Research Article

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Some new fixed point theorems of α-partially nonexpansive mappings

https://doi.org/10.1515/dema-2023-0148
received May 22, 2023; accepted January 22, 2024

Abstract: In this paper, we introduce a new class of nonlinear mappings and compare it to other classes of nonlinear mappings that have appeared in the literature. We establish various existence and convergence theorems for this class of mappings under different assumptions in Banach spaces, particularly Banach spaces with a normal structure. In addition, we provide examples to substantiate the findings presented in this study. We prove the existence of a common fixed point for a family of commuting α-partially nonexpansive self-mappings. Furthermore, we extend the results reported by Suzuki (*Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl. **340** (2008), no. 2, 1088–1095), Llorens-Fuster (*Partially nonexpansive mappings*, Adv. Theory Nonlinear Anal. Appl. **6** (2022), no. 4, 565–573), and Dhompongsa et al. (*Edelstein's method and fixed point theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl. **350** (2009), no. 1, 12–17). Finally, we present an open problem concerning the existence of fixed points for α-partially nonexpansive mappings in the context of uniformly nonsquare Banach spaces.

Keywords: nonexpansive mapping, condition (E), uniformly convex space

MSC 2020: 47H10, 54H25, 47H09

1 Introduction

Let $(\Omega, \|\cdot\|)$ be a Banach space, and let Υ be a nonempty subset of Ω. A mapping $\Psi : \Upsilon \to \Upsilon$ is considered nonexpansive if it satisfies the inequality $\|\Psi(\zeta) - \Psi(\varrho)\| \leq \|\zeta - \varrho\|$ for all $\zeta, \varrho \in \Upsilon$. A point $z \in \Upsilon$ is considered a fixed point of Ψ if $\Psi(z) = z$. It is commonly known that in a general Banach space, a nonexpansive mapping may not necessarily possess a fixed point. Nevertheless, in 1965, Browder [1], Göhde [2], and Kirk [3] independently established fixed point theorems for nonexpansive mappings that satisfy certain geometric conditions such as uniform convexity or normal structure.

Nonexpansive mappings have been significant in nonlinear functional analysis, being connected to variational inequalities and the theory of monotone and accretive operators. They can be viewed as a variant of the traditional Banach contractions. The investigation of the existence of fixed points for nonexpansive mappings and their behavior over time has evolved since the mid-1960s. It has been mostly studied in the context of closed convex subsets of Banach spaces and is now a specialized area of metric fixed point theory. Subsequently, numerous authors have made advancements in the field of nonexpansive mappings, resulting in several generalizations and extensions of their properties. A compilation of some of the significant expansions and generalizations of nonexpansive mappings can be found in the study by Pant et al. [4], see also [5]. Shukla and Wiśnicki [6] offer intriguing insights into Iterative methods applied to monotone nonexpansive mappings within ordered Banach spaces. Demma et al. [7] delved into the intricacies of fixed point results in

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b-metric spaces, utilizing Picard sequences and b-simulation functions to meticulously analyze convergence dynamics. Subsequently, Jantakarn and Kaewcharoen [8] established robust convergence theorems for mixed equilibrium problems and Bregman relatively nonexpansive mappings in reflexive Banach spaces. Sharma et al. [9] presented an empirical study that unveiled the effectiveness of a novel iterative algorithm for generalized nonexpansive operators. See also studies by Al-Sa'di et al. [10] and Kriketa and Boudeliou [11].

In 2008, Suzuki [12] introduced a class of mappings known as mappings that satisfy condition (C), which fall under the category of nonexpansive type mappings. Suzuki's work on this class of mappings led to the development of crucial fixed point theorems.

Definition 1.1. [12]. Let Ω be a Banach space and Υ a nonempty subset of Ω. A mapping $\Psi : \Upsilon \to \Upsilon$ is said to satisfy condition (C) if

$$\frac{1}{2}\|\zeta - \Psi(\zeta)\| \le \|\zeta - \varrho\| \quad \text{implies } \|\Psi(\zeta) - \Psi(\varrho)\| \le \|\zeta - \varrho\| \quad \forall \zeta, \varrho \in \Upsilon.$$

This condition lies between nonexpansiveness and quasi-nonexpansiveness in terms of strength. The author demonstrated the existence of fixed points for mappings that satisfy condition (C) and presented various fixed point theorems and convergence theorems for such mappings. In addition, the author established a strong convergence theorem that is related to Ishikawa's theorem [13] and a weak convergence theorem that is connected to Edelstein and O'Brien's work [14].

Building upon the concept introduced by Kirk and Massa [15] and Dhompongsa et al. [16] extended the scope by proving a fixed point theorem for mappings exhibiting condition (C) in a Banach space. This theorem holds true when the asymptotic center of the Banach space is both nonempty and compact within a bounded, closed, and convex subset of every bounded sequence. Furthermore, they derived fixed point theorems for this particular class of mappings defined on weakly compact convex subsets of Banach spaces that satisfy property (D).

García-Falset et al. [17] further generalized condition (C) into the following two class of mappings.

Definition 1.2. [17]. For $\lambda \in (0, 1)$, a mapping $\Psi : \Upsilon \to \Omega$ is said to satisfy condition (C_λ) if

$$\lambda\|\zeta - \Psi(\zeta)\| \le \|\zeta - \varrho\| \quad \text{implies } \|\Psi(\zeta) - \Psi(\varrho)\| \le \|\zeta - \varrho\| \quad \forall \zeta, \ \varrho \in \Upsilon.$$

Definition 1.3. [17] Let Υ be a nonempty subset of a Banach space Ω. A mapping $\Psi : \Upsilon \to \Upsilon$ is said to fulfill condition (E_μ) if there exists $\mu \ge 1$ such that

$$\|\zeta - \Psi(\varrho)\| \le \mu\|\zeta - \Psi(\zeta)\| + \|\zeta - \varrho\| \quad \forall \zeta, \ \varrho \in \Upsilon.$$

We say that Ψ satisfies condition (E) if it satisfies (E_μ) for some $\mu \ge 1$.

