Research Article

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Geometric invariants properties of osculating curves under conformal transformation in Euclidean space $\mathbb{R}^3$

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Abstract: An osculating curve is a type of curve in space that holds significance in the study of differential geometry. In this article, we investigate certain geometric invariants of osculating curves on smooth and regularly immersed surfaces under conformal transformations in Euclidean space $\mathbb{R}^3$. The primary objective of this article is to explore conditions sufficient for the conformal invariance of the osculating curve under both conformal transformations and isometries. We also compute the tangential and normal components of the osculating curves, demonstrating that they remain invariant under the isometry of the surfaces in $\mathbb{R}^3$.

Keywords: osculating curves, conformal transformation, homothetic transformation, isometry of surfaces, normal and tangential components

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1 Introduction

Differential geometry is the branch of mathematics that studies the geometry of smooth curves and surfaces. The concepts of regular and smooth curves are fundamental in the field of differential geometry. The smoothness of a curve is a property measured by the number of continuous derivatives. A curve is said to be smooth if it is differentiable everywhere and is regular if it has non-zero derivative at every point of the domain. For more information regarding regular and smooth curves, readers can refer to [1–5].

Throughout this article, the terms transformation, motion, and map have the same meanings. There are numerous methods that exist for categorizing motions, but we study only those that preserve certain geometric properties. Transformations can be categorized into three equivalence classes based on the invariant nature of Gaussian curvatures (K) and mean curvatures (H), namely, conformal, isometric, and non-conformal or general motion [6,7]. Isometry refers to a type of movement that preserves both the length and the angle between curves on surfaces. In the area of differential geometry, under isometry, the Gaussian curvature (K) remains invariant, while the mean curvature (H) undergoes alteration. A movement that preserves the angle between two directed curves is termed conformal motion.

In conformal motion, length may not be preserved. In the field of cartography, conformal transformations play an important role. One well-known instance of a conformal transformation is stereographic projection, a process that maps a sphere onto a plane. In 1569, Gerardus Mercator was the first to use conformal maps for

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the creation of the first angle-preserving map, which he named "Mercator's world map." In 2018, Gunn and Bobenko released an animated film on conformal mapping as part of the Springer Video MATH series [8]. For general motion, neither length nor angles are conserved between any intersecting pairs of curves on the surface.

The study of normal, rectifying, and osculating curves is a familiar topic discussed in classical texts on the differential geometry of curves and surfaces [1]. Within the Euclidean three-dimensional space $\mathbb{R}^3$, Chen [2] conducted a study on the dynamics of rectifying curves, exploring properties associated with such curves. Chen [2] also examined the conditions for osculating and rectifying curves on smooth surfaces to maintain invariance under the surface isometry.

Chen [2] came across the following query concerning rectifying curves. *What occurs when the position vector of a space curve consistently resides in its rectifying plane.* It is also observed that the component of a space curve's position vector along the normal to the surface remains invariant with respect to the isometry of surfaces. Chen and Dillen [5] investigated rectifying curves serving as centrodes and extremal curves. Subsequently, Chen et al. [3] made specific findings regarding rectifying curves by dilating unit speed curves on the unit sphere S^2 in Euclidean space $\mathbb{R}^3$. They also derived the necessary and sufficient conditions for the centrode of a unit speed curve in $\mathbb{R}^3$ to be a rectifying curve.

Shaikh and Ghosh [9] investigated specific geometric characteristics of rectifying curves on smooth surfaces that maintain their invariance under surface isometry. Additionally, they examined the invariant properties of osculating curves under surface isometry [6]. Thereafter, Lone [7] studied some invariant properties of normal curves in smooth surfaces.

Conformal transformations hold significant importance in the area of differential geometry. In the past, Shaikh et al. [10] provided some interesting results regarding rectifying curves under conformal transformations. They also demonstrated that, for a rectifying curve on smooth surfaces, the normal and geodesic curvatures cannot remain conformally and homothetically invariant under conformal and homothetic transformations, respectively. For further details, one can refer to [10–13].

This article extends the findings presented in [6,7], where, in addition to various results on normal, rectifying, and osculating curves, several other properties of these curves were explored. The primary objective of this article is to build upon the work of Lone [7], who investigated the geometric invariants of normal curves under conformal transformations in $\mathbb{R}^3$. Also, in [7], the author investigated the invariant properties of normal curves undergoing conformal transformations. Furthermore, the study examined the normal and tangential components of these curves within the same motion.

