Research Article

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Nonlinear nonlocal elliptic problems in $\mathbb{R}^3$: existence results and qualitative properties

https://doi.org/10.1515/dema-2023-0147
received June 28, 2023; accepted January 23, 2024

Abstract: We consider the following nonlinear nonlocal elliptic problem:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla\psi|^2\mathrm{d}x\right)\Delta\psi+\lambda\psi=\left(\int_{\mathbb{R}^3}\frac{G(\psi(y))}{|x-y|^{\alpha}}\mathrm{d}y\right)G'(\psi),\quad x\in\mathbb{R}^3,$$

where $a, b > 0$ are constants, $\lambda > 0$ is a parameter, $\alpha \in (0, 3)$, and $G \in C^1(\mathbb{R}, \mathbb{R})$. By using variational methods, we establish the existence of least energy solutions for the above equation under conditions on the nonlinearity G we believe to be almost necessary. Some qualitative properties of the least energy solutions are also obtained

Keywords: nonlocal elliptic problem, variational method, least energy solution, qualitative property

MSC 2020: 35J60, 35J20, 35Q55

1 Introduction

The aim of this article is to study the following Kirchhoff-type equation with convolution nonlinearities:

$$\begin{cases}-\left(a+b\int_{\mathbb{R}^3}|\nabla\psi|^2\mathrm{d}x\right)\Delta\psi+\lambda\psi=(|x|^{-\alpha}*G(\psi))g(\psi),\\ \psi\in H^1(\mathbb{R}^3),\end{cases}\qquad(\mathcal{P})$$

where $a, b > 0$ are constants, $\lambda > 0$ is a parameter, $\alpha \in (0, 3)$, and $*$ is a notation for the convolution of two functions in $\mathbb{R}^3$, $G \in C^1(\mathbb{R}, \mathbb{R})$, and $g = G'$. We assume that nonlinearity g satisfies the following hypotheses:

(g_1) $g \in C(\mathbb{R}, \mathbb{R})$, and there exists $C > 0$ such that $|g(t)| \le C(|t|^{\frac{3-\alpha}{3}} + |t|^{5-\alpha})$ for all $t \in \mathbb{R}$.

(g_2) Let $G(t) = \int_0^t g(\tau)\mathrm{d}\tau$ and suppose

$$\lim_{t\to0}\frac{G(t)}{|t|^{\frac{6-\alpha}{3}}}=0,\quad \lim_{|t|\to\infty}\frac{G(t)}{|t|^{6-\alpha}}=0.$$

(g_3) There exists $\zeta \ne 0$ such that $G(\zeta) \ne 0$.

Problem $(\mathcal{P})$ is often referred to as nonlocal since the terms $(\int_{\mathbb{R}^3}|\nabla\psi|^2\mathrm{d}x)\Delta\psi$ and $(|x|^{-\alpha}*G(\psi))g(\psi)$ involving in the equation, which implies that problem $(\mathcal{P})$ is no longer a pointwise identity. It is the difficulties generated

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by the phenomenon that make us more interested in the study of such problems. Besides, such a problem has some physical motivations. For example, problem $(\mathcal{P})$ is related to the stationary analogue of the equation

$$\rho\frac{\partial^2\psi}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial\psi}{\partial x}\right|^2\mathrm{d}x\right)\frac{\partial^2\psi}{\partial x^2}=f(x,\psi), \tag{1.1}$$

which was proposed by Kirchhoff [1], where ρ, P_0, h, E, L are constants. Equation (1.1) extends the classical d'Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations. After the pioneer work of Lions [2], where a functional analysis approach was proposed, Kirchhoff-type problems began to call attention of several researchers. In recent years, problems involving Kirchhoff-type operators have been studied in many articles. Some interesting results were obtained, see, for example, [3–7] and the references therein.

We also note that a special case of $(\mathcal{P})$, relevant in physical applications, is the Choquard equation:

$$-\Delta\psi+\psi=(|x|^{-1}*|\psi|^2)\psi,\quad \psi\in H^1(\mathbb{R}^3), \tag{1.2}$$

which models an electron trapped in its own hole and was proposed by Choquard in 1976 as an approximation to Hartree-Fock theory for a one-component plasma [8]. Equation (1.2) arises in various branches of mathematical physics, such as the quantum theory of large systems of nonrelativistic bosonic atoms and molecules, physics of multiple-particle systems, etc., see, for example, [9]. It was also proposed by Penrose [10] as a model for the self-gravitational collapse of a quantum mechanical wave function. Lieb [8] proved the existence and uniqueness (modulo translations) of a minimizer to equation (1.2) by using symmetric decreasing rearrangement inequalities. Later, Lions [11] showed the existence of infinitely many radially symmetric solutions to (1.2). Recently, Ma and Zhao [12] considered the generalized Choquard equation:

$$-\Delta\psi+\psi=(|x|^{-\alpha}*|\psi|^p)|\psi|^{p-2}\psi,\quad \psi\in H^1(\mathbb{R}^N), \tag{1.3}$$

where $p\geq 2$. Under some assumptions on N, α, and p, they proved that every positive solution of (1.3) is radially symmetric and monotone decreasing about some point. Subsequently, Moroz and Van Schaftingen [13] showed the regularity, positivity, and radial symmetry of the ground state solutions for the optimal range of parameters and also obtained decay asymptotics at infinity for them. Further results for related problems may be found in [14–20] and the references therein.

For the nonlocal Kirchhoff-type equations on the whole space $\mathbb{R}^3$ involving convolution-type nonlinearities, little result is known. Motivated by the works described above, in this article, we study the existence of least energy solutions for the doubly nonlocal equation $(\mathcal{P})$ under the very general assumptions (g_1)–(g_3).

The corresponding energy functional of problem $(\mathcal{P})$ is defined by

$$E(\psi)=\frac{1}{2}\int_{\mathbb{R}^3}(a\,|\nabla\psi|^2+\lambda\,|\psi|^2)\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla\psi|^2\mathrm{d}x\right)^2-\frac{1}{2}\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(\psi))G(\psi)\mathrm{d}x. \tag{1.4}$$

Under the hypotheses (g_1)–(g_3), by the Hardy-Littlewood-Sobolev inequality (see [21], Theorem 4.3)

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\phi(x)f(y)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\right|\leq C(\alpha,r)\|\phi\|_{L^r}\|f\|_{L^s},\quad \forall\phi\in L^r(\mathbb{R}^3),\quad \forall f\in L^s(\mathbb{R}^3),$$

where $0<\alpha<3$, $1<r,s<\infty$, and $\frac{1}{r}+\frac{1}{s}+\frac{\alpha}{3}=2$, it is clear that the functional $E(\psi)$ is well-defined and $E(\psi)\in C^1(H^1(\mathbb{R}^3),\mathbb{R})$. Hence, the solutions of equation $(\mathcal{P})$ are the critical points of the energy functional $E(\psi)$.