The class of mappings satisfying condition (E) properly contains many important classes of generalized nonexpansive mappings. Many important results concerning nonexpansive mappings have been proved for the class of mappings satisfying condition (E) [18,19].

Recently, Llorens-Fuster [20] introduced the class of partially nonexpansive (PNE) mappings, which properly contains the class of Suzuki-nonexpansive mappings. PNE mappings may not possess fixed points, despite being defined on compact convex sets. Nonetheless, he demonstrated that the combination of both properties (PNE and condition (E)) ensures the existence of fixed points in Banach spaces with a suitable geometrical property in their norm.

Motivated by Llorens-Fuster [20], we generalize the class of PNE mappings and consider a new class of mappings known α-PNE mappings. In fact, we introduce a class of mapping, which contains both the PNE mappings and the class of mappings satisfying condition (C_λ). We present some examples to illustrate our claims. We present a number of fixed points theorems for α-PNE mappings. In this way, results presented in previous studies [12,16,17,20] are extended, generalized, and complemented.

2 Preliminaries

Let Ω be a Banach space and Υ a closed convex subset of Ω such that $\Upsilon \neq \emptyset$. We denote $F(\Psi)$ the set of all fixed points of mapping Ψ, i.e., $F(\Psi) = \{z \in \Upsilon : \Psi(z) = z\}$. Suppose $\{\zeta_n\}$ is a bounded sequence in Ω. For $\zeta \in \Omega$, the asymptotic radius of $\{\zeta_n\}$ at ζ can be defined as follows:

$$r(\zeta, \{\zeta_n\}) = \limsup_{n\to\infty} \|\zeta_n - \zeta\|.$$

Let

$$r \equiv r(\Upsilon, \{\zeta_n\}) \coloneqq \inf\{r(\zeta, \{\zeta_n\}) : \zeta \in \Upsilon\}$$

and

$$A \equiv A(\Upsilon, \{\zeta_n\}) \coloneqq \{\zeta \in \Upsilon : r(\zeta, \{\zeta_n\}) = r\}.$$

The number r is asymptotic radius, and the set A is asymptotic center of $\{\zeta_n\}$ relative to Υ.

The sequence $\{\zeta_n\}$ is called regular relative to Υ if for every subsequence $\{\zeta_{n'}\}$ of $\{\zeta_n\}$ the following equality holds

$$r(\Upsilon, \{\zeta_n\}) = r(\Upsilon, \{\zeta_{n'}\}).$$

Lemma 2.1. [21, 22]. *Let Υ and $\{\zeta_n\}$ be the same as earlier. Then there exists a subsequence of $\{\zeta_n\}$, which is regular relative to Υ.*

Definition 2.2. [23]. Let Υ be a nonempty subset of a Banach space Ω. A sequence $\{\zeta_n\}$ in Υ is said to be approximate fixed point sequence (in short, a.f.p.s.) for a mapping $\Psi : \Upsilon \to \Upsilon$ if $\lim_{n\to\infty} \|\zeta_n - \Psi(\zeta_n)\| = 0$.

Definition 2.3. [23]. A Banach space $(\Omega, \|\cdot\|)$ is said to have normal structure, if for each bounded closed convex subset C of Ω consisting of more than one point, there is a point $\zeta \in C$ such that

$$\sup\{\|\varrho - \zeta\| : \varrho \in C\} < \operatorname{diam}_{\|.\|}(C) = \sup\{\|\varrho - \zeta\| : \zeta, \varrho \in C\}.$$

Definition 2.4. [24]. A mapping $\Psi : \Upsilon \to \Upsilon$ is said to be a quasi-nonexpansive (QNE) if

$$\|\Psi(\zeta) - z\| \leq \|\zeta - z\| \quad \forall \zeta \in \Upsilon \quad \text{and} \quad z \in F(\Psi).$$

A mapping satisfying condition (E) with a fixed point is QNE.

Lemma 2.5. [25] *Let Ω be a Banach space and Υ a closed subset of Ω such that $\Upsilon \neq \emptyset$. Let $\Psi : \Upsilon \to \Upsilon$ be a QNE. Then $F(\Psi)$ is closed in Υ. If Υ is convex and Ω is strictly convex then $F(\Psi)$ is convex.*

Proposition 2.6. [17] *Let Ω be a Banach space having the Opial property and Υ be a subset of Ω such that $\Upsilon \neq \emptyset$. Let $\Psi : \Upsilon \to \Upsilon$ be a mapping satisfying condition (E). Let $\{\zeta_n\}$ be a sequence in Υ such that $\{\zeta_n\}$ converges weakly to ζ and $\lim_{n\to\infty} \|\zeta_n - \Psi(\zeta_n)\| = 0$. Then $\Psi(\zeta) = \zeta$.*

3 α-PNE mapping

Recently, Llorens-Fuster [20] introduced the following class of mappings:

Definition 3.1. Let $\Psi : \Upsilon \to \Upsilon$ be a mapping. A mapping Ψ is called as PNE, if

$$\left\| \Psi\left(\frac{1}{2}(\zeta + \Psi(\zeta))\right) - \Psi(\zeta) \right\| \leq \frac{1}{2}\|\zeta - \Psi(\zeta)\|$$

for all $\zeta \in \Upsilon$.

Remark 3.2.
- If $\Psi : \Upsilon \to \Upsilon$ satisfies condition (C), then Ψ is PNE but converse need not be true.
- Having fixed points for a mapping is not necessarily implied by either condition PNE or condition (E).