Our motivation is derived from the comprehensive study conducted by Shaikh and Chen, in which they investigated the geometric properties of rectifying curves on smooth surfaces that remain invariant under surface isometry. Moreover, they explored the invariant properties of osculating curves under surface isometry. Recently, Sharma and Singh [15] studied the characterizations of rectifying curves and proved that Christoffel symbols for rectifying curves remain invariant under surface isometry. Furthermore, Lone studied the invariant properties of normal curves under conformal and isometric transformation and obtained a condition for the conformal image of normal curves. However, a natural question arises: *What will happen to the geometric properties of osculating curves on smooth and regular surfaces concerning conformal and homothetic transformations in the Euclidean space* $\mathbb{R}^3$?

In this article, we try to investigate the conditions for the conformal image of an osculating curve on regular and smooth surfaces under conformal, homothetic, and isometric transformations in the Euclidean space $\mathbb{R}^3$. By using these conditions, we obtained the expression for the normal and tangential components of an osculating curve under these transformations. Furthermore, we investigate that these components are conformally invariant under conformal transformations and invariant under isometric transformations.

2 Preliminaries

This section presents information on orthonormal frames, rectifying, normal, and osculating curves. It includes discussions on their first fundamental form, geodesic and normal curvatures, as well as the introduction of some fundamental definitions.

Consider two smooth and regular surfaces Q and $\tilde{Q}$ immersed in the Euclidean space $\mathbb{R}^3$, and let $\mathcal{G} : Q \to \tilde{Q}$ be a smooth map. The dilation function, proportionate to the area elements of Q and $\tilde{Q}$, is represented as $\zeta(x, y)$. For detailed information about dilation functions, [4,5,14] can be consulted. The conformal transformation, as defined in the following manner [4], represents a generalized class of specific motions:

- If the dilation function $\zeta(x, y) = c$, where c is a constant with $c \neq 0, 1$, then $\mathcal{G}$ constitutes a homothetic transformation.
- When the function $\zeta(x, y) = 1$, $\mathcal{G}$ transforms into an isometry.

Let E, F, and G denote the coefficients of the first fundamental form of the surface Q, and $\tilde{E}, \tilde{F}$, and $\tilde{G}$ represent the coefficients of the first fundamental form of $\tilde{Q}$. For surfaces Q and $\tilde{Q}$ to be isometric, it is necessary and sufficient that the first fundamental form's coefficients remain invariant [7], i.e., $\tilde{E} = E$, $\tilde{F} = F$, and $\tilde{G} = G$.

Suppose $\delta : I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed smooth parametrized curve with arc length (r) and having at least fourth-order continuous derivative. The tangent, normal, and binormal vectors of the curve δ are denoted by $\vec{t}, \vec{n}$, and $\vec{b}$, respectively. At each point on the curve $\delta(r)$, these vectors $\vec{t}, \vec{n}$, and $\vec{b}$ are mutually perpendicular, forming an orthonormal frame denoted as $\{\vec{t}, \vec{n}, \vec{b}\}$.

Suppose $\vec{t}'(r) \neq 0$, and consider the unit normal vector $\vec{n}$ along the tangent at a point on the curve δ. We can express $\vec{t}'(r) = \kappa(r)\vec{n}(r)$, where $\vec{t}'(r)$ denotes the derivative of $\vec{t}$ with respect to the arc parameter r, and $\kappa(r)$ is the curvature of $\delta(r)$. Furthermore, the binormal vector field $\vec{b}$ is determined as $\vec{b} = \vec{t} \times \vec{n}$, and we can express $\vec{b}'(r) = \tau(r)\vec{n}(r)$, with $\tau(r)$ representing another curvature function known as the torsion of the curve $\delta(r)$. In [1,2,6,9], the Serret-Frenet equations are provided as

$$\begin{aligned} \vec{t}'(r) &= \kappa(r)\vec{n}(r), \\ \vec{n}'(r) &= -\kappa(r)\vec{t}(r) + \tau(r)\vec{b}(r), \\ \vec{b}'(r) &= -\tau(r)\vec{n}(r), \end{aligned}$$

where κ and τ represent the curvature and torsion function of the curve δ, respectively, and satisfy the following:

$$\vec{t}(r) = \delta'(r), \quad \vec{n}(r) = \frac{\vec{t}'(r)}{\kappa(r)}, \text{ and } \vec{b}(r) = \vec{t}(r) \times \vec{n}(r).$$

