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Research Article

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On a generalization of the Opial inequality

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Abstract: Inequalities are essential in pure and applied mathematics. In particular, Opial's inequality and its generalizations have been playing an important role in the study of the existence and uniqueness of initial and boundary value problems. In this work, some new Opial-type inequalities are given and applied to generalized Riemann-Liouville-type integral operators.

Keywords: Opial-type inequalities, fractional derivatives and integrals, fractional integral inequalities

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1 Introduction

Integral inequalities are used in countless mathematical problems such as approximation theory and spectral analysis, statistical analysis, and the theory of distributions. Studies involving integral inequalities play an important role in several areas of science and engineering.

In recent years, there has been a growing interest in the study of many classical inequalities applied to integral operators associated with different types of fractional derivatives, since integral inequalities and their applications play a vital role in the theory of differential equations and applied mathematics. Some of the inequalities studied are Gronwall, Chebyshev, Jensen-type, Hermite-Hadamard-type, Ostrowski-type, Grüss-type, Hardy-type, Gagliardo-Nirenberg-type, Opial-type, reverse Minkowski, and reverse Hölder inequalities (see, e.g., [1–14]).

In this work, we obtain new Opial-type inequalities, and we apply them to the generalized Riemann-Liouville-type integral operators defined in [15], which include most of the known Riemann-Liouville-type integral operators.

2 Preliminaries

One of the first operators that can be called fractional is the Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{C}$, with $\mathrm{Re}(\alpha) > 0$, defined as follows (see [16]).

Definition 1. Let $a < b$ and $f \in L^1((a, b); \mathbb{R})$. The right and left side Riemann-Liouville fractional integrals of order α, with $\mathrm{Re}(\alpha) > 0$, are defined, respectively, by

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$$^{\mathrm{RL}}J^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\mathrm{d}s \tag{1}$$

and

$$^{\mathrm{RL}}J^{\alpha}_{b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)\mathrm{d}s, \tag{2}$$

with $t \in (a, b)$.

When $\alpha \in (0, 1)$, their corresponding Riemann-Liouville fractional derivatives are given by

$$(^{\mathrm{RL}}D^{\alpha}_{a^+}f)(t) = \frac{\mathrm{d}}{\mathrm{d}t}(^{\mathrm{RL}}J^{1-\alpha}_{a^+}f(t)) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \frac{f(s)}{(t-s)^{\alpha}}\mathrm{d}s,$$

$$(^{\mathrm{RL}}D^{\alpha}_{b^-}f)(t) = -\frac{\mathrm{d}}{\mathrm{d}t}(^{\mathrm{RL}}J^{1-\alpha}_{b^-}f(t)) = -\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^b \frac{f(s)}{(s-t)^{\alpha}}\mathrm{d}s.$$

Other definitions of fractional operators are the following ones.

Definition 2. Let $a < b$ and $f \in L^1((a, b); \mathbb{R})$. The *right and left side Hadamard fractional integrals of order* α, with $\mathrm{Re}(\alpha) > 0$, are defined, respectively, by

$$H^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s)}{s}\mathrm{d}s \tag{3}$$

and

$$H^{\alpha}_{b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\log\frac{s}{t}\right)^{\alpha-1}\frac{f(s)}{s}\mathrm{d}s, \tag{4}$$

with $t \in (a, b)$.

When $\alpha \in (0, 1)$, *Hadamard fractional derivatives* are given by the following expressions:

$$(^{\mathrm{H}}D^{\alpha}_{a^+}f)(t) = t\frac{\mathrm{d}}{\mathrm{d}t}(H^{1-\alpha}_{a^+}f(t)) = \frac{1}{\Gamma(1-\alpha)}t\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \left(\log\frac{t}{s}\right)^{-\alpha}\frac{f(s)}{s}\mathrm{d}s,$$

$$(^{\mathrm{H}}D^{\alpha}_{b^-}f)(t) = -t\frac{\mathrm{d}}{\mathrm{d}t}(H^{1-\alpha}_{b^-}f(t)) = \frac{-1}{\Gamma(1-\alpha)}t, \frac{\mathrm{d}}{\mathrm{d}t}\int_t^b \left(\log\frac{s}{t}\right)^{-\alpha}\frac{f(s)}{s}\mathrm{d}s,$$

with $t \in (a, b)$.