Motivated by the aforementioned definition, we generalize PNE as follows:

Definition 3.3. Let Ω be a Banach space and Υ be a convex subset of Ω such that $\Upsilon \neq \emptyset$. A mapping $\Psi : \Upsilon \to \Upsilon$ is said to be α-PNE if there exists $\alpha \in (0, 1)$ such that

$$\|\Psi((1 - \alpha)\zeta + \alpha\Psi(\zeta)) - \Psi(\zeta)\| \leq \alpha\|\zeta - \Psi(\zeta)\|$$

for all $\zeta \in \Upsilon$.

Remark 3.4. The class of α-PNE mappings properly contains PNE. Indeed, PNE is $\frac{1}{2}$-PNE mapping.

The converse of the above remark is not true.

Example 3.5. Let $\Upsilon = [0, 3] \subset \mathbb{R}$ with the usual norm. Define $\Psi : \Upsilon \to \Upsilon$ by

$$\Psi(\zeta) = \begin{cases} 0, & \text{if } \zeta \neq 2 \quad \text{and} \quad \zeta \neq 3 \\ \dfrac{1}{2}, & \text{if } \zeta = 2 \\ \dfrac{11}{10}, & \text{if } \zeta = 3. \end{cases}$$

First, we show that Ψ is α-PNE mapping for $\alpha = \frac{10}{19}$. We consider the following cases:
(1) If $\zeta \neq 2$ and $\zeta \neq 3$, then the condition is trivially satisfied.
(2) If $\zeta = 2$, then

$$\begin{aligned} \|\Psi((1 - \alpha)\zeta + \alpha\Psi(\zeta)) - \Psi(\zeta)\| &= \left\| \Psi\left(\left(1 - \frac{10}{19}\right) \times 2 + \frac{10}{19}\Psi(2)\right) - \Psi(2) \right\| \\ &= \left\| \Psi\left(\frac{23}{19}\right) - \frac{1}{2} \right\| = \frac{1}{2} < \frac{15}{19} = \frac{10}{19}\left\| 2 - \frac{1}{2} \right\| \\ &= \alpha\|\zeta - \Psi(\zeta)\|. \end{aligned}$$

(3) If $\zeta = 3$, then

$$\begin{aligned} \|\Psi((1 - \alpha)\zeta + \alpha\Psi(\zeta)) - \Psi(\zeta)\| &= \left\| \Psi\left(\left(1 - \frac{10}{19}\right) \times 3 + \left(1 - \frac{10}{19}\right)\Psi(3)\right) - \Psi(3) \right\| \\ &= \left\| \Psi\left(\frac{38}{19}\right) - \frac{11}{10} \right\| = \left\| \Psi(2) - \frac{11}{10} \right\| \\ &= \frac{6}{10} < 1 = \frac{10}{19}\left\| 3 - \frac{11}{10} \right\| \\ &= \alpha\|\zeta - \Psi(\zeta)\|. \end{aligned}$$

On the other hand, Ψ is not PNE mapping. Indeed, at $\zeta = 3$,

$$\begin{aligned}\left\|\Psi\left(\frac{\zeta+\Psi(\zeta)}{2}\right)-\Psi(\zeta)\right\| &= \left\|\Psi\left(\frac{3+\Psi(3)}{2}\right)-\Psi(3)\right\| \\ &= \left\|\Psi\left(\frac{41}{20}\right)-\frac{11}{10}\right\| \\ &= \frac{11}{10} > \frac{19}{20} = \frac{1}{2}\left\|3-\frac{11}{10}\right\| \\ &= \frac{1}{2}\|\zeta-\Psi(\zeta)\|.\end{aligned}$$

Example 3.6. Let $\Upsilon = [0,2] \subset \mathbb{R}$ with the norm. Define $\Psi : \Upsilon \to \Upsilon$ by

$$\Psi(\zeta) = \begin{cases} 1, & \text{if } \zeta = 0 \\ \dfrac{\zeta}{2}, & \text{if } \zeta \in (0,2]. \end{cases}$$

We shall show that Ψ is α-PNE mapping for $\alpha = \frac{3}{4}$.
Case 1. If $\zeta = 0$, then

$$\begin{aligned}\|\Psi((1-\alpha)\zeta+\alpha\Psi(\zeta))-\Psi(\zeta)\| &= \left\|\Psi\left(\left(1-\frac{3}{4}\right)\times 0+\frac{3}{4}\Psi(0)\right)-\Psi(0)\right\| \\ &= \left\|\Psi\left(\frac{3}{4}\right)-1\right\| = \left\|\frac{3}{8}-1\right\| = \frac{5}{8} < \frac{3}{4} \\ &= \frac{3}{4}\times\|0-\Psi(0)\| = \alpha\|\zeta-\Psi(\zeta)\|.\end{aligned}$$

Case 2. If $\zeta \in (0,2]$, then

$$\begin{aligned}\|\Psi((1-\alpha)\zeta+\alpha\Psi(\zeta))-\Psi(\zeta)\| &= \left\|\Psi\left(\frac{1}{4}\zeta+\frac{3}{4}\Psi(\zeta)\right)-\Psi(\zeta)\right\| \\ &= \left\|\Psi\left(\frac{1}{4}\zeta+\frac{3}{8}\zeta\right)-\frac{\zeta}{2}\right\| \\ &= \left\|\frac{5}{16}\zeta-\frac{\zeta}{2}\right\| = \frac{3}{16}\zeta < \frac{6}{16}\zeta = \frac{3}{4}\left\|\zeta-\frac{\zeta}{2}\right\| = \alpha\|\zeta-\Psi(\zeta)\|.\end{aligned}$$