At any arbitrary point $\delta(r)$ on the curve δ, the plane formed by $\{\vec{t}, \vec{n}\}$ is termed the osculating plane, while the plane constituted by $\{\vec{t}, \vec{b}\}$ is referred to as the rectifying plane. Similarly, the plane defined by $\{\vec{n}, \vec{b}\}$ is known as the normal plane. This terminology is particularly relevant when discussing the position vector of the curve, which defines various types of curves [11]:

- A curve is identified as a normal curve if its position vector resides in the normal plane.
- A curve is identified as a rectifying curve if its position vector resides in the rectifying plane.
- If the position vector of a curve lies in the osculating plane, the curve is classified as an osculating curve.

First, we will try to investigate the following: *Which properties of an osculating curve on regular and smooth surfaces remain invariant under different transformations*?

If a curve's position vector resides in the osculating plane, the curve is referred to as an osculating curve [11], i.e.,

$$\delta(r) = \mu_1(r)\vec{t}(r) + \mu_2(r)\vec{n}(r), \tag{1}$$

where μ_1 and μ_2 are the differentiable functions.

Consider the coordinate chart map $\sigma : U \to Q$ on a regular and smooth surface Q, along with the unit speed parametrized curve $\delta(r) : I \to Q$, where $I = (a, b) \subset \mathbb{R}$ and $U \subset \mathbb{R}^2$.

The curve $\delta(r)$ is thus defined by

$$\delta(r) = \sigma(x(r), y(r)). \tag{2}$$

By applying the chain rule to differentiate (2) with respect to r, we obtain

$$\delta'(r) = \sigma_x x' + \sigma_y y'. \tag{3}$$

Now, $\vec{t}(r) = \delta'(r)$. Substituting this into equation (3), we obtain

$$\vec{t}(r) = \sigma_x x' + \sigma_y y'. \tag{4}$$

Again, differentiating equation (4) with respect to r, we obtain

$$\vec{t}'(r) = x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}.$$

Suppose that $\mathbb{N}$ is the normal to the surface Q and $\kappa(r)$ is the curvature of the curve $\delta(r)$. Then, the normal vector $\vec{n}(r)$ is

$$\vec{n}(r) = \frac{1}{\kappa(r)}(x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}). \tag{5}$$

Now, the binormal vector $\vec{b}(r)$ is

$$\vec{b}(r) = \vec{t}(r) \times \vec{n}(r).$$

By putting the value of $\vec{t}(r)$ and $\vec{n}(r)$ from (4) and (5), we obtain

$$\begin{aligned}\vec{b}(r) &= \frac{1}{\kappa(r)}[(\sigma_x x' + \sigma_y y') \times (x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})],\\ &= \frac{1}{\kappa(r)}[(y''x' - y'x'')N + x'^3\sigma_x \times \sigma_{xx} + 2x'^2y'\sigma_x \times \sigma_{xy} + x'y'^2\sigma_x \times \sigma_{yy} + x'^2y'\sigma_y \times \sigma_{xx}\\ &\quad + 2x'y'^2\sigma_y \times \sigma_{xy} + y'^3\sigma_y \times \sigma_{yy}].\end{aligned} \tag{6}$$

Definition 1. Consider two regular and smooth surfaces, Q and $\tilde{Q}$, in $\mathbb{R}^3$. A diffeomorphism $\mathcal{G} : Q \to \tilde{Q}$ is termed an isometry if it maps curves of the same length from Q to $\tilde{Q}$.

Definition 2. [6] If Q and $\tilde{Q}$ are two regular and smooth surfaces in the Euclidean space $\mathbb{R}^3$, and let $\delta(r)$ be a curve parametrized by arc length lying on the surface Q. Then, $\delta'(r)$ is perpendicular to the unit surface normal $\mathbb{N}$, and also, $\delta'(r)$ and $\delta''(r)$ are perpendicular. Thus, δ'' can be represented as the linear combination of $\mathbb{N}$ and $\mathbb{N} \times \delta'$, i.e.,

$$\delta'' = \kappa_n\mathbb{N} + \kappa_g\mathbb{N} \times \delta',$$

where the parameters κ_n and κ_g represent the normal and geodesic curvatures of the curve δ and are defined by

$$\begin{aligned}\kappa_n &= \delta'' \cdot \mathbb{N},\\ \kappa_g &= \delta'' \cdot (\mathbb{N} \times \delta').\end{aligned}$$