Definition 3. Let $0 < a < b$, $g : [a, b] \to \mathbb{R}$ an increasing positive function on $(a, b]$ with continuous derivative on (a, b), $f : [a, b] \to \mathbb{R}$ an integrable function, and $\alpha \in (0, 1)$ a fixed real number. The right and left side fractional integrals in [17] of order α of f with respect to g are defined, respectively, by

$$I^{\alpha}_{g,a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{g'(s)f(s)}{(g(t)-g(s))^{1-\alpha}}\mathrm{d}s \tag{5}$$

and

$$I^{\alpha}_{g,b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \frac{g'(s)f(s)}{(g(s)-g(t))^{1-\alpha}}\mathrm{d}s, \tag{6}$$

with $t \in (a, b)$.

There are other definitions of integral operators in the global case, but they are slight modifications of the previous ones.

3 General fractional integral of Riemann-Liouville type

Now, we give the definition of a general fractional integral in [15].

Definition 4. Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a, b] \to \mathbb{R}$ be a positive function on $(a, b]$ with continuous positive derivative on (a, b), and $G : [0, g(b) - g(a)] \times (0, \infty) \to \mathbb{R}$ a continuous function which is positive on $(0, g(b) - g(a)] \times (0, \infty)$. Let us define the function $T : [a, b] \times [a, b] \times (0, \infty) \to \mathbb{R}$ by

$$T(t, s, \alpha) = \frac{G(|g(t) - g(s)|, \alpha)}{g'(s)}.$$

The *right and left integral operators*, denoted, respectively, by J^{α}_{T,a^+} and J^{α}_{T,b^-}, are defined for each measurable function f on $[a, b]$ as

$$J^{\alpha}_{T,a^+} f(t) = \int_a^t \frac{f(s)}{T(t, s, \alpha)} \mathrm{d}s, \tag{7}$$

$$J^{\alpha}_{T,b^-} f(t) = \int_t^b \frac{f(s)}{T(t, s, \alpha)} \mathrm{d}s, \tag{8}$$

with $t \in [a, b]$.

We say that $f \in L^1_T[a, b]$ if $J^{\alpha}_{T,a^+}|f|(t), J^{\alpha}_{T,b^-}|f|(t) < \infty$ for every $t \in [a, b]$.

Note that these operators generalize the integral operators in Definitions 1–3:

(1) If we choose

$$g(t) = t, \quad G(x, \alpha) = \Gamma(\alpha)x^{1-\alpha}, \quad T(t, s, \alpha) = \Gamma(\alpha)\, |t - s|^{1-\alpha},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left Riemann-Liouville fractional integrals ${}^{\mathrm{RL}}J^{\alpha}_{a^+}$ and ${}^{\mathrm{RL}}J^{\alpha}_{b^-}$ in (1) and (2), respectively. Its corresponding right and left Riemann-Liouville fractional derivatives are

$$({}^{\mathrm{RL}}D^{\alpha}_{a^+}f)(t) = \frac{\mathrm{d}}{\mathrm{d}t}({}^{\mathrm{RL}}J^{1-\alpha}_{a^+}f(t)), \quad ({}^{\mathrm{RL}}D^{\alpha}_{b^-}f)(t) = -\frac{\mathrm{d}}{\mathrm{d}t}({}^{\mathrm{RL}}J^{1-\alpha}_{b^-}f(t)).$$