Hence, the mapping Ψ is $\frac{3}{4}$-PNE mapping. The mapping Ψ is not PNE. Indeed, at $\zeta = 0$,

$$\begin{aligned}\left\|\Psi\left(\frac{\zeta+\Psi(\zeta)}{2}\right)-\Psi(\zeta)\right\| &= \left\|\Psi\left(\frac{1}{2}\right)-1\right\| \\ &= \left\|\frac{1}{4}-1\right\| = \frac{3}{4} > \frac{1}{2} = \frac{1}{2}\|0-\Psi(0)\| \\ &= \frac{1}{2}\|\zeta-\Psi(\zeta)\|.\end{aligned}$$

Proposition 3.7. *It $\Psi : \Upsilon \to \Upsilon$ satisfies condition (C_λ) for any $\lambda \in (0,1)$, then Ψ is λ-PNE.*

Proof. Since, for all $\zeta \in \Upsilon$,

$$\lambda\|\zeta-\Psi(\zeta)\| = \|(1-\lambda)\zeta+\lambda\Psi(\zeta)-\zeta\|$$

then, from condition (C_λ),

$$\|\Psi((1-\lambda)\zeta+\lambda\Psi(\zeta))-\Psi(\zeta)\| \le \|(1-\lambda)\zeta+\lambda\Psi(\zeta)-\zeta\| = \lambda\|\zeta-\Psi(\zeta)\|,$$

and we obtain the desired result. □

Thus, every nonexpansive mapping $\Psi : \Upsilon \to \Upsilon$ is PNE. However, it should be noted that the converse of Proposition 3.7 is not necessarily true.

Example 3.8. [20] Let $(\mathbb{R}^2, \|\cdot\|_\infty)$ be a Banach space. Consider $\Psi : B_\infty[0_\Omega, 2] \to B_\infty[0_\Omega, 2]$ is the mapping

$$\Psi(\zeta) = \begin{cases} \dfrac{\zeta}{\|\zeta\|_\infty} & \zeta \in B_\infty[0_\Omega, 2] \backslash B_\infty[0_\Omega, 1] \\ \zeta & \zeta \in B_\infty[0_\Omega, 1]. \end{cases}$$

It is shown by Llorens-Fuster [20] that mapping Ψ is PNE. Now, let $s \in (0,1)$, take $\zeta \coloneqq (1,1)$ and $\varrho_s \coloneqq (1-s, 1+s)$. It can be noted that ζ is a fixed point of Ψ. Indeed,

$$\lambda\|\zeta - \Psi(\zeta)\|_\infty = 0 \le \|\zeta - \varrho_s\|_\infty,$$

$$\|\Psi(\zeta) - \Psi(\varrho_s)\|_\infty = \|\zeta - \Psi(\varrho_s)\|_\infty = \left\|(1,1) - \left(\frac{1-s}{1+s}, 1\right)\right\|_\infty = 1 - \frac{1-s}{1+s} = \frac{2s}{1+s},$$

while

$$\|\zeta - \varrho_s\|_\infty = \|(s, -s)\|_\infty = s.$$

Thus,

$$\|\zeta - \Psi(\varrho_s)\|_\infty = \frac{2}{1+s}\|\zeta - \varrho_s\|_\infty > \|\zeta - \varrho_s\|_\infty.$$

Hence, Ψ fails to satisfy condition (C_λ) (with respect to the norm $\|\cdot\|_\infty$).

Example 3.9. [20] Suppose that $\Psi : B[0_\Omega, 1] \to B[0_\Omega, 1]$ is the mapping given by the following definition:

$$\Psi(\zeta) = \begin{cases} \dfrac{1}{2}\dfrac{\zeta}{\|\zeta\|} & \zeta \in B[0_\Omega, 1] \setminus B\left[0_\Omega, \dfrac{1}{2}\right] \\ 0_\Omega & \zeta \in B\left[0_\Omega, \dfrac{1}{2}\right]. \end{cases}$$

It is shown by Llorens-Fuster [20] that the mapping Ψ is PNE. The mapping Ψ does not satisfy condition (C).

Lemma 3.10. *Let Ω be a Banach space and Υ a bounded convex subset of Ω such that $\Upsilon \neq \varnothing$. Let $\Psi : \Upsilon \to \Upsilon$ be an α-PNE mapping. Let $\zeta_0 \in \Upsilon$ and define,*

$$\zeta_{n+1} = (1-\alpha)\zeta_n + \alpha\Psi(\zeta_n) \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.1}$$

Then $\lim_{n\to\infty}\|\zeta_n - \Psi(\zeta_n)\| = 0$.

Proof. From (3.1), we have

$$\zeta_{n+1} - \zeta_n = \alpha(\zeta_n - \Psi(\zeta_n)). \tag{3.2}$$

From (3.1), taking into account that Ψ is α-PNE,

$$\|\Psi(\zeta_{n+1}) - \Psi(\zeta_n)\| = \|\Psi((1-\alpha)\zeta_n + \alpha\Psi(\zeta_n)) - \Psi(\zeta_n)\| \le \alpha\|\zeta_n - \Psi(\zeta_n)\|.$$

From (3.2),

$$\|\Psi(\zeta_{n+1}) - \Psi(\zeta_n)\| \le \|\zeta_{n+1} - \zeta_n\|.$$

Thus, in view of [12, Lemma 3] [26], it follows that $\lim_{n\to\infty}\|\zeta_n - \Psi(\zeta_n)\| = 0$. □

Theorem 3.11. *Let Ω be a Banach space having normal structure. Let Υ be a weakly compact convex subset of Ω such that $\Upsilon \neq \varnothing$. Let $\Psi : \Upsilon \to \Upsilon$ be an α-PNE mapping and Ψ satisfies condition (E). Then Ψ admits a fixed point.*

Proof. First, we show that Ψ has an a.f.p. sequence in each nonempty closed convex Ψ-invariant subset C of Υ. Let $\zeta_0 \in C$ and define,

$$\zeta_{n+1} = (1-\alpha)\zeta_n + \alpha\Psi(\zeta_n) \quad \forall n \in \mathbb{N} \cup \{0\}.$$