Now, according to the Serret-Frenet equations, we obtain

$$\vec{t}'(r) = \delta''(r) = \kappa(r)\vec{n}(r).$$

Now,

$$\begin{aligned}\kappa_n &= \delta'' \cdot \mathbb{N},\\ &= \kappa(r)\vec{n}(r)\cdot \mathbb{N},\\ &= (x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})\cdot \mathbb{N},\\ &= (x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})\cdot(\sigma_x \times \sigma_y).\end{aligned}$$

By solving the aforementioned expression and applying the properties of vector algebra, we obtain

$$\kappa_n = x'^2X + 2x'y'Y + y'^2Z,$$

where X, Y, and Z denote the magnitudes of the second fundamental form [10].

From this, the conclusion has been obtained that the curve $\delta(r)$ is asymptotic on the surface Q if and only if its normal curvature $\kappa_n = 0$ [12].

3 Osculating curve under a conformal map

Consider Q and $\tilde{Q}$ as two regular and smooth immersed surfaces in the Euclidean space $\mathbb{R}^3$, and let $\delta(r)$ be an osculating curve that is located on the surface Q. Thus, the expression for $\delta(r)$ can be formulated as:

$$\delta(r) = \mu_1(r)\vec{t}(r) + \mu_2(r)\vec{n}(r),$$

for some smooth function $\mu_1(r)$ and $\mu_2(r)$.

Now, employing equations (4) and (5), we obtain

$$\delta(r) = \mu_1(r)(\sigma_x x' + \sigma_y y') + \mu_2(r)\frac{1}{\kappa(r)}(x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}). \tag{7}$$

We will consider the derivative map $\mathcal{G}_*(\delta(r))$ of the curve $\delta(r)$, which arises from the multiple of the 3×3 matrix $\mathcal{G}_*$ and the 3×1 matrix $\delta(r)$ [10].

Theorem 1. *Let Q and $\tilde{Q}$ be two regular and smooth immersed surfaces in the Euclidean space $\mathbb{R}^3$ and $\mathcal{G} : Q \to \tilde{Q}$ be a conformal map. Suppose $\delta(r)$ is an osculating curve that lies on the surface Q. Then, its conformal image $\tilde{\delta}(r)$ is also an osculating curve that lies on the surface $\tilde{Q}$, if the following condition holds*

$$\tilde{\delta}(r) = \frac{\mu_2(r)}{\kappa(r)}\{x'^2(\zeta\mathcal{G}_*)_x\sigma_x + 2x'y'(\zeta\mathcal{G}_*)_x\sigma_y + y'^2(\zeta\mathcal{G}_*)_y\sigma_y\} + \zeta\mathcal{G}_*(\delta(r)), \tag{8}$$

where ζ is a dilation function and $\kappa(r)$ is the curvature of the curve $\delta(r)$.

Proof. Suppose that $\tilde{Q}$ is the image of Q. Consider $\sigma(x, y)$ and $\tilde{\sigma}(x, y)$ to be the coordinate chart map of the surface Q and $\tilde{Q}$ respectively, such that $\tilde{\sigma}(x, y) = \mathcal{G} \circ \sigma(x, y)$.

Since $\mathcal{G} : Q \to \tilde{Q}$ is a conformal map, the differential map $d\mathcal{G} = \mathcal{G}_*$ of $\mathcal{G}$ takes vectors of tangent space $T_{\mathcal{G}}(Q)$ to dilated tangent vectors of $T_{\mathcal{G}(q)}(\tilde{Q})$ with a dilation factor ζ, where "q" is a point on Q.

$$\tilde{\sigma}_x(x, y) = \zeta(x, y)\mathcal{G}_*(\sigma(x, y))\sigma_x, \tag{9}$$

$$\tilde{\sigma}_y(x, y) = \zeta(x, y)\mathcal{G}_*(\sigma(x, y))\sigma_y. \tag{10}$$