(2) If we choose

$$g(t) = \log t, \quad G(x, \alpha) = \Gamma(\alpha)x^{1-\alpha}, \quad T(t, s, \alpha) = \Gamma(\alpha)t \left| \log \frac{t}{s} \right|^{1-\alpha},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left Hadamard fractional integrals $H^{\alpha}_{a^+}$ and $H^{\alpha}_{b^-}$ in (3) and (4), respectively. Its corresponding right and left Hadamard fractional derivatives are

$$({}^{\mathrm{H}}D^{\alpha}_{a^+}f)(t) = t\frac{\mathrm{d}}{\mathrm{d}t}(H^{1-\alpha}_{a^+}f(t)), \quad ({}^{\mathrm{H}}D^{\alpha}_{b^-}f)(t) = -t\frac{\mathrm{d}}{\mathrm{d}t}(H^{1-\alpha}_{b^-}f(t)).$$

(3) If we choose a function g with the properties in Definition 4 and

$$G(x, \alpha) = \Gamma(\alpha)x^{1-\alpha}, \quad T(t, s, \alpha) = \Gamma(\alpha)\frac{|g(t) - g(s)|^{1-\alpha}}{g'(s)},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left fractional integrals I^{α}_{g,a^+} and I^{α}_{g,b^-} in (5) and (6), respectively.

Definition 5. Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a, b] \to \mathbb{R}$ be a positive function on $(a, b]$ with continuous positive derivative on (a, b), and $G : [0, g(b) - g(a)] \times (0, \infty) \to \mathbb{R}$ a continuous function which is positive on $(0, g(b) - g(a)] \times (0, \infty)$. For each function $f \in L^1_T[a, b]$, its *right and left generalized derivatives of order* α are defined, respectively, by

$$D^{\alpha}_{T,a^+} f(t) = \frac{1}{g'(t)} \frac{\mathrm{d}}{\mathrm{d}t} (J^{1-\alpha}_{T,a^+} f(t)),$$
$$D^{\alpha}_{T,b^-} f(t) = \frac{-1}{g'(t)} \frac{\mathrm{d}}{\mathrm{d}t} (J^{1-\alpha}_{T,b^-} f(t)), \tag{9}$$

for each $t \in (a, b)$.

Note that if we choose

$$g(t) = t, \quad G(x, \alpha) = \Gamma(\alpha) x^{1-\alpha}, \quad T(t, s, \alpha) = \Gamma(\alpha)|t - s|^{1-\alpha},$$

then $D^{\alpha}_{T,a^+} f(t) = {}^{\mathrm{RL}}D^{\alpha}_{a^+} f(t)$ and $D^{\alpha}_{T,b^-} f(t) = {}^{\mathrm{RL}}D^{\alpha}_{b^-} f(t)$. Also, we can obtain Hadamard and other fractional derivatives as particular cases of this generalized derivative.

4 Opial-type inequality

In 1960, Opial [18] proved the following inequality:

If $f \in C^1[0, h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0, h)$, then

$$\int_0^h |f(x) f'(x)| \mathrm{d}x \le \frac{h}{4} \int_0^h |f'(x)|^2 \mathrm{d}x.$$

Opial's inequality and its generalizations play a main role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations [19–23]. For an extensive survey on these Opial-type inequalities, see [19,23].

We need the following result in [24, p. 44] (see the original proof in [25]). Although the result in [24, p. 44] deals with measures on $(0, \infty)$, it can be reformulated for measures on a compact interval (see, e.g., [26, Theorem 3.1]).