By the convexity of set C, the sequence $\{\zeta_n\}$ in C. In view of Lemma 3.10, it follows that $\|\zeta_n - \Psi(\zeta_n)\| \to 0$ as $n \to \infty$. If $\zeta \in \Upsilon$ and $\{\zeta_n\}$ is an a.f.p. sequence for Ψ, by the assumption that Ψ satisfies condition (E), we obtain

$$\limsup\|\zeta_n - \Psi(\zeta)\| \leq \limsup(\mu\|\zeta_n - \Psi(\zeta_n)\| + \|\zeta_n - \zeta\|) = \limsup\|\zeta_n - \zeta\|.$$

Therefore, Ψ is an L-type mapping and from [27, Theorem 4.4], we can conclude that Ψ admits a fixed point in Υ. □

Corollary 3.12. *Let Ω be a uniformly convex in every direction (in short, UCED) Banach space and Υ a weakly compact convex subset of Ω such that $\Upsilon \neq \emptyset$. Let $\Psi : \Upsilon \to \Upsilon$ be an α-PNE mapping and satisfies condition (E). Then Ψ has a fixed point.*

The following result is a generalization of [12, Theorem 2].

Theorem 3.13. *Let Ω be Banach space and Υ a compact convex subset of Ω such that $\Upsilon \neq \emptyset$. Assume that Ψ is same as in Theorem 3.11. Define a sequence $\{\zeta_n\}$ in Υ by $\zeta_0 \in \Upsilon$ and*

$$\zeta_{n+1} = (1-\alpha)\zeta_n + \alpha\Psi(\zeta_n) \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Then $\{\zeta_n\}$ converges strongly to a fixed point of Ψ.

Proof. From Lemma 3.10, it follows that $\lim_{n\to\infty}\|\Psi(\zeta_n) - \zeta_n\| = 0$. Due to the compactness of Υ, it is guaranteed that there exists a subsequence $\{\zeta_{n_j}\}$ of $\{\zeta_n\}$ and an element $z \in \Upsilon$ such that $\{\zeta_{n_j}\}$ converges to z. Since Ψ satisfies condition (E)

$$\|\zeta_{n_j} - \Psi(z)\| \leqslant \mu\|\Psi(\zeta_{n_j}) - \zeta_{n_j}\| + \|\zeta_{n_j} - z\|$$

for all $j \in \mathbb{N}$. Thus, $\{\zeta_{n_j}\}$ converges to $\Psi(z)$, which implies $\Psi(z) = z$, that is, $z \in F(\Psi)$. Again,

$$\|\zeta_{n+1} - z\| \leq \alpha\|\Psi(\zeta_n) - z\| + (1-\alpha)\|\zeta_n - z\| \leq \|\zeta_n - z\|$$

for $n \in \mathbb{N}$ and $\lim_{n\to\infty}\|\zeta_n - z\|$ exists. Hence, $\{\zeta_n\}$ converges to z. □

The following result is a generalization of [12, Theorem 3].

Theorem 3.14. *Let Ω be a Banach space having the Opial property and Υ a weakly compact convex subset of Ω such that $\Upsilon \neq \emptyset$. Assume that Ψ and $\{\zeta_n\}$ are same as in Theorem 3.13. Then $\{\zeta_n\}$ converges weakly to a fixed point of Ψ.*

Proof. From Lemma 3.10, it follows that $\lim_{n\to\infty}\|\Psi(\zeta_n) - \zeta_n\| = 0$. Since Υ is weakly compact, there exist a subsequence $\{\zeta_{n_j}\}$ of $\{\zeta_n\}$ and $z \in \Upsilon$ such that $\{\zeta_{n_j}\}$ converges weakly to z. In view of Proposition 2.6, $\Psi(z) = z$. Following the same proof technique as in Theorem 3.13, one can show that $\{\|\zeta_n - z\|\}$ is a nonincreasing sequence. Arguing by contradiction, suppose $\{\zeta_n\}$ does not converge weakly to z. Then $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ and $w \in \Upsilon$ such that $\{\zeta_{n_k}\}$ converges weakly to w and $z \neq w$. We note $\Psi(w) = w$. By Opial property,

$$\begin{aligned}\lim_{n\to\infty}\|\zeta_n - z\| &= \lim_{j\to\infty}\|\zeta_{n_j} - z\| < \lim_{j\to\infty}\|\zeta_{n_j} - w\| \\ &= \lim_{n\to\infty}\|\zeta_n - w\| = \lim_{k\to\infty}\|\zeta_{n_k} - w\| \\ &< \lim_{k\to\infty}\|\zeta_{n_k} - z\| = \lim_{n\to\infty}\|\zeta_n - z\|.\end{aligned}$$

This is a contradiction. This completes the proof. □

In the next example, we show that all the assumptions of Theorem 3.13 are satisfied and [12, Theorem 3] is not applicable.

Example 3.15. Let $\Upsilon = [0,4] \subset \mathbb{R}$ with the usual norm. Define $\Psi : \Upsilon \to \Upsilon$ by

$$\Psi(\zeta) = \begin{cases} 0, & \text{if } \zeta \in [0,2] \\ 0.9, & \text{if } \zeta \in (2,4]. \end{cases}$$

First, we show that Ψ is α-PNE mapping for $\alpha = 0.9$. We consider the following cases:
(i) If $\zeta \in [0,2]$, then the condition trivially satisfied.
(ii) If $\zeta \in (2,4]$, then

$$\begin{aligned} \|\Psi((1-\alpha)\zeta + \alpha\Psi(\zeta)) - \Psi(\zeta)\| &= \|\Psi((1-0.9)\zeta + 0.9\Psi(\zeta)) - \Psi(\zeta)\| \\ &= \|\Psi(0.1\zeta + 0.9 \times 0.9) - 0.9\| \\ &= 0.9 \le 0.9\|\zeta - 0.9\| = \|\zeta - \Psi(\zeta)\|. \end{aligned}$$