As we now partially differentiate equations (9) and (10), with respect to "x" and "y" respectively, we obtain

$$\begin{aligned}\tilde{\sigma}_{xx} &= \zeta_x \mathcal{G}_* \sigma_x + \zeta \frac{\partial \mathcal{G}_*}{\partial x} \sigma_x + \zeta \mathcal{G}_* \sigma_{xx},\\ \tilde{\sigma}_{yy} &= \zeta_y \mathcal{G}_* \sigma_y + \zeta \frac{\partial \mathcal{G}_*}{\partial y} \sigma_y + \zeta \mathcal{G}_* \sigma_{yy},\\ \tilde{\sigma}_{xy} &= \zeta_x \mathcal{G}_* \sigma_y + \zeta \frac{\partial \mathcal{G}_*}{\partial x} \sigma_y + \zeta \mathcal{G}_* \sigma_{xy},\\ &= \zeta_y \mathcal{G}_* \sigma_x + \zeta \frac{\partial \mathcal{G}_*}{\partial y} \sigma_x + \zeta \mathcal{G}_* \sigma_{xy}.\end{aligned} \tag{11}$$

Now,

$$\begin{aligned}&\zeta \mathcal{G}_* \sigma_x \times \left(\zeta_x \mathcal{G}_* \sigma_x + \zeta \frac{\partial \mathcal{G}_*}{\partial x} \sigma_x\right) = \zeta \mathcal{G}_* \sigma_x \times \left(\zeta_x \mathcal{G}_* \sigma_x + \zeta \frac{\partial \mathcal{G}_*}{\partial x} \sigma_x + \zeta \mathcal{G}_* \sigma_{xx}\right) - \zeta \mathcal{G}_*(\sigma_x \times \sigma_{xx}).\\ &\Rightarrow \zeta \mathcal{G}_* \sigma_x \times \left(\zeta_x \mathcal{G}_* \sigma_x + \zeta \frac{\partial \mathcal{G}_*}{\partial x} \sigma_x\right) = \tilde{\sigma}_x \times \tilde{\sigma}_{xx} - \zeta \mathcal{G}_*(\sigma_x \times \sigma_{xx}).\end{aligned} \tag{12}$$

On a similar pattern, we find

$$\begin{aligned}\zeta \mathcal{G}_* \sigma_x \times \left(\zeta_y \mathcal{G}_* \sigma_x + \zeta \frac{\partial \mathcal{G}_*}{\partial y} \sigma_x\right) &= \tilde{\sigma}_x \times \tilde{\sigma}_{xy} - \zeta \mathcal{G}_*(\sigma_x \times \sigma_{xy}),\\ \zeta \mathcal{G}_* \sigma_x \times \left(\zeta_y \mathcal{G}_* \sigma_y + \zeta \frac{\partial \mathcal{G}_*}{\partial y} \sigma_y\right) &= \tilde{\sigma}_x \times \tilde{\sigma}_{yy} - \zeta \mathcal{G}_*(\sigma_x \times \sigma_{yy}),\\ \zeta \mathcal{G}_* \sigma_y \times \left(\zeta_x \mathcal{G}_* \sigma_x + \zeta \frac{\partial \mathcal{G}_*}{\partial x} \sigma_x\right) &= \tilde{\sigma}_y \times \tilde{\sigma}_{xx} - \zeta \mathcal{G}_*(\sigma_y \times \sigma_{xx}),\\ \zeta \mathcal{G}_* \sigma_y \times \left(\zeta_x \mathcal{G}_* \sigma_y + \zeta \frac{\partial \mathcal{G}_*}{\partial x} \sigma_y\right) &= \tilde{\sigma}_y \times \tilde{\sigma}_{xy} - \zeta \mathcal{G}_*(\sigma_y \times \sigma_{xy}),\\ \zeta \mathcal{G}_* \sigma_y \times \left(\zeta_y \mathcal{G}_* \sigma_y + \zeta \frac{\partial \mathcal{G}_*}{\partial y} \sigma_y\right) &= \tilde{\sigma}_y \times \tilde{\sigma}_{yy} - \zeta \mathcal{G}_*(\sigma_y \times \sigma_{yy}).\end{aligned} \tag{13}$$

