Research Article

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Nilpotent perturbations of m-isometric and m-symmetric tensor products of commuting d-tuples of operators

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Abstract: If $\mathbb{T}_1$ and $\mathbb{T}_2$ are commuting d-tuples of Hilbert space operators in $B(\mathcal{H})^d$ such that $(\mathbb{T}_1^* \otimes I + I \otimes \mathbb{T}_2^*, \mathbb{T}_1 \otimes I + I \otimes \mathbb{T}_2)$ is strictly m-isometric (resp., m-symmetric) for some positive integer m, then there exist a scalar d-tuple λ and positive integers m_i, $1 \le i \le 2$, such that $m = m_1 + 2m_2 - 2$, $(\mathbb{T}_1^* + \overline{\lambda}, \mathbb{T}_1 + \lambda)$ is m_1 isometric, and $\mathbb{T}_2 - \lambda$ is m_2-nilpotent (resp., $m = m_1 + m_2 - 1$, $(\mathbb{T}_1^* + \overline{\lambda}, \mathbb{T}_1 + \lambda)$ is m_1-symmetric and $(\mathbb{T}_2^* - \overline{\lambda}, \mathbb{T}_2 - \lambda)$ is m_2 symmetric).

Keywords: Banach space, left/right multiplication operator, m-isometric and m-symmetric commuting d-tuples of operators, tensor product of operators

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1 Introduction

Let $B(\mathcal{X})^d$ (resp., $B(\mathcal{H})^d$) denote the product of d-copies of the algebra of operators, i.e., bounded linear transformations, on an infinite dimensional complex Banach space $\mathcal{X}$ into itself (resp., on an infinite dimensional complex Hilbert space $\mathcal{H}$ into itself). A d-tuple $\mathbb{A} = (A_1, \ldots, A_d) \in B(\mathcal{X})^d$ is a commuting d-tuple if $[A_i, A_j] = A_iA_j - A_jA_i = 0$ for all $1 \le i, j \le d$. Recall from Gleeson and Richter [1] that a commuting d-tuple $\mathbb{A} \in B(\mathcal{H})^d$ is m-isometric if

$$\triangle^m_{\mathbb{A}^*,\mathbb{A}}(I) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\sum_{|\beta|=j}\frac{j!}{\beta!}\mathbb{A}^{*\beta}\mathbb{A}^{\beta} = 0,$$

where $\beta = (\beta_1, \ldots, \beta_d)$, β_i are non-negative integers for all $1 \le i \le d$, $|\beta| = \sum_{i=1}^d \beta_i$, $\beta! = \prod_{i=1}^d \beta_i!$, $\mathbb{A}^\beta = \prod_{i=1}^d A_i^{\beta_i}$, and $\mathbb{A}^{*\beta} = \prod_{i=1}^d A_i^{*\beta_i}$. Generalizing this to commuting d-tuple pairs $(\mathbb{A}, \mathbb{B})$ of Banach space operators, the pair $(\mathbb{A}, \mathbb{B})$ is said to be m-isometric if

$$\triangle^m_{\mathbb{A},\mathbb{B}}(I) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(\sum_{|\beta|=j}\frac{j!}{\beta!}\mathbb{A}^{\beta}\,\mathbb{B}^{\beta}\right) = 0$$

[1–3]. Analogously [3], the commuting d-tuple $(\mathbb{A}, \mathbb{B})$ is said to be m-symmetric if

$$\delta^m_{\mathbb{A},\mathbb{B}}(I) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(\sum_{i=1}^d A_i\right)^{m-j}\left(\sum_{i=1}^d B_i\right)^j = 0.$$

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The classes consisting of m-isometric and m-symmetric commuting d-tuples have been considered by a number of authors in the recent past (see [1–6] for further references), and it is known that these classes share a number of, though not all, properties with their single operator (i.e., 1-tuple) kin. For example, if $\mathbb{A},\mathbb{B},\mathbb{S}$ and $\mathbb{T}$ are commuting d-tuples in $B(\mathcal{X})^d$, $\mathbb{A}$ commutes with $\mathbb{S}$, $\mathbb{B}$ commutes with $\mathbb{T}$, and $\triangle^m_{\mathbb{A},\mathbb{B}}(I)=\triangle^n_{\mathbb{S},\mathbb{T}}(I)=0$ (resp., $\delta^m_{\mathbb{A},\mathbb{B}}(I)=\delta^n_{\mathbb{S},\mathbb{T}}(I)=0$) for some positive integers m and n, then $\triangle^{m+n-1}_{\mathbb{A}\mathbb{S},\mathbb{B}\mathbb{T}}(I)=0$ (resp., $\delta^{m+n-1}_{\mathbb{A}\mathbb{S},\mathbb{B}\mathbb{T}}(I)=0$); see [3, Corollary 4.2] for the general case and [7] for the 1-tuple case. The converse fails, i.e., given commuting d-tuples $\mathbb{A},\mathbb{B},\mathbb{S}$, and $\mathbb{T}$ such that $\mathbb{A}$ commutes with $\mathbb{S}$, $\mathbb{B}$ commutes with $\mathbb{T}$, and $\triangle^{m+n-1}_{\mathbb{A}\mathbb{S},\mathbb{B}\mathbb{T}}(I)=0$ (resp., $\delta^{m+n-1}_{\mathbb{A}\mathbb{S},\mathbb{B}\mathbb{T}}(I)=0$) it does not follow that $\mathbb{A},\mathbb{B}$ and $\mathbb{S},\mathbb{T}$ (or, scalar multiples thereof) satisfy $\triangle^m_{\mathbb{A},\mathbb{B}}(I)=\triangle^n_{\mathbb{S},\mathbb{T}}(I)=0$ (resp., $\delta^m_{\mathbb{A},\mathbb{B}}(I)=\delta^n_{\mathbb{S},\mathbb{T}}(I)=0$). A case where there is a positive answer is that of the tensor product pairs $(\mathbb{A}\otimes\mathbb{S},\mathbb{B}\otimes\mathbb{T})$: if $\mathbb{A},\mathbb{B},\mathbb{S},\mathbb{T}$ are commuting d-tuples such that $(\mathbb{A}\otimes\mathbb{S},\mathbb{B}\otimes\mathbb{T})$ is m-isometric (resp., m-symmetric), then there exist integers $m_i>0$ and a non-zero scalar c such that $m=m_1+m_2-1$, $(\mathbb{A},c\mathbb{B})$ is strict m_1-isometric, and $(\mathbb{S},\frac{1}{c}\mathbb{T})$ is strict m_2-isometric (resp., $(\mathbb{A},c\mathbb{B})$ is strict m_1-symmetric, and $(\mathbb{S},\frac{1}{c}\mathbb{T})$ is strict m_2-symmetric). Recall here that the pair $(\mathbb{A},\mathbb{B})$ is strict m-isometric (resp., strict m-symmetric) if $\triangle^m_{\mathbb{A},\mathbb{B}}(I)=0$ and $\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I)\neq 0$ (resp., $\delta^m_{\mathbb{A},\mathbb{B}}(I)=0$ and $\delta^{m-1}_{\mathbb{A},\mathbb{B}}(I)\neq 0$). This article considers tensor sum pairs $(\mathbb{S}\otimes I+I\otimes\mathbb{Q},\mathbb{T}\otimes I)$, $\mathbb{S},\mathbb{T}$, and $\mathbb{Q}$ commuting d-tuples in $B(\mathcal{X})^d$, and $(\mathbb{T}_1^*\otimes I+I\otimes\mathbb{T}_2^*,\mathbb{T}_1\otimes I+I\otimes\mathbb{T}_2)$, $\mathbb{T}_1$ and $\mathbb{T}_2$ commuting d-tuples in $B(\mathcal{H})^d$. We prove that if $(\mathbb{S}\otimes I+I\otimes\mathbb{Q},\mathbb{T}\otimes I)$ is strictly m-isometric, then there exist positive integers m_i and a scalar d-tuple $\lambda=(\lambda_1,\ldots,\lambda_d)$ such that $m=m_1+m_2-1$, $(\mathbb{S}+\lambda,\mathbb{T})$ is m_1-isometric and $\mathbb{Q}-\lambda$ is m_2-nilpotent; if $(\mathbb{T}_1^*\otimes I+I\otimes\mathbb{T}_2^*,\mathbb{T}_1\otimes I+I\otimes\mathbb{T}_2)$ is strict m-symmetric (resp., strict m-isometric), then there exist positive integers m_i and a scalar d-tuple $\lambda=(\lambda_1,\ldots,\lambda_d)$ such that $m=m_1+m_2-1$, $(\mathbb{T}_1^*+\overline{\lambda},\mathbb{T}_1+\lambda)$ is m_1-symmetric and $(\mathbb{T}_2^*-\overline{\lambda},\mathbb{T}_2-\lambda)$ is m_2-symmetric (resp., $m=m_1+2m_2-2$, $(\mathbb{T}_1^*+\overline{\lambda},\mathbb{T}_1+\lambda)$ is m_1-isometric and $\mathbb{T}_2-\lambda$ is m_2-nilpotent). This generalizes [8, Theorem 22] and [9, Theorem 1.3] to the case of commuting d-tuples of operators. The argument that we use to prove these results is, in spirit, similar to the argument used in proving related results in [5,7] and depends upon reducing the d-tuple problem to a 1-tuple case.