Hence, Ψ is $\frac{9}{10}$-PNE mapping. Now, we show that Ψ satisfies condition (E) for $\mu = 4$. For this, we consider the following cases:
(a) If $\zeta, \varrho \in [0,2]$, then

$$\|\zeta - \Psi(\varrho)\| = \zeta \le 4\zeta + \|\varrho - \zeta\| = 4\|\zeta - \Psi(\zeta)\| + \|\varrho - \zeta\|.$$

(b) If $\zeta \in [0, 0.9)$ and $\varrho \in (2,4]$, then $\|0.9 - \zeta\| < \|\varrho - \zeta\|$ and

$$\|\zeta - \Psi(\varrho)\| = \|\zeta - 0.9\| \le 4\|\zeta - \Psi(\zeta)\| + \|\varrho - \zeta\|.$$

(c) If $\zeta \in [0.9, 2]$ and $\varrho \in (2,4]$, then $\|\zeta - 0.9\| \le 1.1$ and

$$\|\zeta - \Psi(\varrho)\| = \|\zeta - 0.9\| \le 4\zeta \le 4\|\zeta - \Psi(\zeta)\| + \|\varrho - \zeta\|.$$

(d) If $\zeta, \varrho \in (2,4]$, then

$$\|\zeta - \Psi(\varrho)\| = \|\zeta - 0.9\| \le 4\|\zeta - 0.9\| + \|\varrho - \zeta\| = 4\|\zeta - \Psi(\zeta)\| + \|\varrho - \zeta\|.$$

(e) If $\zeta \in (2,4]$ and $\varrho \in [0,2]$, then $\|\zeta - 0.9\| \ge 1.1$ and

$$\|\zeta - \Psi(\varrho)\| = \|\zeta\| \le 4\|\zeta - 0.9\| \le 4\|\zeta - \Psi(\zeta)\| + \|\varrho - \zeta\|.$$

Finally, we show that Ψ does not satisfy condition (C). Indeed at $\zeta = 1.4$ and $\varrho = 2.1$, we have

$$\frac{1}{2}\|\varrho - \Psi(\varrho)\| = \frac{1}{2}\|2.1 - 0.9\| = 0.6 < 0.7 = \|\zeta - \varrho\|$$

and

$$\|\Psi(\zeta) - \Psi(\varrho)\| = 0.9 > 0.7 = \|\zeta - \varrho\|.$$

It can be seen that Υ is a compact convex subset of $\mathbb{R}$, Ψ is α-PNE mapping, and Ψ satisfies condition (E) and 0 is a fixed point of Ψ.

Remark 3.16. As demonstrated in Example 3.6, the absence of condition (E) can result in a deficiency of fixed points, regardless of the domain's compactness. We can show that the mapping Ψ (considered in Example 3.6) does not satisfy condition (E). Let $\zeta_n = \frac{1}{n}$ and $\varrho_n = 0$ for all $n \in \mathbb{N}\setminus\{1,2\}$, then,

$$\begin{aligned} \frac{\|\zeta_n - \Psi(\varrho_n)\| - \|\zeta_n - \varrho_n\|}{\|\zeta_n - \Psi(\zeta_n)\|} &= \frac{\left\|\frac{1}{n} - 1\right\| - \left\|\frac{1}{n} - 0\right\|}{\left\|\frac{1}{n} - \frac{1}{2n}\right\|} \\ &= \frac{1 - \frac{2}{n}}{\frac{1}{2n}} \\ &= 2n - 4 \to +\infty. \end{aligned}$$

The following result is a generalization of [12, Theorem 6]

Theorem 3.17. *Let* Ω *and* Υ *be same as in Corollary* 3.12. *Let* S *be a family of commuting* α*-PNE self-mappings on* Υ *satisfying* (E). *Then* S *has a common fixed point.*

Proof. Let $\Psi_1, \Psi_2, \ldots, \Psi_l \in S$. By Corollary 3.12, Ψ_1 admits a fixed point in Υ and $F(\Psi_1) \neq \varnothing$. In view of Lemma 2.5 $F(\Psi_1)$ is convex and closed. Let $A := \bigcap_{j=1}^{k-1} F(\Psi_j)$ is nonempty, closed, and convex for some $k \in \mathbb{N}$ with $1 < k \leq l$. For $\zeta \in A$ and $j \in \mathbb{N}$ with $1 \leqslant j < k$, since $\Psi_k \circ \Psi_j = \Psi_j \circ \Psi_k$,

$$\Psi_k(\zeta) = \Psi_k \circ \Psi_j(\zeta) = \Psi_j \circ \Psi_k(\zeta),$$

thus, $\Psi_k(\zeta)$ is a fixed point of Ψ_j, this follows $\Psi_k(\zeta) \in A$. Thus, we $\Psi_k(A) \subset A$. By Corollary 3.12, Ψ_k admits a fixed point in A, and

$$A \cap F(\Psi_k) = \bigcap_{j=1}^{k} F(\Psi_j) \neq \varnothing.$$

From Lemma 2.5, the set $A \cap F(\Psi_k)$ is convex and closed. From induction, it follows that $\bigcap_{j=1}^{\ell} F(\Psi_j) \neq \varnothing$. That is, $\{F(\Psi) : \Psi \in S\}$ has the finite intersection property. Since Υ is weakly compact, and $F(\Psi)$ is weakly closed for every $\Psi \in S$, and it follows that $\bigcap_{\Psi \in S} F(\Psi) \neq \varnothing$. □

Lemma 3.18. *Let* Ω *be a Banach space and* Υ *a subset of* Ω *such that* $\Upsilon \neq \varnothing$. *Let* $\Psi : \Upsilon \to \Upsilon$ *be a mapping satisfying condition (E). Let* $\{\zeta_n\}$ *be a bounded a.f.p. sequence for* Ψ. *Then* $A(\Upsilon, \{\zeta_n\})$ *is invariant under* Ψ.

