1)e_{ii} + I)d(ae_{ii}),$$

for $a \in R$ and $i \in \mathbb{N}$, which follows that $2(I - e_{ii})d(ae_{ii}) = 0$, since

$$2ae_{ii}d(ae_{ii}) = d((ae_{ii})^2) = d((ae_{ii} + e_{1i})^2) = 2(ae_{ii} + e_{1i})d(ae_{ii} + e_{1i})$$

for $a \in R$ and $i > 1$, and since

$$2(ae_{ii} + e_{1i})d(ae_{ii} + e_{1i}) = 2(ae_{ii} + e_{1i})d(ae_{ii}),$$

we gain $2e_{1i}d(ae_{ii}) = 0$ for $a \in R$ and $i > 1$. Note that $R$ is 2-torsion free which implies $(I - e_{ii})d(ae_{ii}) = 0$ for $i \in \mathbb{N}$ and $e_{1i}d(ae_{ii}) = 0$ for $i > 1$. Hence,

$$d(ae_{ii}) = 0, a \in R, i > 1. \tag{3.7}$$

Let $A = (a_{ij}) \in \mathcal{TF}$. Then, $A = \sum_{1 \le i \le j \le n} a_{ij}e_{ij}$ for some positive integer $n$. Then, (3.6) and (3.7) imply

$$d(A) = \sum_{1 \le i \le j \le n} d(a_{ij}e_{ij}) = d(a_{11}e_{11}).$$

Note that $(I - e_{11})d(a_{11}e_{11}) = 0$. Then, $d$ determines mappings $d_j : R \to R$ such that $d(a_{11}e_{11}) = (b_{ij})$, where

$$b_{ij} = \begin{cases} d_j(a_{11}), & i = 1, \\ 0, & i \neq 1. \end{cases}$$

It is easy to check that $d_j : R \to R$ $(j = 1, 2, \ldots)$ are Jordan left derivations. □

we establish the new representation of Jordan left derivations.

**Theorem 3.4.** *Let $d : \mathcal{T} \to \mathcal{T}$ be a Jordan left derivation, and let R be 2-torsion free. Then, there exist Jordan left derivations $d_j : R \to R$ $(j = 1, 2, \ldots)$ such that $d((a_{ij})) = (b_{ij})$,* where *$(a_{ij}) \in \mathcal{T}$ and*

$$b_{ij} = \begin{cases} d_j(a_{11}), & i = 1, \\ 0, & i \neq 1. \end{cases}$$

**Proof.** By Lemma 2.2, we obtain $d(e_{ii}A + Ae_{ii}) = 2e_{ii}d(A)$, where $A \in \mathcal{T}$ and $i = 1, 2, \ldots$. Using Lemma 3.3, we have $(I - e_{ii})d(Ae_{ii}) = 0$, and thus,

$$(I - e_{ii})d(e_{ii}A) = 0. \tag{3.8}$$

Let $\alpha = e_{ii}A - e_{ii}Ae_{ii}$, $\beta = Ae_{ii} - e_{ii}Ae_{ii}$, and $\gamma = (I - e_{ii})A(I - e_{ii})$. Combining

$$d((I - e_{ii})A + A(I - e_{ii})) = d(\alpha) + d(\beta) + 2d(\gamma)$$

and $d((I - e_{ii})A + A(I - e_{ii})) = 2(I - e_{ii})d(A)$ follows

$$d(\alpha) + d(\beta) + 2d(\gamma) = 2(I - e_{ii})d(A) = 2(I - e_{ii})(d(\alpha) + d(\gamma) + d(\beta) + d(e_{ii}Ae_{ii})). \tag{3.9}$$

Using Lemma 3.3, we derive $d(\beta) = 0$ for $i = 1, 2, \ldots$ and $d(e_{ii}Ae_{ii}) = 0$ for all $i > 1$. By (3.8), we obtain

$$2(I - e_{ii})(d(\alpha) + d(\gamma)) = 2(I - e_{ii})d(\gamma). \tag{3.10}$$

Then, (3.9) and (3.10) imply $d(\alpha) + 2d(\gamma) = 2(I - e_{ii})d(\gamma)$, *i.e.*,