The plan of the article is as follows. We introduce all relevant terminology, notation, and complementary results in Section 2. The main results are proved in Section 3, and the final section lists the references.

2 Preliminaries

Given operators $A,B\in B(\mathcal{X})$, let L_A and $R_B\in B(B(\mathcal{X}))$ denote, respectively, the operators $L_A(X)=AX$ and $R_B(X)=XB$ of left multiplication by A and right multiplication by B. For commuting d-tuples $\mathbb{A}=(A_1,\ldots,A_d)$, $\mathbb{B}=(B_1,\ldots,B_d)$ in $B(\mathcal{X})^d$ and a scalar d-tuple $\alpha=(\alpha_1,\ldots,\alpha_d)$ such that α_i is a non-negative integer for all $1\le i\le d$, let $\mathbb{L}^\alpha_{\mathbb{A}},\mathbb{R}^\alpha_{\mathbb{B}}$, the convolution $(\mathbb{L}_{\mathbb{A}}*\mathbb{R}_{\mathbb{B}})^j(X)$ and the multiplication $(\mathbb{L}_{\mathbb{A}}\times\mathbb{R}_{\mathbb{B}})(X)$, $X\in B(\mathcal{X})$, be defined by

$$\mathbb{L}^\alpha_{\mathbb{A}}=\prod_{i=1}^d L^{\alpha_i}_{A_i},\quad \mathbb{R}^\alpha_{\mathbb{B}}=\prod_{i=1}^d R^{\alpha_i}_{B_i},$$

$$(\mathbb{L}_{\mathbb{A}}*\mathbb{R}_{\mathbb{B}})^j(X)=\left(\sum_{|\alpha|=j}\frac{j!}{\alpha!}\mathbb{L}^\alpha_{\mathbb{A}}\mathbb{R}^\alpha_{\mathbb{B}}\right)(X)=\left(\sum_{i=1}^d L_{A_i}R_{B_i}\right)^j(X)$$

$$(\text{all integers } j\ge 0,\ \alpha!=\alpha_1!\ldots\alpha_d!),\ |\alpha|=\sum_{i=1}^d\alpha_i \text{ and}$$

$$(\mathbb{L}_{\mathbb{A}}\times\mathbb{R}_{\mathbb{B}})(X)=\left(\sum_{i=1}^d L_{A_i}\right)\left(\sum_{i=1}^d R_{B_i}\right)(X).$$

Define the operator $\lfloor\sum_{i=1}^d A_iXB_i\rfloor^n$ by

$$\left\lfloor\sum_{i=1}^d A_iXB_i\right\rfloor^n=\sum_{i=1}^d A_i\left(\left\lfloor\sum_{i=1}^d A_iXB_i\right\rfloor^{n-1}\right)B_i\quad\text{for all positive integers } n.$$

(Thus, $\lfloor A\rfloor^n = A\lfloor A\rfloor^{n-1}I = I\lfloor A\rfloor^{n-1}A = \ldots = A^n, \lfloor AB\rfloor^n = A\lfloor AB\rfloor^{n-1}B = \ldots = A^nB^n$, and $\lfloor\sum_{i=1}^d A_iB_i\rfloor^n = \sum_{i=1}^d A_i\lfloor\sum_{i=1}^d A_iB_i\rfloor^{n-1} B_i$.) The commuting d-tuples $\mathbb{A}$ and $\mathbb{B}$ commute, $[\mathbb{A},\mathbb{B}] = 0$, if

$$[A_i, B_j] = A_iB_j - B_jA_i = 0 \quad \text{for all } 1 \le i, j \le d.$$

It is clear that

$$[\mathbb{L}_{\mathbb{A}}, \mathbb{R}_{\mathbb{B}}] = 0$$

and if $[\mathbb{A},\mathbb{B}] = 0$, then

$$[\mathbb{L}_{\mathbb{A}}, \mathbb{L}_{\mathbb{B}}] = [\mathbb{R}_{\mathbb{A}}, \mathbb{R}_{\mathbb{B}}] = 0.$$

A pair $(\mathbb{A},\mathbb{B})$ of commuting d-tuples of $B(\mathcal{X})^d$ operators is m-isometric if

$$\begin{aligned}
0 = \triangle^m_{\mathbb{A},\mathbb{B}}(I) &= \sum_{j=0}^m (-1)^j \binom{m}{j}\left(\sum_{|\beta|=j}\frac{j!}{\beta!}\mathbb{A}^\beta\,\mathbb{B}^\beta\right)\\
&= \sum_{j=0}^m (-1)^j \binom{m}{j}\left\lfloor\sum_{i=1}^d A_iB_i\right\rfloor^j\\
&= \sum_{j=0}^m (-1)^j \binom{m}{j}\left(\sum_{i=1}^d L_{A_i}R_{B_i}\right)^j(I)\\
&= \sum_{j=0}^m (-1)^j \binom{m}{j}(\mathbb{L}_{\mathbb{A}} * \mathbb{R}_{\mathbb{B}})^j(I)\\
&= (I - \mathbb{L}_{\mathbb{A}} * \mathbb{R}_{\mathbb{B}})^m(I)
\end{aligned}$$

and the pair $(\mathbb{A},\mathbb{B})$ is m-symmetric if

$$\begin{aligned}
0 = \delta^m_{\mathbb{A},\mathbb{B}}(I) &= \sum_{j=0}^m (-1)^j \binom{m}{j}\left(\sum_{i=1}^d A_i\right)^{m-j}\left(\sum_{i=1}^d B_i\right)^j\\
&= \left(\sum_{j=0}^m (-1)^j \binom{m}{j}\left(\sum_{i=1}^d L_{A_i}\right)^{m-j}\left(\sum_{i=1}^d R_{B_i}\right)^j\right)(I)\\
&= \left(\sum_{j=0}^m (-1)^j \binom{m}{j}\mathbb{L}_{\mathbb{A}}^{m-j} \times \mathbb{R}_{\mathbb{B}}^j\right)(I)\\
&= (\mathbb{L}_{\mathbb{A}} - \mathbb{R}_{\mathbb{B}})^m(I).
\end{aligned}$$

Commuting tuples of m-isometric, similarly n-symmetric, operators are known to share a large number of properties with their single operator counterparts: thus, just as for the 1-tuple case, $\triangle^m_{\mathbb{A},\mathbb{B}}(I) = 0$ implies $\triangle^t_{\mathbb{A},\mathbb{B}}(I) = 0$, similarly $\delta^m_{\mathbb{A},\mathbb{B}}(I) = 0$ implies $\delta^t_{\mathbb{A},\mathbb{B}}(I) = 0$, for integers $t \ge m$. However, there are instances where a property holds for the single operator version but fails for the d-tuple version: for example, whereas

$$\triangle^m_{A,B}(I) = 0 \Leftrightarrow \triangle^m_{A^{-1},B^{-1}}(I) = 0 \quad \text{for all invertible } A \text{ and } B$$

(similarly, for n-symmetric (A, B)), these properties fail for commuting d-tuples (see [3]).

If $(\mathbb{A},\mathbb{B}) \in (X, m)$-isometric (i.e., $\triangle^m_{\mathbb{A},\mathbb{B}}(X) = (I - \mathbb{L}_{\mathbb{A}} * \mathbb{R}_{\mathbb{B}})^m(X) = 0$, $X \in B(\mathcal{X})$), then

$$\begin{aligned}
\triangle^m_{\mathbb{A},\mathbb{B}}(X) = 0 &\Leftrightarrow (I - \mathbb{L}_{\mathbb{A}} * R_{\mathbb{B}})(\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(X)) = 0\\
&\Leftrightarrow (\mathbb{L}_{\mathbb{A}} * \mathbb{R}_{\mathbb{B}})\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(X) = \triangle^{m-1}_{\mathbb{A},\mathbb{B}}(X)\\
&\Rightarrow \cdots\\
&\Rightarrow (\mathbb{L}_{\mathbb{A}} * \mathbb{R}_{\mathbb{B}})^t\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(X) = \triangle^{m-1}_{\mathbb{A},\mathbb{B}}(X),
\end{aligned}$$

and if $(\mathbb{A},\mathbb{B}) \in (X, n)$-symmetric (i.e., $\delta^n_{\mathbb{A},\mathbb{B}}(X) = (\mathbb{L}_{\mathbb{A}} - \mathbb{R}_{\mathbb{B}})^n(X) = 0$, $X \in B(\mathcal{X})$), then