Now, we can state our main results.

Theorem 1.1. *Let $\alpha\in(0,3)$ and suppose that $(g_1)-(g_3)$ hold, then for every $\lambda>0$ the equation $(\mathcal{P})$ has a least energy solution. More precisely, there exists a nontrivial solution $\psi_*\in H^1(\mathbb{R}^3)$ such that $E(\psi_*)=m>0$, where*

$$m=\inf\{E(\psi):\psi\in H^1(\mathbb{R}^3)\backslash\{0\}\quad\text{and}\quad E'(\psi)=0\}.$$

In the proof of the existence result, we found critical points of $E(\psi)$ basing on a minimax framework. However, in our problem $(\mathcal{P})$, the nonlinearity g does not satisfy the Ambrosetti-Rabinowitz type condition and the corresponding Palais-Smale ((PS) for short) condition seems to be difficult to obtain. Moreover, since g does not have any homogeneity property, we cannot use the usual arguments as in the pure power case. For example, the Nehari manifold method cannot be applied in our case. To overcome these difficulties, we employ a scaling technique to construct a special (PS) sequence that satisfies the corresponding Pohozaev-type identity. More precisely, we consider an auxiliary functional $\widetilde{E}(\theta, v)$ in space $\mathbb{R} \times H^1(\mathbb{R}^3)$:

$$\widetilde{E}(\theta, v) = E(v(e^{-\theta}x)) = \frac{ae^{\theta}}{2}\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x + \frac{\lambda e^{3\theta}}{2}\int_{\mathbb{R}^3}|v|^2\mathrm{d}x + \frac{be^{2\theta}}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)^2 - \frac{e^{(6-\alpha)\theta}}{2}\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(v))G(v)\mathrm{d}x.$$

In Section 2, it will be seen that $\widetilde{E}(\theta, v)$ has a property related to the (PS) condition, which is a key of our argument to show the existence of critical points.

In the last, we give some qualitative properties of least energy solutions to problem $(\mathcal{P})$.

Theorem 1.2. *Let $\alpha \in (0, 3)$ and suppose that $(g_1) - (g_3)$ hold. Moreover, if g is odd or even and has fixed sign on $(0, +\infty)$, then every least energy solution of $(\mathcal{P})$ has fixed sign and is radially symmetric with respect to some point in $\mathbb{R}^3$.*

The rest of this article is organized as follows. In Section 2, we introduce some notations, set the variational framework for problem $(\mathcal{P})$, and present some preliminary results. In Section 3, we obtain the existence of least energy solutions to problem $(\mathcal{P})$. The qualitative properties of the least energy solutions of Theorem 1.2 are established in Section 4.

2 Variational setting and preliminaries

In this article, we will use the following notations:

- $C, C_1, C_2, \ldots$ denote various positive constants whose exact values are not important.
- $\to$ (respectively $\rightharpoonup$) denotes strong (respectively weak) convergence.
- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard inner product

$$\langle \psi, \varphi \rangle = \int_{\mathbb{R}^3}(\nabla\psi\nabla\varphi + \psi\varphi)\mathrm{d}x,$$

and the associated norm $\|\psi\|_{H^1(\mathbb{R}^3)} = \sqrt{\langle \psi, \psi \rangle}$. For fixed $a > 0, \lambda > 0$, we also use the notation

$$\|\psi\|^2 = \int_{\mathbb{R}^3}(a\,|\nabla\psi|^2 + \lambda\psi^2)\mathrm{d}x,$$

which is a norm equivalent to $\|\psi\|_{H^1(\mathbb{R}^3)}$.

First, we show that the energy functional $E(\psi)$ defined in (1.4) satisfies the mountain-pass geometry.

Lemma 2.1. *Let $\alpha \in (0, 3)$ and $(g_1) - (g_3)$ hold. Then, for any given $\lambda > 0$, there exist $\delta_0, \rho_0 > 0$ and $\psi_0 \in H^1(\mathbb{R}^3)$* with $\|\psi_0\| > \rho_0$ *such that $E(\psi) \geq \delta_0 > 0$ for all $\psi \in H^1(\mathbb{R}^3)$ with $\|\psi\| = \rho_0$ and $E(\psi_0) < 0$.*

Proof. For every $\psi \in H^1(\mathbb{R}^3)$, by (g_1) and the Hardy-Littlewood-Sobolev inequality, we have that

$$\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(\psi))G(\psi)\mathrm{d}x \leq C\left(\int_{\mathbb{R}^3}|G(\psi)|^{\frac{6}{6-\alpha}}\mathrm{d}x\right)^{\frac{6-\alpha}{3}}$$

$$\begin{aligned}
&\le C_1\left(\int_{\mathbb{R}^3}(|\psi|^{\frac{6-\alpha}{3}}+|\psi|^{6-\alpha})^{\frac{6}{6-\alpha}}\mathrm{d}x\right)^{\frac{6-\alpha}{3}}\\
&\le C_2\left(\int_{\mathbb{R}^3}(|\psi|^2+|\psi|^6)\mathrm{d}x\right)^{\frac{6-\alpha}{3}}\\
&\le C_3\left(\left(\int_{\mathbb{R}^3}|\psi|^2\mathrm{d}x\right)^{\frac{6-\alpha}{3}}+\left(\int_{\mathbb{R}^3}|\nabla\psi|^2\mathrm{d}x\right)^{6-\alpha}\right).
\end{aligned} \tag{2.1}$$

Note that $\alpha \in (0,3)$, then $\frac{6-\alpha}{3} > 1$, from (2.1) and by the Sobolev embedding, there exists $\rho > 0$ small such that if $\|\psi\| \le \rho$, then

$$\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(\psi))G(\psi)\mathrm{d}x \le \frac{1}{2}\|\psi\|^2,$$

and therefore

$$E(\psi) = \frac{1}{2}\|\psi\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla\psi|^2\mathrm{d}x\right)^2 - \frac{1}{2}\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(\psi))G(\psi)\mathrm{d}x \ge \frac{1}{4}\|\psi\|^2.$$

Hence, we can choose some $\delta_0, \rho_0 > 0$ such that $E(\psi) \ge \delta_0 > 0$ for all $\|\psi\| = \rho_0$.