4.1 Muckenhoupt inequality

Let us consider $1 \le p \le q < \infty$ and measures μ_0, μ_1 on $[a, b]$ with $\mu_0(\{b\}) = 0$. Then there exists a positive constant C such that

$$\left\| \int_a^x u(t) \mathrm{d}t \right\|_{L^q([a,b],\mu_0)} \le C \|u\|_{L^p([a,b],\mu_1)}$$

for any measurable function u on $[a, b]$, if and only if

$$B := \sup_{a<x<b} \mu_0([x, b))^{1/q} \|(\mathrm{d}\mu_1/\mathrm{d}x)^{-1}\|^{1/p}_{L^{1/(p-1)}([a,x])} < \infty, \tag{10}$$

where we use the convention $0 \cdot \infty = 0$. Moreover, we can choose

$$C = \begin{cases} B\left(\dfrac{q}{q-1}\right)^{(p-1)/p} q^{1/q}, & \text{if } p > 1, \\ B, & \text{if } p = 1. \end{cases} \tag{11}$$

Muckenhoupt inequality will play a crucial role to prove the next result, which improves the classical Opial inequality in several ways:

(1) It allows us to integrate with respect to very general measures.
(2) The hypotheses $f(b) = 0$ and $f > 0$ on (a, b) are no longer needed.
(3) The hypothesis $f \in C^1[a, b]$ is replaced by a weaker one: it is sufficient to require f to be absolutely continuous on $[a, b]$.

Theorem 1. *Let us consider* $1 \le p \le q < \infty$ *and two measures* μ_0, μ_1 *on* $[a, b]$ *with* $\mu_0(\{b\}) = 0$. *Assume that the constant B defined as follows is finite*:

$$B := \sup_{a<x<b} \mu_0([x, b))^{1/q} \|(\mathrm{d}\mu_1/\mathrm{d}x)^{-1}\|^{1/p}_{L^{1/(p-1)}([a,x])}.$$

Then, for every absolutely continuous function f *on* $[a, b]$ *with* $f(a) = 0$,

$$\|ff'\|_{L^1([a,b],\mu_0)} \le C\|f'\|_{L^p([a,b],\mu_1)} \|f'\|_{L^{q/(q-1)}([a,b],\mu_0)},$$

where the constant C can be chosen as

$$C := \begin{cases} B\left(\dfrac{q}{q-1}\right)^{(p-1)/p} q^{1/q}, & \text{if } p > 1, \\ B, & \text{if } p = 1. \end{cases}$$

Proof. By the Muckenhoupt inequality, the constant C satisfies

$$\left\| \int_a^x u(t)\mathrm{d}t \right\|_{L^q([a,b],\mu_0)} \le C\|u\|_{L^p([a,b],\mu_1)}$$

for any measurable function u on $[a, b]$. For each absolutely continuous function f on $[a, b]$ with $f(a) = 0$, we have that there exists f' a.e. on $[a, b]$, $f' \in L^1[a, b]$, and

$$f(x) = \int_a^x f'(t)\mathrm{d}t$$

for every $x \in [a, b]$. Consequently,

$$\|f\|_{L^q([a,b],\mu_0)} \le C\|f'\|_{L^p([a,b],\mu_1)}.$$

Hence, the Hölder inequality gives

$$\|ff'\|_{L^1([a,b],\mu_0)} \le \|f\|_{L^q([a,b],\mu_0)} \|f'\|_{L^{q/(q-1)}([a,b],\mu_0)} \le C\|f'\|_{L^p([a,b],\mu_1)} \|f'\|_{L^{q/(q-1)}([a,b],\mu_0)}.$$ □

Remark 1. For each absolutely continuous function f on $[a, b]$ the set

$$S = \{x \in [a, b] : \nexists f'(x)\}$$

has zero Lebesgue measure, but it is possible to have $\mu_0(S) > 0$ and/or $\mu_1(S) > 0$. The argument in the proof of Theorem 1 gives that the inequality holds for any fixed choice of values of f' on S.

Theorem 1 has the following direct consequence.