$$\begin{aligned}
\delta^n_{\mathbb{A},\mathbb{B}}(X) = 0 &\Leftrightarrow (\mathbb{L}_{\mathbb{A}} - \mathbb{R}_{\mathbb{B}})(\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X)) = 0\\
&\Leftrightarrow \mathbb{L}_{\mathbb{A}}(\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X)) = \mathbb{R}_{\mathbb{B}}(\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X))
\end{aligned}$$

$$\Rightarrow \cdots$$
$$\Rightarrow \mathbb{L}^t_{\mathbb{A}}(\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X)) = \mathbb{R}^t_{\mathbb{B}}(\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X))$$

for all integers $t \geq 0$, where

$$\begin{aligned}\mathbb{L}_{\mathbb{A}}(\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X)) &= \mathbb{L}_{\mathbb{A}}\left(\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\mathbb{L}_{\mathbb{A}}^{n-1-j}\times\mathbb{R}_{\mathbb{B}}^j\right)(X)\\ &= \left(\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\mathbb{L}_{\mathbb{A}}^{n-j}\times\mathbb{R}_{\mathbb{B}}^j\right)(X)\end{aligned}$$

and

$$\mathbb{R}_{\mathbb{B}}(\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X)) = \left(\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\mathbb{L}_{\mathbb{A}}^{n-1-j}\times\mathbb{R}_{\mathbb{B}}^{j+1}\right)(X).$$

Let $\mathcal{X}\,\overline{\otimes}\,\mathcal{X}$ denote the completion, endowed with a reasonable cross norm, of the algebraic tensor product of $\mathcal{X}$ with $\mathcal{X}$ has overline. Let $S \otimes T$ denote the tensor product of $S, \mathbb{T} \in B(\mathcal{X})$. The tensor product of the d-tuples $\mathbb{A} = (A_1, \ldots, A_d)$ and $\mathbb{B} = (B_1, \ldots, B_d)$ is the d^2-tuple

$$\mathbb{A} \otimes \mathbb{B} = (A_1 \otimes B_1, \ldots, A_1 \otimes B_d, \ldots, A_d \otimes B_1, \ldots, A_d \otimes B_d).$$

(Thus, $\mathbb{A} \otimes I = (A_1 \otimes I, \ldots, \mathbb{A}_d \otimes I)$.) A commuting d-tuple $\mathbb{Q} = (Q_1, \ldots, Q_d)$ is n-nilpotent for some positive integer n if $\mathbb{Q}^n = \prod_{i=1}^d Q_i^{\alpha_i} = 0$ for all d-tuples $\alpha = (\alpha_i, \ldots, \alpha_d)$ of non-negative integers α_i such that $|\alpha| = \sum_{i=1}^d \alpha_i = n$ and $\mathbb{Q}^\alpha$ is non-zero for at least one α with $|\alpha| \leq n-1$. Clearly, $\mathbb{Q}$ is n-nilpotent if and only if the tensor product operators $\mathbb{Q} \otimes I$ and $I \otimes \mathbb{Q}$ are n-nilpotent. Furthermore, $(\mathbb{A}, \mathbb{B})$ is m-isometric (resp., m-symmetric) if and only if $(\mathbb{A} \otimes I, \mathbb{B} \otimes I)$ and $(I \otimes \mathbb{A}, I \otimes \mathbb{B})$ are m-isometric (resp., m-symmetric).

Given a commuting d-tuple $\mathbb{A} = (A_1, \ldots, A_d)$ and a scalar $\lambda = (\lambda_1, \ldots, \lambda_d)$, we abbreviate $(\mathbb{A} - \lambda I) = (A_1 - \lambda_1 I, \ldots, A_d - \lambda_d I)$ to $\mathbb{A} - \lambda$.

3 Results

We start with a technical lemma. Recall that a pair $(\mathbb{A}, \mathbb{B})$ of commuting d-tuples in $B(\mathcal{X})^d$ is strict m-isometric if $\triangle^m_{\mathbb{A},\mathbb{B}}(I) = 0$ and $\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I) \neq 0$. Since $\triangle^m_{\mathbb{A},\mathbb{B}}(I) = 0$ implies $\triangle^t_{\mathbb{A},\mathbb{B}}(I) = 0$ for integers $t \geq m$, $(\mathbb{A}, \mathbb{B})$ strict m-isometric implies the linear independence of the sequence $\{\triangle^r_{\mathbb{A},\mathbb{B}}(I)\}_{r=0}^{m-1}$. A similar statement holds for strict n-symmetric operators, i.e., if $\delta^n_{\mathbb{A},\mathbb{B}}(I) = 0$ and $\delta^{n-1}_{\mathbb{A},\mathbb{B}}(I) \neq 0$, then the sequence $\{\delta^r_{\mathbb{A},\mathbb{B}}(I)\}_{r=0}^{n-1}$ is linearly independent.

Lemma 3.1. *Given commuting d-tuples $\mathbb{A}, \mathbb{B} \in B(\mathcal{X})^d$, if $(\mathbb{A}, \mathbb{B})$ is strict m-isometriic, then the sequence*

$$\{\triangle^r_{\mathbb{A},\mathbb{B}}(I)\mathbb{B}^\beta\}_{r=0}^{m-1}, \quad |\beta| = r,$$

is linearly independent.

Proof. The proof is by contradiction. Suppose that there exist scalars a_j, not all zero, such that

$$M = \sum_{j=0}^{m-1} a_j \triangle^j_{\mathbb{A},\mathbb{B}}(I)\mathbb{B}^\beta = 0, \quad |\beta| = j.$$

The hypothesis $(\mathbb{A}, \mathbb{B})$ is m-isometric implies

$$\begin{aligned}(\mathbb{L}_{\mathbb{A}} * \mathbb{R}_{\mathbb{B}})\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I) = \triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I) &\Rightarrow (\mathbb{L}_{\mathbb{A}} * \mathbb{R}_{\mathbb{B}})^t \triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I) = \triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I)\\ &\Leftrightarrow \sum_{|\beta|=t}\frac{t!}{\beta!}\mathbb{A}^\beta\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I)\mathbb{B}^\beta = \triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I)\end{aligned}$$

for all positive integers t. We have

$$\begin{aligned}\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(M) = \sum_{j=0}^{m-1} a_j \triangle^{m-1+j}_{\mathbb{A},\mathbb{B}}(I)\mathbb{B}^{\beta} = 0 &\Rightarrow a_0\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I) = 0\\
&\Rightarrow a_0 = 0,\\
\triangle^{m-2}_{\mathbb{A},\mathbb{B}}(M) = \sum_{j=1}^{m-1} a_j \triangle^{m-2+j}_{\mathbb{A},\mathbb{B}}(I)\mathbb{B}^{\beta} = 0 &\Rightarrow a_1\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I)\mathbb{B}^{\beta} = 0, \quad |\beta| = 1\\
&\Rightarrow a_1 \sum_{|\beta|=1} \mathbb{A}^{\beta}\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I)\mathbb{B}^{\beta} = 0\\
&\Leftrightarrow a_1\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I) = 0\\
&\Leftrightarrow a_1 = 0.\end{aligned}$$

Repeating the argument, we eventually have

$$\begin{aligned}&a_0 = a_1 = \ldots = a_{m-2} = 0, \quad M = a_{m-1}\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I)\mathbb{B}^{\beta}, \quad |\beta| = m-1\\
\Rightarrow\ & a_{m-1} \sum_{|\beta|=m-1} \frac{(m-1)!}{\beta!}\mathbb{A}^{\beta}\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I)\mathbb{B}^{\beta} = 0\\
\Leftrightarrow\ & a_{m-1}\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I) = 0 \Leftrightarrow a_{m-1} = 0.\end{aligned}$$

This being a contradiction, the lemma is proved. □

If, for commuting d-tuples $\mathbb{S}$, $\mathbb{T}$, and $\mathbb{Q}$, the pair $(\mathbb{S}, \mathbb{T})$ is m_1-isometric and $\mathbb{Q}$ is m_2-nilpotent, then $(\mathbb{S} \otimes I, \mathbb{T} \otimes I)$ is m_1-isometric, $I \otimes \mathbb{Q}$ is m_2-nilpotent, and $(\mathbb{S} \otimes I + I \otimes \mathbb{Q}, \mathbb{T} \otimes I)$ is $(m_1 + m_2 - 1)$-isometric [3]. The converse has a solution, as the following theorem, proved for the single linear operators case in [8] (see also [9]), proves.