If we choose ζ of assumption (g_3) such that $G(\zeta) \ne 0$ and set $w = \zeta\chi_{B_1(0)}$, we obtain

$$\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(w))G(w)\mathrm{d}x = (G(\zeta))^2\int_{B_1(0)}\int_{B_1(0)}|x-y|^{-\alpha}\mathrm{d}x\mathrm{d}y > 0.$$

By (g_1), we know that $\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(\psi))G(\psi)\mathrm{d}x$ is continuous in $L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$, and since $H^1(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$, there exists $v \in H^1(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(v))G(v)\mathrm{d}x > 0.$$

Let $\psi_t(x) = v(\frac{x}{t})$, then for every $t > 0$, by a direct computation, we obtain that

$$E(\psi_t) = \frac{at}{2}\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x + \frac{\lambda t^3}{2}\int_{\mathbb{R}^3}|v|^2\mathrm{d}x + \frac{bt^2}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)^2 - \frac{t^{6-\alpha}}{2}\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(v))G(v)\mathrm{d}x.$$

Since $6-\alpha > 3$, we have that $E(\psi_t) \to -\infty$ as $t \to +\infty$. Hence, we can obtain $E(\psi_t) < 0$ for $t > 0$ large enough, taking $\psi_0 = \psi_t$ then we complete the proof of Lemma 2.1. □

By Lemma 2.1, we know that

$$\Gamma \coloneqq \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, E(\gamma(1)) < 0\} \ne \emptyset \tag{2.2}$$

and

$$c^* \coloneqq \inf_{\gamma\in\Gamma}\max_{0\le t\le 1}E(\gamma(t)) > 0. \tag{2.3}$$

Thus, by Ekeland's principle, there exists a (PS) sequence at the mountain-pass level c^* for E. Namely a sequence $\{\psi_n\} \subset H^1(\mathbb{R}^3)$ such that

$$E(\psi_n) \to c^*, \quad E'(\psi_n) \to 0 \quad \text{as } n \to \infty, \tag{2.4}$$

where c^* is the minimax level of functional E given in (2.3).

We will make use of the following Pohožaev type identity.

Lemma 2.2. *Let $\alpha \in (0,3)$, $\lambda > 0$ and suppose g satisfies $(g_1)-(g_3)$. If $\psi \in H^1(\mathbb{R}^3)$ is a weak solution to problem $(\mathcal{P})$, then the following Pohožaev type identity holds*:

$$\frac{a}{2}\int_{\mathbb{R}^3}|\nabla\psi|^2\mathrm{d}x + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla\psi|^2\mathrm{d}x\right)^2 + \frac{3\lambda}{2}\int_{\mathbb{R}^3}|\psi|^2\mathrm{d}x = \frac{6-\alpha}{2}\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(\psi))G(\psi)\mathrm{d}x. \tag{2.5}$$

Proof. The proof is standard, the details are hence omitted (see, e.g., [13,22]). □

By Lemma 2.2, we know that if ψ is a critical point of E, then ψ satisfies $P(\psi) = 0$, where

$$P(\psi) = \frac{a}{2}\int_{\mathbb{R}^3}|\nabla\psi|^2\mathrm{d}x + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla\psi|^2\mathrm{d}x\right)^2 + \frac{3\lambda}{2}\int_{\mathbb{R}^3}|\psi|^2\mathrm{d}x - \frac{6-\alpha}{2}\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(\psi))G(\psi)\mathrm{d}x.$$

It is natural to ask whether there exists a (PS) sequence $\{\psi_n\}$ satisfying (2.4) and $P(\psi_n) \to 0$. In fact, we have the following result.

Lemma 2.3. *Let $\alpha \in (0,3)$, $\lambda > 0$ and suppose that $(g_1)-(g_3)$ hold, then there exists a sequence $\{\psi_n\} \subset H^1(\mathbb{R}^3)$ such that, as $n \to \infty$, there hold $E(\psi_n) \to c^*$, $E'(\psi_n) \to 0$, and $P(\psi_n) \to 0$.*

Following the idea of [23], we consider the map $\Phi : \mathbb{R} \times H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ given by $\Phi(\theta, v) = v(e^{-\theta}x)$, and define an auxiliary functional $\widetilde{E} : \mathbb{R} \times H^1(\mathbb{R}^3) \to \mathbb{R}$:

$$\widetilde{E}(\theta, v) = E(\Phi(\theta, v)) = \frac{ae^{\theta}}{2}\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x + \frac{\lambda e^{3\theta}}{2}\int_{\mathbb{R}^3}|v|^2\mathrm{d}x + \frac{be^{2\theta}}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x\right)^2 - \frac{e^{(6-\alpha)\theta}}{2}\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(v))G(v)\mathrm{d}x.$$

We endow $\mathbb{R} \times H^1(\mathbb{R}^3)$ with the usual norm $\|(\theta, v)\|_{\mathbb{R}\times H^1(\mathbb{R}^3)} = \sqrt{|\theta|^2 + \|v\|^2}$.

In view of (g_1), the auxiliary functional $\widetilde{E}(\theta, v)$ is continuously Fréchet differentiable on $\mathbb{R} \times H^1(\mathbb{R}^3)$ and satisfies

$$\widetilde{E}(0, v) = E(v), \quad \widetilde{E}(\theta, v) = E(v(e^{-\theta}x)) \tag{2.6}$$

for all $v \in H^1(\mathbb{R}^3)$, $\theta \in \mathbb{R}$ and $x \in \mathbb{R}^3$. We define

$$\widetilde{\Gamma} := \{\widetilde{\gamma} \in C([0,1], \mathbb{R} \times H^1(\mathbb{R}^3)) : \widetilde{\gamma}(0) = (0,0), \widetilde{E}(\widetilde{\gamma}(1)) < 0\}.$$

It is easy to see that $\widetilde{\Gamma} \neq \emptyset$, and thus we know that

$$d^* = \inf_{\widetilde{\gamma}\in\widetilde{\Gamma}} \sup_{t\in[0,1]} \widetilde{E}(\widetilde{\gamma}(t))$$

is well defined. The next lemma shows that d^* coincide with c^*.

Lemma 2.4. $d^* = c^*$.