Proof. Suppose $z \in A(\Upsilon, \{\zeta_n\})$ and by the definition of asymptotic radius of $\{\zeta_n\}$

$$\begin{aligned} r(\Psi(z), \{\zeta_n\}) &= \limsup_{n\to\infty} \|\zeta_n - \Psi(z)\| \\ &\leq \mu \limsup_{n\to\infty} \|\Psi(\zeta_n) - \zeta_n\| + \limsup_{n\to\infty} \|\zeta_n - z\| \\ &= \limsup_{n\to\infty} \|\zeta_n - z\| \\ &= r(z, \{\zeta_n\}). \end{aligned}$$

It follows that $\Psi(z) \in A(\Upsilon, \{\zeta_n\})$. □

Lemma 3.19. *Let* Ω, Υ, *and* Ψ *be same as in Lemma* 3.18. *Then the following holds*:

$$\|\zeta - \Psi(\zeta)\| \leqslant 2\|\zeta - \varrho\| + \mu\|\Psi(\varrho) - \varrho\|, \forall \zeta, \varrho \in \Upsilon.$$

Proof. Let $\zeta, \varrho \in \Upsilon$. From the definition of Ψ,

$$\begin{aligned} \|\zeta - \Psi(\zeta)\| &\leqslant \|\zeta - \varrho\| + \|\varrho - \Psi(\zeta)\| \\ &\leqslant \|\zeta - \varrho\| + \mu\|\Psi(\varrho) - \varrho\| + \|\varrho - \zeta\| \\ &= 2\|\zeta - \varrho\| + \mu\|\Psi(\varrho) - \varrho\|. \end{aligned}$$

□

Theorem 3.20. *Let* Ω *be a Banach space and* Υ *a bounded closed convex subset of* Ω *such that* $\Omega \neq \varnothing$. *Let* $\Psi : \Upsilon \to \Upsilon$ *be an* α*-PNE and* Ψ *satisfies condition (E). If every bounded sequence in* Ω *has a non-empty and compact asymptotic center in* Υ, *then* Ψ *must have a fixed point.*

Proof. Suppose $\{\zeta_n\}$ is an a.f.p sequence for Ψ and A is the asymptotic center of $\{\zeta_n\}$ relative to Υ. By the hypothesis, set $A \neq \varnothing$ and compact. In view of Lemma 3.18, set A is invariant under Ψ. The proof is now complete as Theorem 3.13 ensures that Ψ has at least one fixed point in A. □

Remark 3.21. The statement of Theorem 3.20 is applicable to Banach spaces that are either uniformly convex or k-uniformly rotund, as the assumption required in the theorem is satisfied by these spaces.

Definition 3.22. [16,28]. A Banach space Ω is said to have property (D) if there exists $\lambda \in [0,1)$ such that for any nonempty weakly compact convex subset Υ of Ω, any sequence $\{\zeta_n\} \subset \Upsilon$, which is regular relative to Υ, and any sequence $\{\varrho_n\} \subset A(\Upsilon, \{\Omega_n\})$, which is regular relative to Υ, we have

$$r(\Upsilon, \{\varrho_n\}) \leqslant \lambda r(\Upsilon, \{\zeta_n\}).$$

Theorem 3.23. *Let Ω be a Banach space having the property (D) and Υ a weakly compact convex subset of Ω such that $\Upsilon \neq \emptyset$. Let $\Psi : \Upsilon \to \Upsilon$ be an α-*PNE and Ψ *satisfies condition (E). Then Ψ admits a fixed point.*

Proof. Let $\{\zeta_n^0\}$ be an a.f.p. sequence for Ψ in Υ. In view of boundedness of $\{\zeta_n^0\}$ and from Lemma 2.1, one can suppose that $\{\zeta_n^0\}$ is regular relative to Υ. Assume $A^0 = A(\Upsilon, \{\zeta_n^0\})$. In view of Lemma 3.18, A^0 is invariant under Ψ. Therefore, an a.f.p. sequence $\{\zeta_n^1\}$ for Ψ in A^0 can be obtained using Lemma 3.10. Let $\{\zeta_n^1\}$ is regular relative to A^0. Since Ω has property (D), for some $\lambda \in [0,1)$, we have

$$r(\Upsilon, \{\zeta_n^1\}) \leqslant \lambda r(\Upsilon, \{\zeta_n^0\}).$$

Proceed with the process to generate a regular sequence relative to Υ, we obtain $\{\zeta_n^m\}$ in A^{m-1} such that, for each $m \geqslant 1$,

$$\lim_{n\to\infty} \|\Psi(\zeta_n^m) - \zeta_n^m\| = 0 \tag{3.3}$$

and

$$r(\Upsilon, \{\zeta_n^m\}) \leqslant \lambda r(\Upsilon, \{\zeta_n^{m-1}\}),$$

where $A^m \coloneqq A(\Upsilon, \{\zeta_n^m\})$. Consequently,

$$r(\Upsilon, \{\zeta_n^m\}) \leqslant \lambda r(\Upsilon, \{\zeta_n^{m-1}\}) \leqslant \cdots \leqslant \lambda^m r(\Upsilon, \{\zeta_n^0\}).$$

Let $\{\zeta_m = \zeta_{m+1}^{m+1}\}$ be the diagonal sequence of the sequences $\{\zeta_n^m\}$. We see that $\{\zeta_m\}$ is a Cauchy sequence. Indeed, for each $m \geqslant 1$ and for any $n \in \mathbb{N}$, we have

$$\|\zeta_m^m - \zeta_n^m\| \leqslant \|\zeta_m^m - \zeta_k^{m-1}\| + \|\zeta_k^{m-1} - \zeta_n^m\| \quad \text{for all } k.$$