Theorem 3.2. *If $\mathbb{S}, \mathbb{T}$, and $\mathbb{Q}$ are commuting d-tuples in $B(\mathcal{X})^d$ such that $(\mathbb{S} \otimes I + I \otimes \mathbb{Q}, \mathbb{T} \otimes I)$ is strict m-isometric for some positive integer m, then there exist positive integers m_i, $1 \le i \le 2$, and a scalar d-tuple $\lambda = (\lambda_1, \ldots, \lambda_d)$ such that $m = m_1 + m_2 - 1$, $(\mathbb{S} + \lambda, \mathbb{T})$ is m_1-isometric and $\mathbb{Q} - \lambda$ is m_2-nilpotent.*

Proof. The m-isometric property of $(\mathbb{S} \otimes I + I \otimes \mathbb{Q}, \mathbb{T} \otimes I)$ implies

$$\begin{aligned}\triangle^{m}_{\mathbb{S}\otimes I+I\otimes\mathbb{Q},\mathbb{T}\otimes I}(I \otimes I) &= \{(I - \mathbb{L}_{\mathbb{S}\otimes I}{}^{*}\mathbb{R}_{\mathbb{T}\otimes I}) - \mathbb{L}_{I\otimes\mathbb{Q}}{}^{*}\mathbb{R}_{\mathbb{T}\otimes I}\}^{m}(I \otimes I)\\
&= \left\{\sum_{j=0}^{m}(-1)^j\binom{m}{j}\triangle^{m-j}_{\mathbb{S}\otimes I,\mathbb{T}\otimes I}(\mathbb{L}_{I\otimes\mathbb{Q}}{}^{*}\mathbb{R}_{\mathbb{T}\otimes I})^j\right\}(I \otimes I).\end{aligned}$$

Since

$$\begin{aligned}\triangle^{m-j}_{\mathbb{S}\otimes I,\mathbb{T}\otimes I}(I \otimes I) &= \left\{\sum_{k=0}^{m-j}(-1)^k\binom{m-j}{k}(\mathbb{L}_{\mathbb{S}\otimes I}{}^{*}\mathbb{R}_{\mathbb{T}\otimes I})^k\right\}(I \otimes I)\\
&= \left\{\sum_{k=0}^{m-j}(-1)^k\binom{m-j}{k}\left(\sum_{i=1}^{d} L_{S_i\otimes I}R_{T_i\otimes I}\right)^k\right\}(I \otimes I)\\
&= \left\{\sum_{k=0}^{m-j}(-1)^k\binom{m-j}{k}\left(\sum_{i=1}^{d} L_{S_i}R_{T_i}\right)^k\right\}(I) \otimes I\\
&= \triangle^{m-j}_{\mathbb{S},\mathbb{T}}(I) \otimes I\end{aligned}$$

and

$$\begin{aligned}(\mathbb{L}_{I\otimes\mathbb{Q}} * \mathbb{R}_{\mathbb{T}\otimes I})^j(I\otimes I) &= \left(\sum_{i=1}^{d} L_{I\otimes Q_i}R_{T_i\otimes I}\right)^j (I\otimes I)\\ &= \left[\sum_{i=1}^{d}(I\otimes Q_i)(T_i\otimes I)\right]^j\\ &= \left[\sum_{i=1}^{d} T_i\otimes Q_i\right]^j\\ &= \sum_{|\beta|=j}\frac{j!}{\beta!}\mathbb{T}^\beta\otimes\mathbb{Q}^\beta,\end{aligned}$$

where

$$\mathbb{T}^\beta = \Pi_{i=1}^d T_i^{\beta_i} \quad \text{and} \quad \mathbb{Q}^\beta = \Pi_{i=1}^d Q_i^{\beta_i}.$$

Since $\sigma_a(\mathbb{Q})$ is non-empty, there exists a $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_a(\mathbb{Q})$. Let $\{y_p\} \subset \mathcal{X}$ be a sequence of unit vectors such that

$$\lim_{p\to\infty}(Q_i - \lambda_i)y_p = 0 \quad \text{all} \quad 1 \le i \le d.$$

Let $x \in \mathcal{X}$ be a unit vector, and let $x^*, y_p^* \in \mathcal{X}^*$ be such that

$$x^*(x) = \langle x, x^*\rangle = 1 = \langle y_p, y_p^*\rangle = y_p^*(y_p).$$

Then

$$\lim_{p\to\infty}\langle\mathbb{Q}^\beta y_p, y_p^*\rangle = \lambda^\beta$$

and

$$\begin{aligned}0 &= \triangle^m_{\mathbb{S}\otimes I+I\otimes\mathbb{Q},\mathbb{T}\otimes I}(I\otimes I)\\ &= \sum_{j=0}^m(-1)^j\binom{m}{j}\sum_{|\beta|=j}\frac{j!}{\beta!}(\triangle^{m-j}_{\mathbb{S},\mathbb{T}}(\mathbb{T}^\beta)\otimes\mathbb{Q}^\beta)\end{aligned}$$

imply

$$\begin{aligned}0 &= \lim_{p\to\infty}\triangle^m_{\mathbb{S}\otimes I+I\otimes\mathbb{Q},\mathbb{T}\otimes I}(I\otimes I)\langle x\otimes y_p, x^*\otimes y_p^*\rangle\\ &= \lim_{p\to\infty}\left(\sum_{j=0}^m(-1)^j\binom{m}{j}\sum_{|\beta|=j}\frac{j!}{\beta!}(\triangle^{m-j}_{\mathbb{S},\mathbb{T}}(\mathbb{T}^\beta)\otimes\mathbb{Q}^\beta)\right)\langle x\otimes y_p, x^*\otimes y_p^*\rangle.\end{aligned}$$

Consequently

$$\begin{aligned}&\left\langle\sum_{j=0}^m(-1)^j\binom{m}{j}\sum_{|\beta|=j}\frac{j!}{\beta!}\triangle^{m-j}_{\mathbb{S},\mathbb{T}}(\mathbb{T}^\beta)x, x^*\right\rangle\lim_{p\to\infty}\langle\mathbb{Q}^\beta y_p, y_p^*\rangle = 0\\ \Rightarrow\ &\left\langle\sum_{j=0}^m(-1)^j\binom{m}{j}\sum_{|\beta|=j}\frac{j!}{\beta!}\triangle^{m-j}_{\mathbb{S},\mathbb{T}}(\mathbb{T}^\beta)x, x^*\right\rangle\lambda^\beta = 0\\ \Rightarrow\ &\left\langle\sum_{j=0}^m(-1)^j\binom{m}{j}\sum_{|\beta|=j}\frac{j!}{\beta!}\triangle^{m-j}_{\mathbb{S},\mathbb{T}}(\lambda^\beta\mathbb{T}^\beta)x, x^*\right\rangle = 0\\ \Rightarrow\ &\sum_{j=0}^m(-1)^j\binom{m}{j}\triangle^{m-j}_{\mathbb{S},\mathbb{T}}(\lambda\mathbb{T})^j = 0\\ \Rightarrow\ &\triangle^m_{\mathbb{S}+\lambda,\mathbb{T}}(I) = 0.\end{aligned}$$

Thus, $(\mathbb{S}+\lambda,\mathbb{T})$ is m-isometric, and hence, there exists a positive integer $m_1 \le m$ such that $(\mathbb{S}+\lambda,\mathbb{T})$ is strict m_1-isometric. Let $m_2 = m - m_1 + 1$. Then,

$$\begin{aligned}
0 &= \triangle^m_{\mathbb{S}\otimes I+I\otimes\mathbb{Q},\mathbb{T}\otimes I}(I\otimes I)\\
&= \triangle^m_{(\mathbb{S}+\lambda)\otimes I+I\otimes(\mathbb{Q}-\lambda),\mathbb{T}\otimes I}(I\otimes I)\\
&= \sum_{j=0}^m(-1)^j\binom{m}{j}(\mathbb{L}_{I\otimes(\mathbb{Q}-\lambda)} * \mathbb{R}_{\mathbb{T}\otimes I})^j(\triangle^{m-j}_{(\mathbb{S}+\lambda)\otimes I,\mathbb{T}\otimes I}(I\otimes I))\\
&\quad(\text{since } [\mathbb{L}_{I\otimes(\mathbb{Q}-\lambda)},\mathbb{L}_{(\mathbb{S}+\lambda)\otimes I}]=0)\\
&= \sum_{j=0}^m(-1)^j\binom{m}{j}(\mathbb{L}_{I\otimes(\mathbb{Q}-\lambda)} * \mathbb{R}_{\mathbb{T}\otimes I})^j(\triangle^{m-j}_{(\mathbb{S}+\lambda),\mathbb{T}}(I)\otimes I)\\
&\quad(\text{since } \triangle^{m-j}_{(\mathbb{S}+\lambda)\otimes I,\mathbb{T}\otimes I}(I\otimes I) = \triangle^{m-j}_{(\mathbb{S}+\lambda),\mathbb{T}}(I)\otimes I)\\
&= \sum_{j=0}^m(-1)^j\binom{m}{j}\left(\sum_{i=1}^d L_{I\otimes(Q_i-\lambda_i)}R_{T_i\otimes I}\right)^j(\triangle^{m-j}_{(\mathbb{S}-\lambda),\mathbb{T}}(I)\otimes I)\\
&= \sum_{j=0}^m(-1)^j\binom{m}{j}\sum_{|\beta|=j}\frac{j!}{\beta!}(I\otimes(\mathbb{Q}-\lambda)^\beta)(\triangle^{m-j}_{(\mathbb{S}-\lambda),\mathbb{T}}(I)\otimes I)(\mathbb{T}^\beta\otimes I)\\
&= \sum_{j=0}^m(-1)^j\binom{m}{j}\sum_{|\beta|=j}\frac{j!}{\beta!}(\triangle^{m-j}_{(\mathbb{S}-\lambda),\mathbb{T}}(I)\mathbb{T}^\beta\otimes(\mathbb{Q}-\lambda)^\beta)\\
&= \sum_{j=m_2}^m(-1)^j\binom{m}{j}\sum_{|\beta|=j}\frac{j!}{\beta!}(\triangle^{m-j}_{(\mathbb{S}-\lambda),\mathbb{T}}(I)\mathbb{T}^\beta\otimes(\mathbb{Q}-\lambda)^\beta),
\end{aligned}$$