Proof. First, since for any $\gamma \in \Gamma$, we can see that $(0, \gamma(t)) \in \widetilde{\Gamma}$, that is, $\{0\} \times \Gamma \subset \widetilde{\Gamma}$, then for $\widetilde{E}(0, v) = E(v)$, we have $d^* \le c^*$. In contrast, for any given $\widetilde{\gamma}(t) = (\theta(t), \eta(t)) \in \widetilde{\Gamma}$, setting $\gamma(t)(x) = \eta(t)(e^{-\theta(t)}(x))$, we can verify that $\gamma(t) \in \Gamma$, and by (2.6), $E(\gamma(t)) = \widetilde{E}(\widetilde{\gamma}(t))$, which implies that $d^* \ge c^*$. Hence, $d^* = c^*$. □

To prove Lemma 2.3, we also need the following result which was established in [23] (see Lemma 2.3) by using Ekeland's variational principle.

Lemma 2.5. *Let $\varepsilon > 0$ and $\gamma_0 \in \widetilde{\Gamma}$ such that*

$$\sup_{0\le t\le 1} \widetilde{E}(\gamma_0(t)) \le d^* + \varepsilon.$$

Then, there exists a pair of $(\theta_0, v_0) \in \mathbb{R} \times H^1(\mathbb{R}^3)$ such that
(*a*) $\widetilde{E}(\theta_0, v_0) \in [d^* - \varepsilon, d^* + \varepsilon]$;
(*b*) $\|D\widetilde{E}(\theta_0, v_0)\|_{(\mathbb{R}\times H^1(\mathbb{R}^3))^*} \le 2\sqrt{\varepsilon}$, *where* $D\widetilde{E}(\theta, v) = (\partial_\theta \widetilde{E}(\theta, v), \widetilde{E}'(\theta, v))$;
(*c*) $\min_{0\le t\le 1}\|(\theta_0, v_0) - \gamma_0(t)\|_{\mathbb{R}\times H^1(\mathbb{R}^3)} \le \sqrt{\varepsilon}$.

Proof of Lemma 2.3. By the definition of c^*, we can choose $\{\gamma_n\} \subset \Gamma$ satisfy

$$\sup_{0\le t\le 1} E(\gamma_n(t)) \le c^* + \frac{1}{n}.$$

Set $\tilde{\gamma}_n(t) := (0, \gamma_n(t))$, then $\tilde{\gamma}_n \in \widetilde{\Gamma}$ and we have

$$\sup_{0\le t\le 1} \widetilde{E}(\tilde{\gamma}_n(t)) \le c^* + \frac{1}{n}.$$

Then, by Lemmas 2.5 and 2.4, there exists a sequence $\{(\theta_n, v_n)\}$ in $\mathbb{R} \times H^1(\mathbb{R}^3)$ such that

$$\widetilde{E}(\theta_n, v_n) \to d^* = c^*, \quad \widetilde{E}'(\theta_n, v_n) \to 0 \tag{2.7}$$

and

$$\min_{0\le t\le 1} \|(\theta_n, v_n) - \tilde{\gamma}_n(t)\|_{\mathbb{R}\times H^1(\mathbb{R}^3)} \to 0 \tag{2.8}$$

as $n \to \infty$. Note that for every $(h, w) \in \mathbb{R} \times H^1(\mathbb{R}^3)$, there holds

$$0 \leftarrow \langle \widetilde{E}'(\theta_n, v_n), (h, w)\rangle = \langle E'(\Phi(\theta_n, v_n)), \Phi(\theta_n, w)\rangle + P(\Phi(\theta_n, v_n))h. \tag{2.9}$$

Let $\psi_n = \Phi(\theta_n, v_n)$ and taking $(h, w) = (1, 0)$ in (2.9), we can obtain

$$P(\psi_n) \to 0 \quad \text{as } n \to \infty. \tag{2.10}$$

From (2.8), it follows that

$$|\theta_n| = |\theta_n - 0| \le \min_{0\le t\le 1} \|(\theta_n, v_n) - (0, \gamma_n(t))\|_{\mathbb{R}\times H^1(\mathbb{R}^3)} \to 0,$$

by which for any $v \in H^1(\mathbb{R}^3)$, set $h = 0$, $w(x) = v(e^{\theta_n}x)$ in (2.9), we obtain

$$\langle E'(\psi_n), v\rangle = \langle \widetilde{E}'(\theta_n, v_n), (0, v(e^{\theta_n}x))\rangle = o(1)\|v(e^{\theta_n}x)\| = o(1)\|v\|$$

for $\theta_n \to 0$ as $n \to \infty$, so $E'(\psi_n) \to 0$ as $n \to \infty$. By (2.7), we can obtain that $E(\psi_n) \to c^*$ as $n \to \infty$. Hence, we have got a sequence $\{\psi_n\} \subset H^1(\mathbb{R}^3)$ that satisfies

$$E(\psi_n) \to c^*, \quad E'(\psi_n) \to 0, \quad P(\psi_n) \to 0 \quad \text{as } n \to \infty. \tag{2.11}$$

This completes the proof of Lemma 2.3. □

Lemma 2.6. *Let $\{\psi_n\}$ be the sequences obtained in Lemma 2.3. Then, $\{\psi_n\}$ is bounded in $H^1(\mathbb{R}^3)$.*

Proof. To show the boundedness of $\{\psi_n\}$, we note that

$$\begin{aligned} E(\psi_n) - \frac{1}{6-\alpha}P(\psi_n) &= \frac{a(5-\alpha)}{2(6-\alpha)}\int_{\mathbb{R}^3} |\nabla\psi_n|^2 dx + \frac{\lambda(3-\alpha)}{2(6-\alpha)}\int_{\mathbb{R}^3} |\psi_n|^2 dx + \frac{b(4-\alpha)}{4(6-\alpha)}\left(\int_{\mathbb{R}^3} |\nabla\psi_n|^2 dx\right)^2 \\ &\ge \frac{3-\alpha}{2(6-\alpha)}\int_{\mathbb{R}^3} (a|\nabla\psi_n|^2 + \lambda|\psi_n|^2) dx. \end{aligned} \tag{2.12}$$

By (2.11), we know that $E(\psi_n) - \frac{1}{6-\alpha}P(\psi_n)$ is bounded, so (2.12) implies that the sequence $\{\psi_n\}$ is bounded in $H^1(\mathbb{R}^3)$. □

3 Existence of least energy solutions

In this section, we will prove Theorem 1.1. We start with the following lemma.

Lemma 3.1. *Suppose that g satisfies (g_1) and (g_2). Let $\{\psi_n\} \subset H^1(\mathbb{R}^3)$ such that $\{E(\psi_n)\}$ is bounded and*

$$E'(\psi_n) \to 0, \quad P(\psi_n) \to 0 \quad \text{as } n \to \infty.$$

Then, up to a subsequence, either $\psi_n \to 0$ strongly in $H^1(\mathbb{R}^3)$, or there exists $\psi \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that $E'(\psi) = 0$.