Corollary 2. *Let us consider* $1 \le p \le q < \infty$ *and a measure* μ *on* $[a,b]$ *with* $\mu(\{b\}) = 0$. *Assume that the constant* B *defined as follows is finite*:

$$B \coloneqq \sup_{a<x<b} \mu([x,b))^{1/q} \|(\mathrm{d}\mu/\mathrm{d}x)^{-1}\|^{1/p}_{L^{1/(p-1)}([a,x])}.$$

Then, for every absolutely continuous function f *on* $[a,b]$ *with* $f(a) = 0$,

$$\|ff'\|_{L^1([a,b],\mu)} \le C\|f'\|_{L^p([a,b],\mu)}\|f'\|_{L^{q/(q-1)}([a,b],\mu)},$$

where the constant C *can be chosen as*

$$C \coloneqq \begin{cases} B\left(\dfrac{q}{q-1}\right)^{(p-1)/p} q^{1/q}, & \text{if } p > 1, \\ B, & \text{if } p = 1. \end{cases}$$

Corollary 2 is a tool to obtain the following result.

Theorem 3. *Let us consider* $1 \le p \le 2$ *and a measure* μ *on* $[a,b]$ *with* $\mu(\{b\}) = 0$. *Assume that the constant* B *defined as follows is finite*:

$$B \coloneqq \sup_{a<x<b} \mu([x,b))^{(p-1)/p} \|(\mathrm{d}\mu/\mathrm{d}x)^{-1}\|^{1/p}_{L^{1/(p-1)}([a,x])}.$$

Then, for every absolutely continuous function f *on* $[a,b]$ *with* $f(a) = 0$,

(1) *if* $1 < p \le 2$,

$$\|ff'\|_{L^1([a,b],\mu)} \le B\left(\frac{p^2}{p-1}\right)^{(p-1)/p} \|f'\|^2_{L^p([a,b],\mu)}.$$

(2) *if* $p = 1$ *and* μ *is a finite measure,*

$$\|ff'\|_{L^1([a,b],\mu)} \le B\|f'\|^2_{L^1([a,b],\mu)}.$$

Proof. Assume first that $1 < p \le 2$. Let us consider $q \ge 2$ such that $1/p + 1/q = 1$, and so, $p = q/(q-1)$ and $q = p/(p-1)$. Thus, $1 < p \le 2 \le q < \infty$ and Corollary 2 gives the result in part (a), since

$$B\left(\frac{q}{q-1}\right)^{(p-1)/p} q^{1/q} = Bp^{(p-1)/p}\left(\frac{p}{p-1}\right)^{(p-1)/p} = B\left(\frac{p^2}{p-1}\right)^{(p-1)/p}.$$

Assume now that μ is a finite measure and fix an absolutely continuous function f on $[a,b]$ such that $f(a) = 0$ and $f' \in L^{p_0}([a,b],\mu)$ for some $p_0 > 1$. We have proved that

$$\|ff'\|_{L^1([a,b],\mu)} \le B\left(\frac{p^2}{p-1}\right)^{(p-1)/p} \|f'\|^2_{L^p([a,b],\mu)}$$

for every $1 < p \le \min\{p_0, 2\}$.

Let us consider $B = B(p)$ as a function of p. Thus,

$$B(p) \le \mu([a,b))^{(p-1)/p} \|(\mathrm{d}\mu/\mathrm{d}x)^{-1}\|^{1/p}_{L^{1/(p-1)}([a,b])}.$$

Since μ is a finite measure, we have

$$\begin{aligned} \operatorname{limsup}_{p\to 1^+} B(p) &\le \lim_{p\to 1^+} \mu([a,b))^{(p-1)/p} \|(\mathrm{d}\mu/\mathrm{d}x)^{-1}\|^{1/p}_{L^{1/(p-1)}([a,b])} \\ &= \|(\mathrm{d}\mu/\mathrm{d}x)^{-1}\|_{L^\infty([a,b])} = B(1). \end{aligned}$$

Since

$$|f'|^p \le |f'|^{p_0}\chi_{\{|f'|\ge 1\}} + \chi_{\{|f'|<1\}} \le |f'|^{p_0} + 1 \in L^1([a,b],\mu)$$

for every $1 < p \le p_0$, dominated convergence theorem gives

$$\lim_{p\to 1^+} \|f'\|^2_{L^p([a,b],\mu)} = \|f'\|^2_{L^1([a,b],\mu)}.$$

Finally, we have

$$\lim_{p\to 1^+}\left(\frac{p^2}{p-1}\right)^{(p-1)/p} = 1,$$

and the desired inequality holds if $f' \in L^{p_0}([a,b],\mu)$ for some $p_0 > 1$.