since $\triangle^t_{\mathbb{S}+\lambda,\mathbb{T}}(I)=0$ for all $t\ge m_1$. This, in view of the fact that the sequence $\{\triangle^{m-j}_{\mathbb{S}+\lambda,\mathbb{T}}(I)\mathbb{T}^\beta\}_{j=m_2}^m$ is linearly independent, implies $(\mathbb{Q}-\lambda)^{m_2}=0$. Evidently, $(\mathbb{Q}-\lambda)^{m_2-1}\ne 0$ (for the reason that if it were so, then $((\mathbb{S}+\lambda)\otimes I+I\otimes(\mathbb{Q}-\lambda),\mathbb{T}\otimes I)$, hence $(\mathbb{S}\otimes I+I\times\mathbb{Q},\mathbb{T}\otimes I)$ would be (m_1+m_2-2)-isometric). This completes the proof. □

A similar argument works for m-symmetric operator pairs $(\mathbb{S}\otimes I+I\otimes\mathbb{Q},\mathbb{T}\otimes I)$. Thus, if

$$\begin{aligned}
\delta^m_{\mathbb{S}\otimes I+I\otimes\mathbb{Q},\mathbb{T}\otimes I}(I\otimes I) &= (\delta_{\mathbb{S}\otimes I,\mathbb{T}\otimes I}+\mathbb{L}_{I\otimes\mathbb{Q}})^m(I\otimes I)\\
&= \sum_{j=0}^m\binom{m}{j}\delta^{m-j}_{\mathbb{S}\otimes I,\mathbb{T}\otimes I}\mathbb{L}^j_{I\otimes\mathbb{Q}}(I\otimes I)\\
&= 0
\end{aligned}$$

and if $\lambda\in\sigma_a(\mathbb{Q})$, then

$$\begin{aligned}
\delta^m_{\mathbb{S}\otimes I+I\otimes\mathbb{Q},\mathbb{T}\otimes I}(I\otimes I)=0 &\Leftrightarrow \sum_{j=0}^m\binom{m}{j}\delta^{m-j}_{\mathbb{S}\otimes I,\mathbb{T}\otimes I}\mathbb{L}^j_{I\otimes\lambda}(I\otimes I)=0\\
&\Leftrightarrow \sum_{j=0}^m\binom{m}{j}\delta^{m-j}_{\mathbb{S}\otimes I,\mathbb{T}\otimes I}\mathbb{L}^j_{\lambda\otimes I}(I\otimes I)=0\\
&\Rightarrow \delta^m_{(\mathbb{S}+\lambda)\otimes I,\mathbb{T}\otimes I}(I\otimes I)=0\\
&\Leftrightarrow \delta^m_{\mathbb{S}+\lambda,\mathbb{T}}(I)=0.
\end{aligned}$$

Assuming $(\mathbb{S}+\lambda,\mathbb{T})$ to be strict m_1-symmetric, $m_1\le m$, and setting $m_1+m_2-1=m$, it then follows that

$$\begin{aligned}
0 &= \delta^m_{\mathbb{S}\otimes I+I\otimes\mathbb{Q},\mathbb{T}\otimes I}(I\otimes I)\\
&= \delta^m_{(\mathbb{S}+\lambda)\otimes I+I\otimes(\mathbb{Q}-\lambda),\mathbb{T}\otimes I}(I\otimes I)\\
&= \sum_{j=0}^m\binom{m}{j}\delta^{m-j}_{(\mathbb{S}+\lambda)\otimes I,\mathbb{T}\otimes I}\mathbb{L}^j_{I\otimes(\mathbb{Q}-\lambda)}(I\otimes I)
\end{aligned}$$

$$= \sum_{j=m_2}^{m} \binom{m}{j} \mathbb{L}^{j}_{I\otimes(\mathbb{Q}-\lambda)}(\delta^{m-j}_{(\mathbb{S}+\lambda)\otimes I,\mathbb{T}\otimes I}(I\otimes I))$$

(since $\delta^{t}_{(\mathbb{S}+\lambda)\otimes I,\mathbb{T}\otimes I}(I\otimes I) = 0$ for all $t > m_1$, $[\delta_{(\mathbb{S}+\lambda)\otimes I,\mathbb{T}\otimes I}, \mathbb{L}_{I\otimes(\mathbb{Q}-\lambda)}] = 0$)

$$= \sum_{j=m_2}^{m} \binom{m}{j} \left(I \otimes \sum_{i=1}^{d} L_{Q_i-\lambda_i}\right)^{j} (\delta^{m-j}_{\mathbb{S}+\lambda,\mathbb{T}}(I)\otimes I)$$

$$= \sum_{j=m_2}^{m} \binom{m}{j} (\delta^{m-j}_{\mathbb{S}+\lambda,\mathbb{T}}(I)\otimes \mathbb{L}^{j}_{\mathbb{Q}-\lambda}(I)).$$

The sequence $\{\delta^{m-j}_{\mathbb{S}+\lambda,\mathbb{T}}(I)\}_{m_2}^{m}$ being linearly independent,

$$\mathbb{L}^{m_2}_{\mathbb{Q}-\lambda}(I) = 0 \Leftrightarrow \mathbb{Q}-\lambda \text{ is } m_2\text{-nilpotent}.$$

A more general result, at least for Hilbert space d-tuples, is possible, as the following analogue of a result of Paul and Gu [9] (see also Gu [8]) shows.

Theorem 3.3. *If $\mathbb{T}_i = (T_{i1}, \ldots, T_{id})$, $1 \le i \le 2$, are commuting d-tuples in $B(\mathcal{H})^d$ such that $(\mathbb{T}_1^* \otimes I + I \otimes \mathbb{T}_2^*, \mathbb{T}_1 \otimes I + I \otimes \mathbb{T}_2)$ is strict m-symmetric for some positive integer m, then there exist positive integers m_i and a scalar $\beta = (\beta_1, \ldots, \beta_d)$ such that $m = m_1 + m_2 - 1$, $(\mathbb{T}_1^* + \overline{\beta}, \mathbb{T}_1 + \beta)$ is m_1-symmetric and $(\mathbb{T}_2^* - \overline{\beta}, \mathbb{T}_2 - \beta)$ is m_2-symmetric.*

Proof. For convenience, let

$$\delta_1 = \delta_{\mathbb{T}_1^*\otimes I,\mathbb{T}_1\otimes I}, \quad \delta_2 = \delta_{I\otimes\mathbb{T}_2^*,I\otimes\mathbb{T}_2} \quad \text{and} \quad \delta_{12} = \delta_{\mathbb{T}_1^*\otimes I+I\otimes\mathbb{T}_2^*,\mathbb{T}_1\otimes I+I\otimes\mathbb{T}_2}.$$