Proof. Suppose that the first part of the alternative does not hold, that is, $\psi_n \nrightarrow 0$ strongly in $H^1(\mathbb{R}^3)$, then we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} (a\,|\nabla\psi_n|^2 + \lambda\,|\psi_n|^2)\mathrm{d}x > 0. \tag{3.1}$$

Similar to (2.12), we can obtain that the sequence $\{\psi_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Now, we claim that, for every $p \in (2,6)$, there holds

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |\psi_n|^p \mathrm{d}x > 0. \tag{3.2}$$

First, by (3.1) and the fact $P(\psi_n) \to 0$, we have

$$\begin{aligned}&\liminf_{n\to\infty} \int_{\mathbb{R}^3} (|x|^{-\alpha} * G(\psi_n))G(\psi_n)\mathrm{d}x \\ &= \liminf_{n\to\infty}\left(\frac{a}{6-\alpha}\int_{\mathbb{R}^3}|\nabla\psi_n|^2\mathrm{d}x + \frac{3\lambda}{6-\alpha}\int_{\mathbb{R}^3}|\psi_n|^2\mathrm{d}x + \frac{b}{6-\alpha}\left(\int_{\mathbb{R}^3}|\nabla\psi_n|^2\mathrm{d}x\right)^2\right) > 0.\end{aligned} \tag{3.3}$$

As g is continuous and satisfies (g_2), for every $\varepsilon > 0$, there exists C_ε such that

$$|G(t)|^{\frac{6}{6-\alpha}} \le \varepsilon(|t|^2 + |t|^6) + C_\varepsilon\,|t|^p, \quad \forall t \in \mathbb{R}. \tag{3.4}$$

We note that, for all $n \in \mathbb{N}$, there holds

$$\int_{\mathbb{R}^3} |\psi_n|^p \mathrm{d}x \le C\left(\int_{\mathbb{R}^3} |\nabla\psi_n|^2 + |\psi_n|^2 \mathrm{d}x\right)\left(\sup_{y\in\mathbb{R}^3}\int_{B_1(y)} |\psi_n|^p \mathrm{d}x\right)^{1-\frac{2}{p}}. \tag{3.5}$$

Since the sequence $\{\psi_n\}$ is bounded in $H^1(\mathbb{R}^3)$, by the Sobolev embedding, we know that $\{\psi_n\}$ is also bounded in $L^2(\mathbb{R}^3)$ and $L^6(\mathbb{R}^3)$. Then, by (3.4) and (3.5), we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} |G(\psi_n)|^{\frac{6}{6-\alpha}} \mathrm{d}x \le C\varepsilon + \bar{C}_\varepsilon\left(\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)} |\psi_n|^p \mathrm{d}x\right)^{1-\frac{2}{p}}.$$

Since $\varepsilon > 0$ is arbitrary, if $\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}|\psi_n|^p\mathrm{d}x = 0$, then

$$\liminf_{n\to\infty}\int_{\mathbb{R}^3} |G(\psi_n)|^{\frac{6}{6-\alpha}}\mathrm{d}x = 0,$$

and by the Hardy-Littlewood-Sobolev inequality, we obtain that

$$\liminf_{n\to\infty}\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(\psi_n))G(\psi_n)\mathrm{d}x \le C\liminf_{n\to\infty}\left(\int_{\mathbb{R}^3}|G(\psi_n)|^{\frac{6}{6-\alpha}}\mathrm{d}x\right)^{\frac{6-\alpha}{3}} = 0,$$

which is in contradiction with (3.3) and so (3.2) holds. Then, up to a translation, we can assume that for some $p \in (2, 6)$, $\liminf_{n\to\infty}\int_{B_1}|\psi_n|^p\mathrm{d}x > 0$. By Rellich's theorem, this implies that up to a subsequence, there exists $\psi \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that

$$\psi_n \rightharpoonup \psi \quad \text{in } H^1(\mathbb{R}^3) \quad \text{and} \quad \int_{\mathbb{R}^3}|\nabla\psi_n|^2\mathrm{d}x \to \mathscr{A} \quad \text{for some } \mathscr{A} \ge 0. \tag{3.6}$$

Then, for any function $\varphi \in C_0^\infty(\mathbb{R}^3)$, by the weak convergence of the sequence $\{\psi_n\}$, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}(a\nabla\psi_n\nabla\varphi + \lambda\psi_n\varphi)\mathrm{d}x = \int_{\mathbb{R}^3}(a\nabla\psi\nabla\varphi + \lambda\psi\varphi)\mathrm{d}x. \tag{3.7}$$

As $\{\psi_n\}$ is bounded in $H^1(\mathbb{R}^3)$, by the Sobolev embedding, it is also bounded in $L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$. Therefore, by (g_1), we have that the sequence $\{G(\psi_n)\}$ is bounded in $L^{\frac{6}{6-\alpha}}(\mathbb{R}^3)$. Since $\psi_n \rightharpoonup \psi$ in $H^1(\mathbb{R}^3)$, up to a subsequence, $\psi_n \to \psi$ a.e. in $\mathbb{R}^3$. By continuity of g, then $G(\psi_n) \to G(\psi)$ a.e. in $\mathbb{R}^3$. This implies that

$$G(\psi_n) \rightharpoonup G(\psi) \quad \text{in } L^{\frac{6}{6-\alpha}}(\mathbb{R}^3).$$

As the Riesz potential $|x|^{-\alpha}$ defines a linear continuous map from $L^{\frac{6}{6-\alpha}}(\mathbb{R}^3)$ to $L^{\frac{6}{\alpha}}(\mathbb{R}^3)$, we obtain

$$|x|^{-\alpha}*G(\psi_n) \rightharpoonup |x|^{-\alpha}*G(\psi) \quad \text{in } L^{\frac{6}{\alpha}}(\mathbb{R}^3). \tag{3.8}$$

Moreover, in view of (g_1) and by Rellich's theorem, we know that

$$g(\psi_n) \to g(\psi) \quad \text{strongly in } L^r_{\mathrm{loc}}(\mathbb{R}^3) \quad \text{for every } r \in \left[1, \frac{6}{5-\alpha}\right). \tag{3.9}$$

Therefore, by (3.8) and (3.9), for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, we can obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(\psi_n))g(\psi_n)\varphi\mathrm{d}x = \int_{\mathbb{R}^3}(|x|^{-\alpha}*G(\psi))g(\psi)\varphi\mathrm{d}x. \tag{3.10}$$