Thus,

$$\|\zeta_m^m - \zeta_n^m\| \leq \limsup_{n\to\infty} \|\zeta_m^m - \zeta_k^{m-1}\| + \limsup_{n\to\infty} \|\zeta_k^{m-1} - \zeta_n^m\| \leq 2r(\Upsilon, \{\zeta_k^{m-1}\}).$$

Therefore, for each n,

$$\begin{aligned}\|\zeta_{m-1} - \zeta_m\| &\leqslant \|\zeta_{m-1} - \zeta_n^m\| + \|\zeta_n^m - \zeta_m\| = \|\zeta_m^m - \zeta_n^m\| + \|\zeta_n^m - \zeta_{m+1}^{m+1}\| \\ &\leq 2r(\Upsilon, \{\zeta_k^{m-1}\}) + \|\zeta_n^m - \zeta_{m+1}^{m+1}\|.\end{aligned}$$

Taking upper limit as $n \to \infty$, we have

$$\|\zeta_{m-1} - \zeta_m\| \leqslant 2r(\Upsilon, \{\zeta_n^{m-1}\}) + r(\Upsilon, \{\zeta_n^m\}) \leqslant 3\lambda^{m-1} r(\Upsilon, \{\zeta_n^0\}).$$

Since $\lambda < 1$, $\{\zeta_m\}$ is a Cauchy sequence. Hence, there exists $\zeta \in \Upsilon$ such that $\{\zeta_m\}$ converges to ζ. Further, we claim that ζ is a fixed point of Ψ. For each $m \geqslant 1$, Lemma 3.19 leads to

$$\|\zeta_m - \Psi(\zeta_m)\| \leqslant 2\|\zeta_n^m - \zeta_{m+1}^{m+1}\| + \mu\|\Psi(\zeta_n^m) - \zeta_n^m\|.$$

Taking upper limit as $n \to \infty$, and (3.3), we have

$$\|\zeta_m - \Psi(\zeta_m)\| \leqslant 2\limsup_{n\to\infty} \|\zeta_n^m - \zeta_{m+1}^{m+1}\| = 2r(\Upsilon, \{\zeta_n^m\}) \leqslant 2\lambda^m r(\Upsilon, \{\zeta_n^0\}).$$

Thus,

$$\lim_{m\to\infty} \|\zeta_m - \Psi(\zeta_m)\| = 0. \tag{3.4}$$

Finally, from (3.4) and by Lemma 3.19 once again that

$$\|\zeta - \Psi(\zeta)\| \leqslant 2\|\zeta - \zeta_m\| + \mu\|\zeta_m - \Psi(\zeta_m)\|,$$

thus, $\|\zeta - \Psi(\zeta)\| = 0$, that is, ζ is a fixed point of Ψ. □

Let us remember that the James constant (also known as the uniformly nonsquare constant) was introduced by Gao and Lau [29] defined below:

$$J(\Omega) = \sup\{\|\zeta + \varrho\| \wedge \|\zeta - \varrho\| : \zeta, \varrho \in B_\Omega\},$$

where B_Ω is the closed unit ball of Ω.

A Banach space Ω is uniformly nonsquare if and only if $J(\Omega) < 2$ (cf. [30]). Gavira proved [31, Corollary 2] that

$$J(\Omega) < \frac{1+\sqrt{5}}{2} \quad \Rightarrow \quad \text{(DL)-condition}.$$

The (DL)-condition defines the proportion between the Chebyshev radius of a bounded sequence's asymptotic center and its asymptotic radius. This condition was later relaxed by Dhompongsa et al. in [28] with the introduction of a new condition called property (D). Several spaces have been identified to meet the (DL)-condition.

From the above property, we have the following corollary.

Corollary 3.24. *Let Ω be a Banach space with $J(\Omega) < \frac{1+\sqrt{5}}{2}$, and Υ a bounded closed convex subset of Ω such that $\Upsilon \neq \varnothing$. If $\Psi : \Upsilon \to \Upsilon$ be an α PNE and satisfies condition (E), then Ψ admits a fixed point.*

Following same the proof of Theorem 3.17, we can obtain following theorem with property (D).

Theorem 3.25. *Let Ω be a Banach space having property (D) and Υ a weakly compact convex subset of Ω such that $\Upsilon \neq \varnothing$. Let S be a family of commuting α-PNE self-mappings on Υ satisfying condition (E). Then S has a common fixed point.*

If a Banach space Ω satisfies the fixed point property (fpp) for nonexpansive mappings, then every nonempty weakly compact convex subset Υ of Ω will have a fixed point for any self-nonexpansive mapping on Υ. García-Falset et al. [32] show that

$$J(\Omega) < 2 \text{ implies fpp for nonexpansive mappings.}$$

We will conclude the paper by posing the following question.

Question 1. Does $J(\Omega) < 2$ imply fpp for the class of α-PNE mappings satisfying condition (E)?

4 Conclusion

We have introduced a new class of nonlinear mappings, namely, α-PNE mappings, and established various existence and convergence theorems for this class of mappings under different assumptions in Banach spaces. We have also shown that our results extend, generalize, and complement the results in [12,16,17,20]. We have extended the class of PNE mappings and introduced the concept of α-PNE mappings, thereby encompassing both PNE mappings and mappings satisfying condition (C_λ).

Acknowledgement: We express our heartfelt gratitude to the reviewers for their helpful feedback and invaluable input, which contributed to improving the quality of this paper. The author acknowledges the National Research Foundation (NRF), South Africa, for their support, grant number 150858.

Funding information: No funding was received.

Author contributions: The authors contributed equally to this work.

Conflict of interest: The authors declare that they have no competing interests.

Ethical approval: The conducted research is not related to either human or animal use.

Data availability statement: Data sharing is not applicable to this article as no data sets were generated or analyzed during this study.

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