Then

$$\begin{aligned}\delta_{12} &= \mathbb{L}_{T_1^*\otimes I+I\otimes\mathbb{T}_2^*} - \mathbb{R}_{\mathbb{T}_1\otimes I+i\otimes\mathbb{T}_2}\\ &= (\mathbb{L}_{\mathbb{T}_1^*\otimes I} - \mathbb{R}_{\mathbb{T}_1\otimes I}) + (\mathbb{L}_{I\otimes\mathbb{T}_2^*} - \mathbb{R}_{I\otimes\mathbb{T}_2})\\ &= \delta_1 + \delta_2,\end{aligned}$$

where $[\delta_1, \delta_2] = 0$ and

$$\begin{aligned}\delta_{12}^{m}(I\otimes I) &= \left(\sum_{k=0}^{m}\binom{m}{k}\delta_1^{m-k}\delta_2^{k}\right)(I\otimes I)\\
&= \sum_{k=0}^{m}\binom{m}{k}\left[\sum_{j=0}^{m-k}(-1)^j\binom{m-k}{j}(\mathbb{L}^{m-k-j}_{\mathbb{T}_1^*\otimes I}\times\mathbb{R}^{j}_{\mathbb{T}_1\otimes I})\right.\\
&\quad\left.\times\sum_{t=0}^{k}(-1)^t\binom{k}{t}(\mathbb{L}^{k-t}_{I\otimes\mathbb{T}_2^*}\times\mathbb{R}^{t}_{I\otimes\mathbb{T}_2})\right](I\otimes I)\\
&= \sum_{k=0}^{m}\binom{m}{k}\left[\sum_{j=0}^{m-k}(-1)^j\binom{m-k}{j}\left(\sum_{i=1}^{d}T_{1i}^*\otimes I\right)^{m-k-j}\right.\\
&\quad\left.\times\left(\sum_{i=1}^{d}T_{1i}\otimes I\right)^{j}\sum_{t=0}^{k}(-1)^t\binom{k}{t}\left(\sum_{i=1}^{d}I\otimes T_{2i}^*\right)^{k-t}\left(\sum_{i=1}^{d}I\otimes T_{2i}\right)^{t}\right]\\
&= \sum_{k=0}^{m}\binom{m}{k}\left[\sum_{j=0}^{m-k}(-1)^j\binom{m-k}{j}\sum_{t=0}^{k}(-1)^t\binom{k}{t}\right.\\
&\quad\left.\times\left(\left(\sum_{i=1}^{d}T_{1i}^*\right)^{m-k-j}\otimes\left(\sum_{i=1}^{d}T_{2i}^*\right)^{k-t}\right)\left(\left(\sum_{i=1}^{d}T_{1i}\right)^{j}\otimes\left(\sum_{i=1}^{d}T_{2i}\right)^{t}\right)\right].\end{aligned}$$

In the following, we start by proving the existence of a complex scalar $\beta = (\beta_1, \ldots, \beta_d)$, $\overline{\beta} = -\beta$, such that

$$\delta_{12} = \delta_{1(\overline{\beta})} + \delta_{2(\beta)}, \quad \delta_{1(\overline{\beta})} = \delta_{(\mathbb{T}_1^*+\beta)\otimes I, (\mathbb{T}_1+\overline{\beta})\otimes I}, \quad \delta_{2(\beta)} = \delta_{I\otimes(\mathbb{T}_2^*+\overline{\beta}), I\otimes(\mathbb{T}_2+\beta)}.$$

The proof is then completed by proving the existence of an integer $m_2 \le m$ such that $(I\otimes(\mathbb{T}_2^*+\overline{\beta}), I\otimes(\mathbb{T}_2+\beta))$ is strictly m_2-symmetric and $((\mathbb{T}_1^*+\beta)\otimes I, (\mathbb{T}_1+\overline{\beta})\otimes I)$ is $m_1(=m-m_2+1)$ symmetric.

For this, let $\mathbb{T}_i = (T_{i1}, \ldots, T_{id})$, $1 \le i \le 2$, $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_a(\mathbb{T}_1)$ and $\mu = (\mu_1, \ldots, \mu_d) \in \sigma_a(\mathbb{T}_2)$. Then, there exist sequences $\{x_p\}$ and $\{y_p\}$ of unit vectors in $\mathcal{H}$ such that

$$\lim_{p\to\infty}(T_{1i}-\lambda_i)x_p = 0 = \lim_{p\to\infty}(T_{2i}-\mu_i)y_p, \quad 1 \le i \le d,$$

and

$$\begin{aligned}
&\lim_{p\to\infty}\left(\left[\left(\sum_{i=1}^d T_{1i}{}^*\right)^{m-k-j} \otimes \left(\sum_{i=1}^d (T_{2i}^*)^{k-t}\right)\right]\left[\left(\sum_{i=1}^d T_{1i}\right)^{j} \otimes \left(\sum_{i=1}^d T_{2i}\right)^{t}\right]\langle x_p\otimes y_p, x_p\otimes y_p\rangle\right)\\
&= \lim_{p\to\infty}\left(\left\langle\left(\sum_{i=1}^d T_{1i}\right)^{j} x_p, \left(\sum_{i=1}^d T_{1i}\right)^{m-k-j} x_p\right\rangle\left\langle\left(\sum_{i=1}^d T_{2i}\right)^{t} y_p, \left(\sum_{i=1}^d T_{2i}\right)^{k-t} y_p\right\rangle\right)\\
&= \left(\sum_{i=1}^d \lambda_i\right)^{j}\left(\sum_{i=1}^d \overline{\lambda_i}\right)^{m-k-j}\left(\sum_{i=1}^d \mu_i\right)^{t}\left(\sum_{i=1}^d \overline{\mu_i}\right)^{k-t}.
\end{aligned}$$

Since

$$\sum_{j=0}^{m-k}(-1)^j\binom{m-k}{j}\left(\sum_{i=1}^d\lambda_i\right)^j\left(\sum_{i=1}^d\overline{\lambda_i}\right)^{m-k-j} = \left(\sum_{i=1}^d(\overline{\lambda_i}-\lambda_i)\right)^{m-k},$$

similarly,

$$\sum_{t=0}^{k}(-1)^t\binom{k}{t}\left(\sum_{i=1}^d\mu_i\right)^t\left(\sum_{i=1}^d\overline{\mu_i}\right)^{k-t} = \left(\sum_{i=1}^d(\overline{\mu_i}-\mu_i)\right)^{k},$$

$$\begin{aligned}
\delta_{12}^m(I\otimes I)\langle x_p\otimes y_p, x_p\otimes y_p\rangle &= \sum_{k=0}^m\binom{m}{k}\left(\sum_{i=1}^d(\overline{\lambda_i}-\lambda_i)\right)^{m-k}\left(\sum_{i=1}^d(\overline{\mu_i}-\mu_i)\right)^{k}\\
&= \left(\sum_{i=1}^d((\overline{\lambda_i}-\lambda_i)+(\overline{\mu_i}-\mu_i))\right)^m\\
&= 0.
\end{aligned}$$

Let

$$\Re\lambda_i = \alpha_i, \quad i(\Im\lambda_i) = \beta_i, \quad \alpha = (\alpha_1, \ldots, \alpha_d), \quad \beta = (\beta_1, \ldots, \beta_d) \quad \text{and} \quad [\beta] = \sum_{i=1}^d \beta_i.$$

Then,

$$\lambda = \alpha + \beta = (\alpha_1+\beta_1, \ldots, \alpha_d+\beta_d) \quad \text{and} \quad [\beta] = \frac{1}{2}\sum_{i=1}^d(\overline{\mu_i}-\mu_i).$$

For a given complex d-tuple $\tau = (\tau_1, \ldots, \tau_d)$, define $\delta_{i(\tau)}$, $1 \le i \le 2$, by

$$\delta_{1(\tau)} = \delta_{(\mathbb{T}_1^*+\overline{\tau})\otimes I, (\mathbb{T}_1+\tau)\otimes I} = \delta_1 + (\mathbb{L}_{\overline{\tau}\otimes I} - \mathbb{R}_{\tau\otimes I})$$

and

$$\delta_{2(\tau)} = \delta_{I\otimes(\mathbb{T}_2^*+\overline{\tau}), I\otimes(\mathbb{T}_2+\tau)} = \delta_2 + (\mathbb{L}_{I\otimes\overline{\tau}} - \mathbb{R}_{I\otimes\tau}).$$

Then, since $\overline{\beta} = -\beta$,

$$\begin{aligned}\delta_{12} = \delta_1 + \delta_2 &= \delta_{1(\overline{\beta})} + \delta_{2(\beta)} + ((-\mathbb{L}_{\beta\otimes I} + \mathbb{R}_{\overline{\beta}\otimes I}) + (-\mathbb{L}_{I\otimes\overline{\beta}} + \mathbb{R}_{I\otimes\beta}))\\ &= \delta_{1(\overline{\beta})} + \delta_{2(\beta)} + \{(\mathbb{L}_{I\otimes\beta} - \mathbb{L}_{\beta\otimes I}) + (\mathbb{R}_{I\otimes\beta} - \mathbb{R}_{\beta\otimes I})\}\\ &= \delta + (\mathbb{L} + \mathbb{R}),\end{aligned}$$

where we have set

$$\delta_1 + \delta_2 = \delta, \quad \mathbb{L}_{I\otimes\beta} - \mathbb{L}_{\beta\otimes I} = \mathbb{L} \quad \text{and} \quad \mathbb{R}_{I\otimes\beta} - \mathbb{R}_{\beta\otimes I} = \mathbb{R}.$$