Now, by (3.6), (3.7), and (3.10), for every $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned}
0 &= \lim_{n\to\infty}\langle E'(\psi_n), \varphi\rangle \\
&= \lim_{n\to\infty}\left(\int_{\mathbb{R}^3}(a\nabla\psi_n\nabla\varphi + \lambda\psi_n\varphi)\mathrm{d}x + b\int_{\mathbb{R}^3}|\nabla\psi_n|^2\mathrm{d}x\int_{\mathbb{R}^3}\nabla\psi_n\nabla\varphi\mathrm{d}x + \int_{\mathbb{R}^3}(|x|^{-\alpha}*G(\psi_n))g(\psi_n)\varphi\mathrm{d}x\right) \\
&= \int_{\mathbb{R}^3}(a\nabla\psi\nabla\varphi + \lambda\psi\varphi)\mathrm{d}x + b\mathscr{A}\int_{\mathbb{R}^3}\nabla\psi\nabla\varphi\mathrm{d}x + \int_{\mathbb{R}^3}(|x|^{-\alpha}*G(\psi))g(\psi)\varphi\mathrm{d}x \\
&= \langle J'(\psi), \varphi\rangle,
\end{aligned} \tag{3.11}$$

where

$$J(\psi) = \frac{a + b\mathscr{A}}{2}\int_{\mathbb{R}^3}|\nabla\psi|^2\mathrm{d}x + \frac{\lambda}{2}\int_{\mathbb{R}^3}|\psi|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3}(|x|^{-\alpha}*G(\psi))G(\psi)\mathrm{d}x.$$

By Fatou's lemma, we obtain that

$$\int_{\mathbb{R}^3}|\nabla\psi|^2\mathrm{d}x \le \liminf_{n\to\infty}\int_{\mathbb{R}^3}|\nabla\psi_n|^2\mathrm{d}x = \mathscr{A}, \quad \int_{\mathbb{R}^3}|\psi|^2\mathrm{d}x \le \liminf_{n\to\infty}\int_{\mathbb{R}^3}|\psi_n|^2\mathrm{d}x. \tag{3.12}$$

Notice that $E'(\psi_n) \to 0$ and $\int_{\mathbb{R}^3} |\nabla\psi_n|^2 \mathrm{d}x \to \mathscr{A}$, we can obtain that

$$\lim_{n\to\infty} \langle J'(\psi_n), \psi_n \rangle = 0.$$

Then by (3.10), the latter part of (3.12) and (3.11), we have

$$\begin{aligned}\limsup_{n\to\infty}(a + b\mathscr{A})\int_{\mathbb{R}^3} |\nabla\psi_n|^2 \mathrm{d}x &= \limsup_{n\to\infty}\left(\int_{\mathbb{R}^3} (|x|^{-\alpha} * G(\psi_n))g(\psi_n)\psi_n \mathrm{d}x - \lambda \int_{\mathbb{R}^3} |\psi_n|^2 \mathrm{d}x\right)\\ &\le \int_{\mathbb{R}^3} (|x|^{-\alpha} * G(\psi))g(\psi)\psi \mathrm{d}x - \lambda \int_{\mathbb{R}^3} |\psi|^2 \mathrm{d}x \\ &= (a + b\mathscr{A})\int_{\mathbb{R}^3} |\nabla\psi|^2 \mathrm{d}x,\end{aligned} \tag{3.13}$$

so combining (3.13) with the former part of (3.12), we obtain that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla\psi_n|^2 \mathrm{d}x = \int_{\mathbb{R}^3} |\nabla\psi|^2 \mathrm{d}x = \mathscr{A}. \tag{3.14}$$

Therefore, from (3.10) and (3.14), we can deduce that for all $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$0 = \lim_{n\to\infty} \langle E'(\psi_n), \varphi \rangle = \langle E'(\psi), \varphi \rangle,$$

that is, $E'(\psi) = 0$ and ψ is a weak solution of equation $(\mathcal{P})$. □

In order to prove the existence of least energy solutions, we need the following result.

Lemma 3.2. *If g satisfies (g_1) and $\psi \in H^1(\mathbb{R}^3)\setminus\{0\}$ solves equation $(\mathcal{P})$, then there exists a path $\gamma \in C([0,1], H^1(\mathbb{R}^3))$ and $t_0 \in (0,1)$ such that*

$$\gamma(0) = 0; \quad \gamma(t_0) = \psi \quad \text{and} \quad E(\gamma(t)) < E(\psi), \quad \forall t \in [0, t_0) \cup (t_0, 1]; \quad E(\gamma(1)) < 0.$$

Proof. We define the path $\tilde{\gamma} : [0, +\infty) \to H^1(\mathbb{R}^3)$ by

$$\tilde{\gamma}(\tau)(x) = \begin{cases} \psi\left(\frac{x}{\tau}\right) & \text{if } \tau > 0, \\ 0 & \text{if } \tau = 0. \end{cases}$$

Clearly, the function $\tilde{\gamma}$ is continuous on $(0, \infty)$. For all $\tau > 0$, one has

$$\int_{\mathbb{R}^3} (a\, |\nabla\tilde{\gamma}(\tau)|^2 + \lambda\, |\tilde{\gamma}(\tau)|^2) \mathrm{d}x = \tau \int_{\mathbb{R}^3} a\, |\nabla\psi|^2 \mathrm{d}x + \tau^3 \int_{\mathbb{R}^3} \lambda\, |\psi|^2 \mathrm{d}x,$$

so that $\tilde{\gamma}$ is continuous at 0. By the Pohožaev type identity (2.5), for every $\tau > 0$, we conclude that

$$\begin{aligned}E(\tilde{\gamma}(\tau)) &= \frac{a\tau}{2}\int_{\mathbb{R}^3} |\nabla\psi|^2 \mathrm{d}x + \frac{\lambda\tau^3}{2}\int_{\mathbb{R}^3} |\psi|^2 \mathrm{d}x + \frac{b\tau^2}{4}\left(\int_{\mathbb{R}^3} |\nabla\psi|^2 \mathrm{d}x\right)^2 - \frac{\tau^{6-\alpha}}{2}\int_{\mathbb{R}^3} (|x|^{-\alpha} * G(\psi))G(\psi) \mathrm{d}x \\ &= \left(\frac{\tau}{2} - \frac{\tau^{6-\alpha}}{2(6-\alpha)}\right) a \int_{\mathbb{R}^3} |\nabla\psi|^2 \mathrm{d}x + \left(\frac{\tau^3}{2} - \frac{3\tau^{6-\alpha}}{2(6-\alpha)}\right)\lambda \int_{\mathbb{R}^3} |\psi|^2 \mathrm{d}x + \left(\frac{\tau^2}{4} - \frac{\tau^{6-\alpha}}{2(6-\alpha)}\right) b \left(\int_{\mathbb{R}^3} |\nabla\psi|^2 \mathrm{d}x\right)^2.\end{aligned}$$