Let us consider now any absolutely continuous function f on $[a,b]$ such that $f(a) = 0$. Define the measure μ^* on $[a,b]$ by $d\mu^* = d\mu + dx$. Since f is an absolutely continuous function on $[a,b]$, $f' \in L^1[a,b]$. If $f' \notin L^1([a,b],\mu)$, then the inequality is direct. So, we can assume that $f' \in L^1([a,b],\mu)$. Thus, there exists a sequence $\{s_n\}$ of simple functions with

$$\lim_{n\to\infty} \|f' - s_n\|_{L^1([a,b],\mu^*)} = 0.$$

Hence, there exists N such that

$$\|s_n\|_{L^1([a,b],\mu^*)} - \|f'\|_{L^1([a,b],\mu^*)} \le \|f' - s_n\|_{L^1([a,b],\mu^*)} < 1$$

for every $n \ge N$. Therefore,

$$\|s_n\|_{L^1([a,b],\mu)} \le \|s_n\|_{L^1([a,b],\mu^*)} \le \|f'\|_{L^1([a,b],\mu^*)} + 1 \tag{12}$$

for every $n \ge N$.

Since μ is a finite measure, if we define $f_n(x) = \int_a^x s_n(t)dt$, then $f_n \in C[a,b] \subset L^p([a,b],\mu)$ for every $p \ge 1$, and we have proved that

$$\|f_n f_n'\|_{L^1([a,b],\mu)} \le B\|f_n'\|^2_{L^1([a,b],\mu)}. \tag{13}$$

Also, for any $x \in [a,b]$

$$|f(x) - f_n(x)| = \left|\int_a^x (f'(t) - s_n(t))dt\right| \le \int_a^x |f'(t) - s_n(t)|dt \le \|f' - s_n\|_{L^1([a,b],\mu^*)}. \tag{14}$$

Applying inequalities (12), (13), and (14) where appropriate,

$$\begin{aligned}
\|ff' - f_n f_n'\|_{L^1([a,b],\mu)} &= \int_a^b |ff' - f_n f_n'|d\mu \\
&\le \int_a^b |ff' - ff_n'|d\mu + \int_a^b |ff_n' - f_n f_n'|d\mu \\
&\le \|f\|_\infty \int_a^b |f' - f_n'|d\mu + \|s_n\|_{L^1([a,b],\mu^*)} \int_a^b |f' - s_n|d\mu \\
&\le \|f\|_\infty \|f' - s_n\|_{L^1([a,b],\mu^*)} + (\|f'\|_{L^1([a,b],\mu^*)} + 1)\mu([a,b])\|f' - s_n\|_{L^1([a,b],\mu^*)}
\end{aligned}$$

for every $n \ge N$. Hence,

$$\lim_{n\to\infty} \|ff' - f_n f_n'\|_{L^1([a,b],\mu)} = 0$$

and so

$$\|ff'\|_{L^1([a,b],\mu)} \leq B\|f'\|^2_{L^1([a,b],\mu)},$$

which completes part (b). □

If we choose μ as the Lebesgue measure on $[a, b]$, then we obtain the following results.