Evidently, $[\mathbb{L}, \mathbb{R}] = 0$ (= $[\mathbb{L}_{I\otimes\beta}, \mathbb{L}_{\beta\otimes I}] = [\mathbb{R}_{I\otimes\beta}, \mathbb{R}_{\beta\otimes I}]$) and, for all positive integers t,

$$\delta_{12}^t(I\otimes I) = \left(\sum_{s=0}^{t}(-1)^s\binom{t}{s}\delta^{t-s}(\mathbb{L} + \mathbb{R})^s\right)(I\otimes I).$$

Since

$$\begin{aligned}(\mathbb{L} + \mathbb{R})^s(I\otimes I) &= \left(\sum_{k=0}^{s}\binom{s}{k}\mathbb{L}^{s-k}\mathbb{R}^k\right)(I\otimes I)\\ &= \sum_{k=0}^{s}(-1)^k\binom{s}{k}\mathbb{L}^{s-k}\left[\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\sum_{i=1}^{d}I\otimes\beta_i\right)^{k-j}\left(\sum_{i=1}^{d}\beta_i\otimes I\right)^j\right]\\ &= \sum_{k=0}^{s}(-1)^k\binom{s}{k}\mathbb{L}^{s-k}\left[\sum_{j=0}^{k}(-1)^j\binom{k}{j}[\beta]^k(I\otimes I)\right],\end{aligned}$$

$(\mathbb{L} + \mathbb{R})^s(I\otimes I) = 0$ for all $k > 0$ and if $k = 0$, then it equals $\mathbb{L}^s(I\otimes I)$. A similar argument shows that $\mathbb{L}^s(I\otimes I) = 0$ for all $s > 0$; hence,

$$\delta_{12}^t(I\otimes I) = (\delta_1 + \delta_2)^t(I\otimes I) = (\delta_{1(\overline{\beta})} + \delta_{2(\beta)})^t(I\otimes I)$$

for all non-negative integers t. Again, considering

$$\delta_{1(\lambda)} = \delta_1 + (\mathbb{L}_{\overline{\lambda}\otimes I}) - \mathbb{R}_{\lambda\otimes I} = \delta_{1(\overline{\beta})} - (\mathbb{L}_{\alpha\otimes I} - \mathbb{R}_{\alpha\otimes I}),$$

$$\begin{aligned}\delta_{1(\lambda)}^t(I\otimes I) &= \{\delta_{1(\overline{\beta})} + (\mathbb{L}_{\alpha\otimes I} - \mathbb{R}_{\alpha\otimes I})\}^t(I\otimes I)\\ &= \sum_{k=0}^{t}\binom{t}{k}\delta_{1(\overline{\beta})}^{t-k}\{(\mathbb{L}_{\alpha\otimes I} - \mathbb{R}_{\alpha\otimes I})\}^t(I\otimes I)\\ &= \sum_{k=0}^{t}\binom{t}{k}\delta_{1(\overline{\beta})}^{t-k}\left(\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\sum_{i=1}^{d}\alpha_i\otimes I\right)^{k-j}\left(\sum_{i=1}^{d}\alpha_i\otimes I\right)^j\right)\\ &= \sum_{k=0}^{t}\binom{t}{k}\delta_{1(\overline{\beta})}^{t-k}\left(\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\sum_{i=1}^{d}\alpha_i\otimes I\right)^k\right).\end{aligned}$$

Hence, $\delta_{1(\lambda)}^t(I\otimes I) = 0$ for all $k > 0$, and if $k = 0$, then $\delta_{1(\lambda)}^t(I\otimes I) = \delta_{1(\overline{\beta})}^t(I\otimes I)$.

Let y be a unit vector in $\mathcal{H}$ and let $\{x_p\} \subset \mathcal{H}$ be a sequence of unit vectors such that

$$\lim_{p\to\infty}(T_{1i} - \lambda_i)x_p = 0; \quad 1 \le i \le d.$$

Then,

$$\begin{aligned}\delta_{1(\overline{\beta})}^{m-k}(I\otimes I) &= \delta_{1(\overline{\lambda})}^{m-k}(I\otimes I)\\ &= \sum_{j=0}^{m-k}(-1)^j\binom{m-k}{j}\left(\sum_{i=1}^{d}(T_{1i}^* - \overline{\lambda}_i)\otimes I\right)^{m-k-j}\left(\sum_{i=1}^{d}(T_{1i} - \lambda_i)\otimes I\right)^j\\ &\Rightarrow \lim_{p\to\infty}\delta_{1(\overline{\beta})}^{m-k}(I\otimes I)\langle x_p\otimes y, x_p\otimes y\rangle\end{aligned}$$

$$= \sum_{j=0}^{m-k}(-1)^j\binom{m-k}{j}\lim_{p\to\infty}\left\langle\left(\sum_{i=1}^{d}(T_{1i}-\lambda_i)\right)^j x_p, \left(\sum_{i=1}^{d}(T_{1i}-\lambda_i)\right)^{m-k-j} x_p\right\rangle\langle y,y\rangle = 0$$

for all $m-k>0$, and if $m-k=0$, then

$$0 = \delta_{12}^m(I\otimes I) = \left(\sum_{k=0}^{m}\binom{m}{k}\delta_{1(\overline{\beta})}^{m-k}\delta_{2(\beta)}^{k}\right)(I\otimes I) = \sum_{k=0}^{m}\binom{m}{k}\delta_{2(\beta)}^{k}(\delta_{1(\overline{\beta})}^{m-k})(I\otimes I), \quad \text{since } [\delta_{1(\overline{\beta})},\delta_{2(\beta)}] = 0 = \delta_{2(\beta)}^m(I\otimes I).$$

Let $m_2 \le m$ be the least positive integer such that $\delta_{2(\beta)}^{m_2}(I\otimes I) = 0$ and $\delta_{2(\beta)}^{m_2-1}(I\otimes I) \ne 0$. Since

$$\begin{aligned}\delta_{2(\beta)}^{t}(I\otimes I) &= \sum_{j=0}^{t}(-1)^j\binom{t}{j}\left(\sum_{i=1}^{d}I\otimes(T_{2i}^*-\overline{\beta}_i)\right)^{t-j}\left(\sum_{i=1}^{d}I\otimes(T_{2i}+\beta_i)\right)^j\\ &= \sum_{j=0}^{t}(-1)^j\binom{t}{j}\left(I\otimes\sum_{i=1}^{d}(T_{2i}^*-\overline{\beta}_i)^{t-j}\right)\left(I\otimes\sum_{i=1}^{d}(T_{2i}-\beta_i)^j\right)\\ &= I\otimes\delta^t_{\mathbb{T}_2^*+\overline{\beta},\mathbb{T}_2+\beta}(I)\end{aligned}$$

for all positive integers t, the assumption $(\mathbb{T}_2^*+\overline{\beta},\mathbb{T}_2+\beta)$ is strict m_2-symmetric and implies that the sequence $\{\delta^r_{\mathbb{T}_2^*+\overline{\beta},\mathbb{T}_2+\beta}(I)\}_{r=0}^{m_2-1}$ is linearly independent.

To complete the proof, we observe that

$$\begin{aligned}0 &= \delta_{12}^m(I\otimes I)\\ &= \sum_{k=0}^{m}\binom{m}{k}\delta_{1(\overline{\beta})}^{m-k}\delta_{2(\beta)}^{k}(I\otimes I)\\ &= \sum_{k=0}^{m}\binom{m}{k}\delta_{1(\overline{\beta})}^{m-k}(I\otimes\delta_{\mathbb{T}_2^*+\overline{\beta},\mathbb{T}_2+\beta}(I))\\ &= \sum_{k=0}^{m}\binom{m}{k}(\delta^{m-k}_{\mathbb{T}_1^*+\beta,\mathbb{T}_1+\overline{\beta}}(I)\otimes I)(I\otimes\delta^k_{\mathbb{T}_2^*+\overline{\beta},\mathbb{T}_2+\beta}(I))\\ &= \sum_{k=0}^{m}\binom{m}{k}(\delta^{m-k}_{\mathbb{T}_1^*+\beta,\mathbb{T}_1+\overline{\beta}}(I)\otimes\delta^k_{\mathbb{T}_2^*+\overline{\beta},\mathbb{T}_2+\beta}(I)).\end{aligned}$$

Since $\delta^t_{\mathbb{T}_2^*+\overline{\beta},\mathbb{T}_2+\beta}(I)=0$ for all $t\ge m_2$, the linear independence of the sequence $\{\delta^r_{\mathbb{T}_2^*+\overline{\beta},\mathbb{T}_2+\beta}(I)\}_{r=0}^{m_2-1}$ implies that $\delta^t_{\mathbb{T}_1^*+\beta,\mathbb{T}_1+\overline{\beta}}(I)=0$ for all $t\ge m_1 = m-m_2+1$. In particular, $(\mathbb{T}_1^*+\beta,\mathbb{T}_1+\overline{\beta})$ is m_1-symmetric. □

Theorem 3.3 begs the question of whether an anlogous result holds for m-isometric sums $(\mathbb{T}_1^*\otimes I + I\otimes\mathbb{T}_2^*, \mathbb{T}_1\otimes I + I\otimes\mathbb{T}_2)$. The following theorem is a result in this direction.