It can be checked directly that $E(\tilde{\gamma}(\tau))$ achieves strict global maximum at $\tau = 1$, then for all $\tau \ge 0$, $\tau \ne 1$, we obtain $E(\tilde{\gamma}(\tau)) < E(\psi)$. In addition, we note that $\lim_{\tau\to\infty} E(\tilde{\gamma}(\tau)) = -\infty$, then the path γ can be defined by a suitable change of variable and we complete the proof of Lemma 3.2. □

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. From Lemma 2.3, there exists a Pohožaev-Palais-Smale sequence $\{\psi_n\} \subset H^1(\mathbb{R}^3)$ at the mountain-pass level $c^* > 0$, that is,

$$E(\psi_n) \to c^*, \quad E'(\psi_n) \to 0, \quad P(\psi_n) \to 0 \quad \text{as } n \to \infty.$$

By (2.12), we know that $\{\psi_n\}$ is bounded in $H^1(\mathbb{R}^3)$, then going if necessary to a subsequence, $\psi_n \rightharpoonup \psi_*$ in $H^1(\mathbb{R}^3)$ and $\psi_n \to \psi_*$ a.e. in $H^1(\mathbb{R}^3)$. Moreover, by Lemma 3.1, for any $\varphi \in H^1(\mathbb{R}^3)$, we can obtain that

$$0 = \lim_{n\to\infty} \langle E'(\psi_n), \varphi\rangle = \langle E'(\psi_*), \varphi\rangle,$$

so ψ_* solves $(\mathcal{P})$.

Since $P(\psi_n) \to 0$ as $n \to \infty$, by the weak convergence of the sequence $\{\psi_n\}$, the weak lower-semicontinuity of the norm and the Pohožaev type identity (2.5), we have

$$\begin{aligned}
E(\psi_*) = E(\psi_*) - \frac{1}{6-\alpha}P(\psi_*) \\
&= \frac{a(5-\alpha)}{2(6-\alpha)}\int_{\mathbb{R}^3}|\nabla\psi_*|^2\mathrm{d}x + \frac{\lambda(3-\alpha)}{2(6-\alpha)}\int_{\mathbb{R}^3}|\psi_*|^2\mathrm{d}x + \frac{b(4-\alpha)}{4(6-\alpha)}\left(\int_{\mathbb{R}^3}|\nabla\psi_*|^2\mathrm{d}x\right)^2 \\
&\le \liminf_{n\to\infty}\left(\frac{a(5-\alpha)}{2(6-\alpha)}\int_{\mathbb{R}^3}|\nabla\psi_n|^2\mathrm{d}x + \frac{\lambda(3-\alpha)}{2(6-\alpha)}\int_{\mathbb{R}^3}|\psi_n|^2\mathrm{d}x + \frac{b(4-\alpha)}{4(6-\alpha)}\left(\int_{\mathbb{R}^3}|\nabla\psi_n|^2\mathrm{d}x\right)^2\right) \\
&= \liminf_{n\to\infty}\left(E(\psi_n) - \frac{1}{6-\alpha}P(\psi_n)\right) \\
&= \liminf_{n\to\infty} E(\psi_n) = c^*.
\end{aligned}$$

Since $c^* > 0$, ψ_* is a nontrivial solution of $(\mathcal{P})$, we have $E(\psi_*) \ge m$ by the definition of the least energy level m, and hence $m \le c^*$. Let $v \in H^1(\mathbb{R}^3)\setminus\{0\}$ be another solution of $(\mathcal{P})$ such that $E(v) \le E(\psi_*)$. By Lemma 3.2, if we lift v to a path and recall the definition of the mountain-pass level c^* in (2.3), we conclude that $E(v) \ge c^* \ge E(\psi_*)$. We have thus proved that $E(v) = E(\psi_*) = c^* = m$. This completes the proof of Theorem 1.1. □

4 Some properties of least energy solutions

In this section, we will present some qualitative properties of the least energy solutions to equation $(\mathcal{P})$. First, we will use the following elementary property of the paths in the construction of the mountain-pass critical level c^*.

Lemma 4.1. *Assume that g satisfies (g_1) and $\gamma \in \Gamma$, where Γ is defined in (2.2). If for every $t \in [0,1]\setminus\{t_*\}$, there holds $c^* = E(\gamma(t_*)) > E(\gamma(t))$, then $E'(\gamma(t_*)) = 0$.*

Proof. This can be deduced from the quantitative deformation lemma of Willem (see [24], Lemma 2.3). We give a sketchy proof here for readers' convenience. Arguing by contradiction, we assume that $E'(\gamma(t_*)) \ne 0$. By continuity, it is possible to choose $\delta > 0$ and $\varepsilon > 0$ such that

$$\inf\{\|E'(v)\| : \|v - \gamma(t_*)\| \le \delta\} > \frac{8\varepsilon}{\delta}.$$

With Willem's notations, take $X = H^1(\mathbb{R}^3)$, $S = \{\gamma(t_*)\}$, and $m = c^*$. By the deformation lemma, there exists $\eta \in C([0,1], H^1(\mathbb{R}^3))$ such that $\eta(1,\gamma) \in \Gamma$ and $E(\eta(1,\gamma(t_*))) \le c^* - \varepsilon < c^*$, and for all $t \in [0,1]$, there holds

$$E(\eta(1,\gamma(t))) \le E(\gamma(t)) < c^*.$$

Since $[0,1]$ is compact, we conclude with the contradiction that $\sup_{t\in[0,1]} E(\eta(1,\gamma(t))) < c^*$, and hence $E'(\gamma(t_*)) = 0$ holds. □

Lemma 4.2. *Assume that g satisfies* (g_1), g *is odd or even, and does not change sign on* $(0,+\infty)$. *Then, any least energy solution* $\psi \in H^1(\mathbb{R}^3)$ *of* $(\mathcal{P})$ *has fixed sign.*