Corollary 4. *Let us consider* $1 \leq p \leq q < \infty$. *Then*

$$\|ff'\|_{L^1([a,b])} \leq \left(\frac{b-a}{1/q+(p-1)/p}\right)^{1/q+(p-1)/p} \left(\frac{q(p-1)}{p(q-1)}\right)^{(p-1)/p} \|f'\|_{L^p([a,b])}\|f'\|_{L^{q/(q-1)}([a,b])}$$

if $p > 1$, *and*

$$\|ff'\|_{L^1([a,b])} \leq (b-a)^{1/q}\|f'\|_{L^1([a,b])}\|f'\|_{L^{q/(q-1)}([a,b])},$$

for every absolutely continuous function f *on* $[a, b]$ *with* $f(a) = 0$.

Proof. Let us compute

$$B = \sup_{a<x<b} (b-x)^{1/q}(x-a)^{(p-1)/p}.$$

For each $\alpha > 0$ and $\beta \geq 0$, consider the function u defined on $[a, b]$ as

$$u(x) = (b-x)^{\alpha}(x-a)^{\beta}.$$

If $\beta = 0$, then

$$\sup_{a<x<b} u(x) = u(a) = (b-a)^{\alpha}.$$

Assume now that $\beta > 0$. We have for $a < x < b$

$$\begin{aligned} u'(x) &= -\alpha(b-x)^{\alpha-1}(x-a)^{\beta} + \beta(b-x)^{\alpha}(x-a)^{\beta-1} = 0 \\ &\Leftrightarrow \beta(b-x)^{\alpha}(x-a)^{\beta-1} = \alpha(b-x)^{\alpha-1}(x-a)^{\beta} \\ &\Leftrightarrow \beta(b-x) = \alpha(x-a) \\ &\Leftrightarrow x = \frac{a\alpha+b\beta}{\alpha+\beta}. \end{aligned}$$

Since $u(a) = u(b) = 0$, we have

$$\begin{aligned} \sup_{a<x<b} u(x) = \max_{a\leq x\leq b} u(x) &= u\left(\frac{a\alpha+b\beta}{\alpha+\beta}\right) \\ &= \left(\frac{\alpha(b-a)}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta(b-a)}{\alpha+\beta}\right)^{\beta} = \frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}}(b-a)^{\alpha+\beta}. \end{aligned}$$

Thus, $B = (b-a)^{1/q}$ if $p = 1$ and

$$B = \frac{(1/q)^{1/q}((p-1)/p)^{(p-1)/p}}{(1/q+(p-1)/p)^{1/q+(p-1)/p}}(b-a)^{1/q+(p-1)/p},$$

$$B\left(\frac{q}{q-1}\right)^{(p-1)/p} q^{1/q} = \left(\frac{b-a}{1/q+(p-1)/p}\right)^{1/q+(p-1)/p} \left(\frac{q(p-1)}{p(q-1)}\right)^{(p-1)/p},$$

if $p > 1$. Hence, Corollary 2 gives the result. □

Corollary 5. *Let us consider* $1 \le p \le 2$. *Then*

$$\|ff'\|_{L^1([a,b])} \le \left(\frac{p(b-a)}{2(p-1)^{1/2}}\right)^{2(p-1)/p} \|f'\|^2_{L^p([a,b])}$$

if $1 < p \le 2$, *and*

$$\|ff'\|_{L^1([a,b])} \le \|f'\|^2_{L^1([a,b])}$$

for every absolutely continuous function f *on* $[a,b]$ *such that* $f(a) = 0$.

Proof. Assume that $1 < p \le 2$. It suffices to consider $q \ge 2$ such that $1/p + 1/q = 1$ (recall that $p = q/(q-1)$ and $q = p/(p-1)$), and to apply Corollary 4:

$$\begin{aligned}
&\left(\frac{b-a}{1/q + (p-1)/p}\right)^{1/q+(p-1)/p}\left(\frac{q(p-1)}{p(q-1)}\right)^{(p-1)/p}\\
&= \left(\frac{b-a}{2(p-1)/p}\right)^{2(p-1)/p}\left(\frac{p(p-1)}{p}\right)^{(p-1)/p}\\
&= \left(\frac{p(b-a)}{2(p-1)}\right)^{2(p-1)/p}(p-1)^{(p-1)/p} = \left(\frac{p(b-a)}{2(p-1)^{1/2}}\right)^{2(p-1)/p}.
\end{aligned}$$