Theorem 3.4. *If $\mathbb{T}_i$, $1\le i\le 2$, are commuting d-tuples in $B(\mathcal{H})^d$ such that $(\mathbb{T}_1^*\otimes I + I\otimes\mathbb{T}_2^*, \mathbb{T}_1\otimes I + I\otimes\mathbb{T}_2)$ is strict m-isometric for some positive integer m, then there exist a scalar $\lambda = (\lambda_1,\dots,\lambda_d)$ and positive integers $m_i\le m$ such that $\mathbb{T}_2-\mu$ is m_2-nilpotent and $(\mathbb{T}_1^*+\overline{\lambda},\mathbb{T}_1-\lambda)$ is m_1-isometric.*

Proof. The proof below depends in an essential way upon Theorem 3.2. The spectrum of $\mathbb{T}_2$ being non-empty, there exists a scalar $\lambda = (\lambda_1,\dots,\lambda_d)\in\sigma_a(\mathbb{T}_2)$. Let $\{x_p\}$ be a sequence of unit vectors in $\mathcal{H}$ such that $\lim_{p\to\infty}(T_{2i}-\lambda_i)x_p = 0$; $1\le i\le d$. For such a scalar λ,

$$\begin{aligned}\triangle^m_{T_1^*\otimes I+I\otimes\mathbb{T}_2^*,\mathbb{T}_1\otimes I+I\otimes\mathbb{T}_2}(I\otimes I) &= \triangle^m_{T_1^*\otimes I+I\otimes\mathbb{T}_2^*,(\mathbb{T}_1+\lambda)\otimes I+I\otimes(\mathbb{T}_2-\lambda)}(I\otimes I)\\ &= ((I-\mathbb{L}_{\mathbb{T}_1^*\otimes I+I\otimes\mathbb{T}_2^*}*\mathbb{R}_{I\otimes(\mathbb{T}_2-\lambda)}) + (\mathbb{L}_{\mathbb{T}_1^*\otimes I}*\mathbb{R}_{I\otimes(\mathbb{T}_2-\lambda)}+\mathbb{L}_{I\otimes\mathbb{T}_2^*}*\mathbb{R}_{I\otimes(\mathbb{T}_2-\lambda)}))^m(I\otimes I)\\ &= \left(\sum_{j=0}^{m}\binom{m}{j}\nabla^{m-j}(M_1+M_2)^j\right)(I\otimes I),\end{aligned}$$

where we have set

$$(I - \mathbb{L}_{\mathbb{T}_1^*\otimes I+I\otimes\mathbb{T}_2^*} * \mathbb{R}_{I\otimes(\mathbb{T}_2-\lambda)})^{m-j} = \triangle^{m-j}_{\mathbb{T}_1^*\otimes I+I\otimes\mathbb{T}_2^*, I\otimes(\mathbb{T}_2-\lambda)} = \nabla^{m-j},$$
$$\mathbb{L}_{\mathbb{T}_1^*\otimes I} * \mathbb{R}_{I\otimes(\mathbb{T}_2-\lambda)} = M_1, \quad \mathbb{L}_{I\otimes\mathbb{T}_2^*} * \mathbb{R}_{I\otimes(\mathbb{T}_2-\lambda)} = M_2.$$

Let $y \in \mathcal{H}$ be a unit vector and let $\{x_p\} \subset \mathcal{H}$ be a sequence of unit vectors such that

$$\lim_{p\to\infty}(T_{2i} - \lambda_i)x_p = 0.$$

Then,

$$\begin{aligned}
&(M_1 + M_2)^j(I \otimes I)\\
&= \sum_{k=0}^{j}\binom{j}{k}(\mathbb{L}_{\mathbb{T}_1^*\otimes I} * \mathbb{R}_{I\otimes(\mathbb{T}_2-\lambda)})^{j-k}(\mathbb{L}_{I\otimes\mathbb{T}_2^*} * \mathbb{R}_{I\otimes(\mathbb{T}_2^*-\lambda)})^k(I\otimes I)\\
&= \sum_{k=0}^{j}\binom{j}{k}(\mathbb{L}_{\mathbb{T}_1^*\otimes I} * \mathbb{R}_{I\otimes(\mathbb{T}_2-\lambda)})^{j-k}\left(\sum_{|\beta|=k}\frac{k!}{\beta!}\mathbb{T}_2^{*\beta}(\mathbb{T}_2-\lambda)^\beta\right)
\end{aligned}$$

implies

$$\begin{aligned}
&\lim_{p\to\infty}(M_1 + M_2)^j\langle y\otimes x_p, y\otimes x_p\rangle\\
&= \sum_{k=0}^{j}\binom{j}{k}(\mathbb{L}_{\mathbb{T}_1^*\otimes I} * \mathbb{R}_{I\otimes(\mathbb{T}_2-\lambda)})^{j-k}\left(\langle y, y\rangle\sum_{|\beta|=k}\frac{k!}{\beta!}\lim_{p\to\infty}\langle(\mathbb{T}_2-\lambda)^\beta x_p, \mathbb{T}_2^\beta x_p\rangle\right).
\end{aligned}$$

Hence, $\lim_{p\to\infty}(M_1 + M_2)^j\langle y\otimes x_p, y\otimes x_p\rangle = 0$ for all $k > 0$, and if $k = 0$, then

$$\begin{aligned}
\lim_{p\to\infty}(M_1 + M_2)^j\langle y\otimes x_p, y\otimes x_p\rangle &= \lim_{p\to\infty}(\mathbb{L}_{\mathbb{T}_2^*\otimes I} * \mathbb{R}_{I\otimes(\mathbb{T}_2-\lambda)})^j\langle y\otimes x_p, y\otimes x_p\rangle\\
&= \sum_{|\beta|=j}\frac{j!}{\beta!}\langle y, \mathbb{T}_2 y\rangle\lim_{p\to\infty}\langle(\mathbb{T}_2-\lambda)^\beta x_p, x_p\rangle = 0
\end{aligned}$$

for all $j > 0$ (and if $j = 0$ then it equals $I\otimes I$). Conclusion:

$$\begin{aligned}
\triangle^m_{T_1^*\otimes I+I\otimes\mathbb{T}_2^*, \mathbb{T}_1\otimes I+I\otimes\mathbb{T}_2}(I\otimes I) &= \nabla^m(I\otimes I)\\
&= \triangle^m_{T_1^*\otimes I+I\otimes\mathbb{T}_2^*, (\mathbb{T}_1+\lambda)\otimes I}(I\otimes I)\\
&= 0.
\end{aligned}$$

Let $n \le m$ be the least positive integer such that $\triangle^n_{T_1^*\otimes I+I\otimes\mathbb{T}_2^*, (\mathbb{T}_1+\lambda)\otimes I}(I\otimes I) = 0$. Then, an application of Theorem 3.2 proves the existence of a scalar $\overline{\mu} = (\overline{\mu_1}, \ldots, \overline{\mu_d}) \in \sigma_a(\mathbb{T}_2^*)$ and positve integers m_i, $1 \le i \le 2$, such that $n = m_1 + m_2 - 1$, $\mathbb{T}_2^* - \overline{\mu}$ (hence, $T_2 - \mu$) is m_2-nilpotent and $(\mathbb{T}_1^* + \overline{\mu}, \mathbb{T}_1 + \lambda)$ is m_1-isometric. The conclusion that $\mathbb{T}_2 - \mu$ is m_2-nilpotent implies $\sigma(\mathbb{T}_2 - \mu) = \sigma_a(\mathbb{T}_2 - \mu) = \sigma_a(\mathbb{T}_2) - \{\mu\} = \{0\}$; hence, $\lambda \in \sigma_a(\mathbb{T}_2)$ if and only if $\lambda = \mu$. Consequently, $(\mathbb{T}_1^* + \overline{\lambda}, \mathbb{T}_1 + \lambda)$ is strict m_1-isometric, $\mathbb{T}_2 - \lambda$ is m_2-nilpotent, and $(\mathbb{T}_1^* + \overline{\lambda})\otimes I + I\otimes(\mathbb{T}_2^* - \overline{\lambda}), (\mathbb{T}_1 + \lambda)\otimes I + I\otimes(\mathbb{T}_2 - \lambda)$, equivalently $(\mathbb{T}_1^*\otimes I + I\otimes\mathbb{T}_2^*, \mathbb{T}_1\otimes I + I\otimes\mathbb{T}_2)$, is $m_1 + 2m_2 - 2 = n + m_2 - 1 = m$ isometric. □

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