Now, from equations (8), (11), (12), and (13), we obtained that

$$\begin{aligned}\tilde{\delta}(r) &= \frac{\mu_2(r)}{\kappa(r)}\{x'^2(\zeta \mathcal{G}_*)_x \sigma_x + 2x'y'(\zeta \mathcal{G}_*)_x \sigma_y + y'^2(\zeta \mathcal{G}_*)_y \sigma_y\}\\ &\quad + \zeta\left\{\mu_1(r)(\mathcal{G}_* \sigma_x x' + \mathcal{G}_* \sigma_y y') + \mu_2(r)\frac{1}{\kappa(r)}(x'' \mathcal{G}_* \sigma_x + y'' \mathcal{G}_* \sigma_y\right.\\ &\quad \left. + x'^2 \mathcal{G}_* \sigma_{xx} + 2x'y' \mathcal{G}_* \sigma_{xy} + y'^2 \mathcal{G}_* \sigma_{yy})\right\},\\ &= \mu_1(r)[x'\tilde{\sigma}_x + y'\tilde{\sigma}_y] + \frac{\mu_2(r)}{\kappa(r)}\left\{x'^2 \zeta_x \mathcal{G}_* \sigma_x + x'^2 \zeta \frac{\partial \mathcal{G}_*}{\partial x} \sigma_x\right.\\ &\quad + 2x'y' \zeta_x \mathcal{G}_* \sigma_y + 2x'y' \zeta \frac{\partial \mathcal{G}_*}{\partial x} \sigma_y + y'^2 \zeta_y \mathcal{G}_* \sigma_y + y'^2 \zeta \frac{\partial \mathcal{G}_*}{\partial y} \sigma_y\\ &\quad \left. + x'^2 \zeta \mathcal{G}_* \sigma_{xx} + 2x'y' \zeta \mathcal{G}_* \sigma_{xy} + y'^2 \zeta \mathcal{G}_* \sigma_{yy} + x''\tilde{\sigma}_x + y''\tilde{\sigma}_y\right\},\\ &= \mu_1(r)[x'\tilde{\sigma}_x + y'\tilde{\sigma}_y] + \frac{\mu_2(r)}{\kappa(r)}\left\{x'^2\left(\zeta_x \mathcal{G}_* \sigma_x + \zeta \frac{\partial \mathcal{G}_*}{\partial x} \sigma_x + \zeta \mathcal{G}_* \sigma_{xx}\right)\right.\\ &\quad \left. + 2x'y'\left(\zeta_x \mathcal{G}_* \sigma_y + \zeta \frac{\partial \mathcal{G}_*}{\partial x} \sigma_y + \zeta \mathcal{G}_* \sigma_{xy}\right) + y'^2\left(\zeta_y \mathcal{G}_* \sigma_y + \zeta \frac{\partial \mathcal{G}_*}{\partial y} \sigma_y + \zeta \mathcal{G}_* \sigma_{yy}\right) + x''\tilde{\sigma}_x + y''\tilde{\sigma}_y\right\}.\end{aligned} \tag{14}$$

Now, substituting equation (11) into equation (14), we find that

$$\tilde{\delta}(r) = \tilde{\mu}_1(r)\{x'\tilde{\sigma}_x + y'\tilde{\sigma}_y\} + \frac{\tilde{\mu}_2(r)}{\tilde{\kappa}(r)}\{x'^2\tilde{\sigma}_{xx} + 2x'y'\tilde{\sigma}_{xy} + y'^2\tilde{\sigma}_{yy} + x''\tilde{\sigma}_x + y''\tilde{\sigma}_y\},$$
$$\Rightarrow \tilde{\delta}(r) = \tilde{\mu}_1(r)\tilde{\vec{t}}(r) + \frac{\tilde{\mu}_2(r)}{\tilde{\kappa}(r)}\tilde{\vec{n}}(r).$$

Thus, for some smooth functions $\tilde{\mu}_1(r)$ and $\tilde{\mu}_2(r)$, $\tilde{\delta}(r)$ may be expressed as the combination of tangent vector $\tilde{\vec{t}}(r)$ and normal vector $\tilde{\vec{n}}(r)$. Here, we assume that $\tilde{\mu}_1 = \mu_1$ and $\frac{\tilde{\mu}_2}{\tilde{\kappa}} = \frac{\mu_2}{\kappa}$.