$$d(\alpha) + 2e_{ii}d(\gamma) = 0, \tag{3.11}$$

for all $i = 1, 2, \ldots$. On the one hand,

$$d((I - e_{ii})\gamma + \gamma(I - e_{ii})) = 2d(\gamma).$$

On the other hand,

$$d((I - e_{ii})\gamma + \gamma(I - e_{ii})) = 2(I - e_{ii})d(\gamma).$$

Hence, $2e_{ii}d(\gamma) = 0$. Since $R$ is 2-torsion free, it follows that

$$e_{ii}d(\gamma) = 0, \tag{3.12}$$

for all $i = 1, 2, \ldots$. Then, (3.11) and (3.12) imply $d(\alpha) = 0$ for all $i = 1, 2, \ldots$. Observe that $d(Ae_{ii}) = d(\beta) + d(e_{ii}Ae_{ii}) = 0$ for all $i > 1$. We assume $(a_{ij}) \in \mathcal{T}$ and use Lemma 3.3 to obtain $d$ determined mappings $d_j : R \to R$ such that $d((a_{ij})) = (b_{ij})$, where

$$b_{ij} = \begin{cases} d_j(a_{11}), & i = 1, \\ 0, & i \neq 1. \end{cases}$$

It is easy to verify that $d_j : R \to R$ $(j = 1, 2, \ldots)$ are Jordan left derivations. □

**Acknowledgements:** Zhang would like to thank Prof. Xiankun Du and Prof. Xiaowei Xu for their suggestions that greatly improved the original manuscript and is also thankful for the careful reviews of referees and the editor.

**Funding information**: The work as supported by the National Natural Science Foundation of China (NSFC) (No. 11901079), China Postdoctoral Science Foundation (No. 2021M700751), and the Scientific and Technological Research Program Foundation of Jilin Province (No. JJKH20190690KJ; No. 20200401085GX; No. JJKH20220091KJ).

**Author contributions**: All authors have accepted responsibility for the entire content of this manuscript and consented to its submission to the journal, reviewed all the results and approved the final version of the manuscript. Zhang gave the conception of the study and presented results, Ma prepared the manuscript, Hu and Sun checked and revised the manuscript.

**Conflict of interest**: The authors declare that there is no conflict of interest.

# References

[1] M. P. Thomas, *The image of a derivation is contained in the radical*, Ann. of Math. (2) **128** (1988), no. 3, 435–460, DOI: https://doi.org/10.2307/1971432.

[2] M. Brešar and J. Vukman, *On left derivations and related mappings*, Proc. Amer. Math. Soc. **110** (1990), no. 1, 7–16, DOI: https://doi.org/10.2307/2048234.

[3] M. H. Ahmadi Gandomani and M. J. Mehdipour, *Jordan, Jordan right and Jordan left derivations on convolution algebras*, Bull. Iranian Math. Soc. **45** (2019), no. 1, 189–204, DOI: https://doi.org/10.1007/s41980-018-0125-7.

[4] Y. Ahmed and W. A. Dudek, *Left Jordan derivations on certain semirings*, Hacet. J. Math. Stat. **50** (2021), no. 3, 624–633, DOI: https://doi.org/10.15672/hujms.491343.

[5] G. An, Y. Ding, and J. Li, *Characterizations of Jordan left derivations on some algebras*, Banach J. Math. Anal. **10** (2016), no. 3, 466–481, DOI: https://doi.org/10.1215/17358787-3599675.

[6] B. Fadaee and H. Ghahramani, *Jordan left derivations at the idempotent elements on reflexive algebras*, Publ. Math. Debrecen **92** (2018), no. 3-4, 261–275, DOI: https://doi.org/10.5486/PMD.2018.7785.

[7] N. M. Ghosseiri, *On Jordan left derivations and generalized Jordan left derivations of matrix rings*, Bull. Iranian Math. Soc. **38** (2012), no. 3, 689–698, DOI: https://doi.org/10.4153/CMB-2011-089-1.

[8] M. Ashraf, *On Jordan left derivations of Lie ideals in prime rings*, Southeast Asian Bull. Math. **25** (2001), no. 3, 379–382, DOI: https://doi.org/10.1007/s10012-001-0379-4.

[9] J. Vukman, *Jordan left derivations on semiprime rings*, Math. J. Okayama Univ. **39** (1997), 1–6.

[10] M. Aşci and C. şahin, *The commutativity in prime gamma rings with left derivation*, Int. Math. Forum **2** (2007), no. 1–4, 103–108, DOI: http://dx.doi.org/10.12988/imf.2007.07008.

[11] K. K. Dey, A. C. Paul, and B. Davvaz, *On centralizing automorphisms and Jordan left derivations on σ-prime gamma rings*, Hacet. J. Math. Stat. **44** (2015), no. 1, 51–57, DOI: https://doi.org/10.15672/HJMS.2015449098.

[12] A. Ebadian and M. Gordji Eshaghi, *Left Jordan derivations on Banach algebras*, Iran. J. Math. Sci. Inform. **6** (2011), no. 1, 1–6, DOI: http://dx.doi.org/10.7508/ijmsi.2011.01.001.