Let us consider now the case $p = 1$. Since the Lebesgue measure on $[a,b]$ is finite, Corollary 3 gives

$$\|ff'\|_{L^1([a,b])} \le B\|f'\|^2_{L^1([a,b])},$$

with

$$B = \sup_{a<x<b} (b-x)^{(p-1)/p}\|1\|^{1/p}_{L^{1/(p-1)}([a,x])} = \sup_{a<x<b} (b-x)^0\|1\|_{L^\infty([a,x])} = 1.$$ □

Remark 2. Note that in the second inequality in Corollary 5:

$$\|ff'\|_{L^1([a,b])} \le \|f'\|^2_{L^1([a,b])},$$

the constant 1 multiplying $\|f'\|^2_{L^1([a,b])}$ does not depend on the length of the interval $[a,b]$.

Corollary 2 and Theorem 3 have, respectively, the following direct consequences for general fractional integrals of Riemann-Liouville type.

Proposition 6. *Let us consider* $1 \le p \le q < \infty$ *and assume that the constant* B *defined as follows is finite*:

$$B \coloneqq \sup_{a<x<b}\left(\int_x^b \frac{1}{T(b,s,a)}\mathrm{d}s\right)^{1/q}\left(\int_a^x T(b,s,a)^{1/(p-1)}\mathrm{d}s\right)^{(p-1)/p}.$$

Then, for every absolutely continuous function f *on* $[a,b]$ *with* $f(a) = 0$,

$$\int_a^b \frac{|f(s)f'(s)|}{T(b,s,a)}\mathrm{d}s \le B\left(\frac{q}{q-1}\right)^{(p-1)/p} q^{1/q}\left(\int_a^b \frac{|f'(s)|^p}{T(b,s,a)}\mathrm{d}s\right)^{1/p}\left(\int_a^b \frac{|f'(s)|^{q/(q-1)}}{T(b,s,a)}\mathrm{d}s\right)^{(q-1)/q}$$

if $p > 1$, *and*

$$\int_a^b \frac{|f(s)f'(s)|}{T(b,s,a)}\mathrm{d}s \le B\int_a^b \frac{|f'(s)|}{T(b,s,a)}\mathrm{d}s\left(\int_a^b \frac{|f'(s)|^{q/(q-1)}}{T(b,s,a)}\mathrm{d}s\right)^{(q-1)/q}.$$

Proposition 7. *Let us consider* $1 \leq p \leq 2$ *and assume that the constant B defined as follows is finite*:

$$B \coloneqq \sup_{a<x<b} \left(\int_x^b \frac{1}{T(b,s,\alpha)} \mathrm{d}s \right)^{(p-1)/p} \left(\int_a^x T(b,s,\alpha)^{1/(p-1)} \mathrm{d}s \right)^{(p-1)/p}.$$

Then, if $1 < p \leq 2$ *and f is any absolutely continuous function on* $[a,b]$ *with* $f(a) = 0$,

$$\int_a^b \frac{|f(s)f'(s)|}{T(b,s,\alpha)} \mathrm{d}s \leq B \left(\frac{p^2}{p-1} \right)^{(p-1)/p} \left(\int_a^b \frac{|f'(s)|^p}{T(b,s,\alpha)} \mathrm{d}s \right)^{2/p}.$$

Furthermore, if

$$\int_a^b \frac{\mathrm{d}s}{T(b,s,\alpha)} < \infty,$$

then

$$\int_a^b \frac{|f(s)f'(s)|}{T(b,s,\alpha)} \mathrm{d}s \leq B \left(\int_a^b \frac{|f'(s)|}{T(b,s,\alpha)} \mathrm{d}s \right)^2.$$

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