Proof. Without loss of generality, we can assume that $g \geq 0$ on $(0,+\infty)$. By Lemma 3.2, there exists an optimal path $\gamma \in \Gamma$ on which the functional E achieves its maximum at t_0. Since g is odd or even, $(|x|^{-\alpha} * G(\psi))G(\psi)$ is even and thus $E(|\psi|) = E(\psi)$ for every $\psi \in H^1(\mathbb{R}^3)$. Hence, for every $t \in [0,1]\backslash\{t_0\}$,

$$E(|\gamma(t)|) = E(\gamma(t)) < E(\gamma(t_0)) = E(|\gamma(t_0)|).$$

By Lemma 4.1, $|\psi| = |\gamma(t_0)|$ is also a least energy solution. It satisfies the equation

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla|\psi||^2 \mathrm{d}x\right)\Delta|\psi| + \lambda|\psi| = (|x|^{-\alpha} * G(|\psi|))g(|\psi|).$$

Since ψ is continuous, by the strong maximum principle we conclude that $|\psi| > 0$ on $\mathbb{R}^3$ and thus ψ has fixed sign. □

Before proving the symmetry of least energy solutions, we recall some elements of the theory of polarization [25]. Assume that $H \subset \mathbb{R}^N (N \geq 3)$ is a closed half-space and let σ_H denote the reflection with respect to ∂H. The polarization $u^H : \mathbb{R}^N \to \mathbb{R}$ of $u : \mathbb{R}^N \to \mathbb{R}$ is defined for $x \in \mathbb{R}^N$ by

$$u^H(x) = \begin{cases} \max\{u(x), u(\sigma_H(x))\} & \text{if } x \in H, \\ \min\{u(x), u(\sigma_H(x))\} & \text{if } x \in \mathbb{R}^N\backslash H. \end{cases}$$

We will use the following standard property of polarizations (see [25], Lemma 5.3).

Lemma 4.3. *If* $u \in H^1(\mathbb{R}^N)$, *then* $u^H \in H^1(\mathbb{R}^N)$ *and* $\int_{\mathbb{R}^N} |\nabla u^H|^2 \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x$.

Lemma 4.4. *Let* $\tilde{\alpha} \in (0,N)$, $u \in L^{\frac{2N}{2N-\tilde{\alpha}}}(\mathbb{R}^N)$, *and* $H \subset \mathbb{R}^N$ *be a closed half-space. If* $u \geq 0$, *then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u(x)u(y)}{|x-y|^{\tilde{\alpha}}} \mathrm{d}x\mathrm{d}y \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^H(x)u^H(y)}{|x-y|^{\tilde{\alpha}}} \mathrm{d}x\mathrm{d}y,$$

with equality if and only if either $u^H = u$ *or* $u^H = u(\sigma_H)$.

The last tool we need is a characterization of symmetric functions by polarizations (see [13], Lemma 5.4).

Lemma 4.5. *If* $u \in L^2(\mathbb{R}^N)$ *is nonnegative. There exist* $x_0 \in \mathbb{R}^N$ *and a nonincreasing function* $v : (0,+\infty) \to \mathbb{R}$ *such that for almost every* $x \in \mathbb{R}^N$, $u(x) = v(|x - x_0|)$ *if and only if for every closed half-space* $H \subset \mathbb{R}^N$, $u^H = u$ *or* $u^H = u(\sigma_H)$.

Lemma 4.6. *If g satisfies* (g_1) *and g is odd or even and does not change sign on* $(0,+\infty)$, *then any least energy solution* $\psi \in H^1(\mathbb{R}^3)$ of $(\mathcal{P})$ *is radially symmetric about a point.*

Proof. Without loss of generality, we can assume that $g \geq 0$ on $(0,+\infty)$. By Lemma 4.2, we can assume that $\psi > 0$. By Lemma 3.2, there exists an optimal path γ such that $\gamma(t_0) = \psi$ and $\gamma(t) \geq 0$ for every $t \in [0,1]$. For every half-space H define the path $\gamma^H \in [0,1] \to H^1(\mathbb{R}^3)$ by $\gamma^H(t) = (\gamma(t))^H$. By Lemma 4.3, $\gamma^H \in C([0,1], H^1(\mathbb{R}^3))$. Observe that since g is nondecreasing, $G(\psi^H) = (G(\psi))^H$, and therefore, by Lemmas 4.3 and 4.4, we have that $E(\gamma^H(t)) \leq E(\gamma(t))$ for all $t \in [0,1]$. Note that $\gamma^H \in \Gamma$, so we have

$$\max_{t\in[0,1]} E(\gamma^H(t)) \geq c^*.$$

Since for every $t \in [0,1]\setminus\{t_0\}$, there holds $E(\gamma^H(t)) \le E(\gamma(t)) < c^*$, we obtain

$$E(\gamma^H(t_0)) = E(\gamma(t_0)) = c^*.$$

By Lemmas 4.3 and 4.4, we have either $(G(\psi))^H = G(\psi)$ or $G(\psi^H) = G(\psi(\sigma_H))$ in $\mathbb{R}^3$.

If $(G(\psi))^H = G(\psi)$, then for every $x \in H$, we have that

$$\int_{\psi(\sigma_H(x))}^{\psi(x)} g(s)ds = G(\psi(x)) - G(\psi(\sigma_H(x))) \ge 0,$$

which implies that either $\psi(\sigma_H(x)) \le \psi(x)$ or $g = 0$ on $[\psi(x), \psi(\sigma_H(x))]$. In particular, $g(\psi^H) = g(\psi)$ on $\mathbb{R}^3$. Applying Lemma 4.1 to γ^H, we obtain $E'(\psi^H) = 0$, and therefore,

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla\psi^H|^2 dx\right)\Delta\psi^H + \lambda\psi^H = (|x|^{-\alpha} * G(\psi^H))g(\psi^H) = (|x|^{-\alpha} * G(\psi))g(\psi).$$

Since ψ satisfies $(\mathcal{P})$, we conclude that $\psi^H = \psi$.

If $G(\psi^H) = G(\psi(\sigma_H))$, we conclude similarly that $\psi^H = \psi(\sigma_H)$. Since this holds for arbitrary H, we conclude by Lemma 4.5 that ψ is radial and radially decreasing. This completes the proof of Lemma 4.6. □

Proof of Theorem 1.2. By Lemmas 4.2 and 4.6, we conclude the proof of Theorem 1.2. □

Acknowledgements: The authors would like to thank the referees for valuable comments and helpful suggestions. This work is partially supported by National Natural Science Foundation of China (No. 12326408).

Author contributions: The authors contributed equally to this article. All authors read and approved the final version.

Conflict of interest: The authors declare that they have no conflict of interest.

Data availability statement: Data sharing is not applicable to this article as no data sets were generated or analyzed during this study.

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