[13] Y. S. Jung and K. H. Park, *Left Jordan derivations on Banach algebras and related mappings*, Bull. Korean Math. Soc. **47** (2010), no. 1, 151–157, DOI: https://doi.org/10.4134/BKMS.2010.47.1.151.

[14] Y. S. Jung, *On the generalized Hyers-Ulam stability of module left derivations*, J. Math. Anal. Appl. **339** (2008), no. 1, 108–114, DOI: https://doi.org/10.1016/j.jmaa.2007.07.003.

[15] W. G. Park and I. S. Chang, *Stability for Jordan left derivations mapping into the radical of Banach algebras*, Honam Math. J. **34** (2012), no. 1, 55–62, DOI: https://doi.org/10.5831/HMJ.2012.34.1.55.

[16] K. H. Park and Y. S. Jung, *Jordan derivations and Jordan left derivations of Banach algebras*, Commun. Korean Math. Soc. **17** (2002), no. 2, 245–252, DOI: https://doi.org/10.4134/CKMS.2002.17.2.245.

[17] J. Vukman, *On left Jordan derivations of rings and Banach algebras*, Aequationes Math. **75** (2008), no. 3, 260–266, DOI: https://doi.org/10.1007/s00010-007-2872-z.

[18] K. Glazek, *A Guide to Literature on Semirings and their Applications in Mathematics and Information Sciences with Complete Bibliography*, Kluwer Academic Publishers, Dodrecht, 2002.

[19] J. S. Golan, *Semirings and their Applications*, Kluwer Academic Publishers, Dodrecht, 1999.

[20] J. Li and J. Zhou, *Jordan left derivations and some left derivable maps*, Oper. Matrices **4** (2010), no. 1, 127–138, DOI: https://dx.doi.org/10.7153/oam-04-06.

[21] S. Ali, *On generalized left derivations in rings and Banach algebras*, Aequationes Math. **81** (2011), no. 3, 209–226, DOI: https://doi.org/10.1007/s00010-011-0070-5.

[22] M. Ashraf and S. Ali, *On generalized Jordan left derivations in rings*, Bull. Korean Math. Soc. **45** (2008), no. 2, 253–261, DOI: https://doi.org/10.4134/BKMS.2008.45.2.253.

[23] H. Cao, J. Lv, and J. M. Rassias, *Superstability for generalized module left derivations and generalized module derivations on a Banach module, II*, J. Inequal. Pure Appl. Math. **10** (2009), no. 3, Paper No. 85, 8 p.

[24] A. Ghosh and O. Prakash, *Jordan left {g, h}-derivation over Some Algebras*, Indian J. Pure Appl. Math. **51** (2020), no. 4, 1433–1450, DOI: https://doi.org/10.1007/s13226-020-0475-8.

[25] A. Ghosh and O. Prakash, *Characterization of Jordan {g, h}-derivations over Matrix Algebras*, J. Algebr. Syst. **11** (2023), no. 1, 77–95, DOI: https://doi.org/10.22044/JAS.2022.11250.1562.

[26] D. Han and F. Wei, *Generalized Jordan left derivations on semiprime algebras*, Monatsh. Math. **161** (2010), no. 1, 77–83, DOI: https://doi.org/10.1007/s00605-009-0116-0.

[27] S. Y. Kang and I. S. Chang, *Approximation of generalized left derivations*, Abstr. Appl. Anal. (2008), Art. ID 915292, 8pp, DOI: https://doi.org/10.1155/2008/915292.

[28] L. Oukhtite, *Generalized Jordan left derivations in rings with involution*, Demonstr. Math. **45** (2012), no. 4, 807–812, DOI: https://doi.org/10.1515/dema-2013-0420.

[29] N. Rehman and A. Z. Ansari, *On Lie ideals and generalized Jordan left derivations of prime rings*, Ukrainian Math. J. **65** (2014), no. 8, 1247–1256, DOI: https://doi.org/10.1007/s11253-014-0855-5.

[30] J. Zhu and C. Xiong, *Characterization of generalized Jordan *-left derivations on real nest algebras*, Linear Algebra Appl. **404** (2005), 325–344, DOI: https://doi.org/10.1016/j.laa.2005.02.034.

[31] S. Cui and F. Niu, *Derivation on infinite matrix rings, (Chinese)*, J. Math. Res. Exposition **24** (2004), no. 2, 353–358.

[32] X. Xu and H. Zhang, *Jordan left derivations in full and upper triangular matrix rings*, Electron. J. Linear Algebra **20** (2010), 753–759, DOI: https://doi.org/10.13001/1081-3810.1407.