This proves that $\tilde{\delta}(r)$ is an osculating curve in $\tilde{Q}$. □

Corollary 1. *Let Q and $\tilde{Q}$ be two regular and smooth immersed surfaces in the Euclidean space $\mathbb{R}^3$, and let $\mathcal{G} : Q \to \tilde{Q}$ be a homothetic conformal map. Suppose that $\delta(r)$ is an osculating curve on the surface Q. Then, its homothetic conformal image $\tilde{\delta}(r)$ is also an osculating curve that lies on the surface $\tilde{Q}$, if the following condition is true*:

$$\tilde{\delta}(r) = \frac{\mu_2(r)}{\kappa(r)}\{x'^2 c(\mathcal{G}_*)_x\sigma_x + 2x'y'c(\mathcal{G}_*)_x\sigma_y + y'^2 c(\mathcal{G}_*)_y\sigma_y\} + c\mathcal{G}_*(\delta(r)),$$

where $c \neq \{0, 1\}$ is a constant.

Proof. We know that for a homothetic map, the dilation factor $\zeta(x, y) = c \neq \{0, 1\}$. By substituting the value of $\zeta = c$ into equation (8), we obtain the desired result for homothetic conformal maps, and thereby deriving the expression above. □

Corollary 2. *Let Q and $\tilde{Q}$ be two regular and smooth immersed surfaces in the Euclidean space $\mathbb{R}^3$, and let $\mathcal{G} : Q \to \tilde{Q}$ be an isometry. Suppose that $\delta(r)$ is an osculating curve on the surface Q. Then, its isometric image $\tilde{\delta}(r)$ is also an osculating curve on the surface $\tilde{Q}$, if the following condition is true*:

$$\tilde{\delta}(r) = \frac{\mu_2(r)}{\kappa(r)}\{x'^2(\mathcal{G}_*)_x\sigma_x + 2x'y'(\mathcal{G}_*)_x\sigma_y + y'^2(\mathcal{G}_*)_y\sigma_y\} + \mathcal{G}_*(\delta(r)).$$

Proof. As a conformal transformation results from composing an isometry with a dilation function, for an isometry between two surfaces, the dilation function $\zeta(x, y) = c = 1$. If we set $\zeta(x, y) = 1$ in equation (8), we obtain the aforementioned expression for the case of isometry. □

Theorem 2. *If Q and $\tilde{Q}$ are two conformal smooth and regular surfaces immersed in the Euclidean space $\mathbb{R}^3$, and let $\delta(r)$ be an osculating curve on the surface Q, then the component of the curve $\delta(r)$ along the surface normal $\mathbb{N}$, satisfying the following condition*:

$$\tilde{\delta} \cdot \tilde{\mathbb{N}} - \zeta^4\delta \cdot \mathbb{N} = \frac{\mu_2(r)}{\kappa(r)}(\tilde{\kappa}_n - \zeta^4\kappa_n), \tag{15}$$

where ζ is the dilation function and κ_n is the normal curvature of the curve $\delta(r)$.

Proof. Since $\tilde{Q}$ is the conformal image of Q, and σ and $\tilde{\sigma} = \mathcal{G} \circ \sigma$ be the coordinate chart maps of the surfaces Q and $\tilde{Q}$, respectively.

Since $\delta(r)$ is an osculating curve on the surface Q, its position vector is defined as

$$\delta(r) = \mu_1(r)\vec{t}(r) + \mu_2(r)\vec{n}(r),$$
$$= \mu_1(r)(\sigma_x x' + \sigma_y y') + \mu_2(r)\frac{1}{\kappa(r)}(x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}).$$

Now,

$$
\begin{aligned}
\delta(r)\cdot\mathbb{N} &= \{\mu_1(r)(\sigma_x x' + \sigma_y y') + \mu_2(r)\frac{1}{\kappa(r)}(x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})\}\cdot\mathbb{N},\\
&= \{\mu_1(r)(\sigma_x x' + \sigma_y y')\}\cdot\mathbb{N} + \{\mu_2(r)\frac{1}{\kappa(r)}(x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})\}\cdot\mathbb{N},\\
&= \{\mu_1(r)(\sigma_x x' + \sigma_y y')\}\cdot(\sigma_x\times\sigma_y) + \{\mu_2(r)\frac{1}{\kappa(r)}(x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})\}\cdot(\sigma_x\times\sigma_y),\\
&= \frac{\mu_2(r)}{\kappa(r)}\{x''\sigma_x\cdot(\sigma_x\times\sigma_y) + y''\phi_y\cdot(\phi_x\times\sigma_y) + x'^2\sigma_{xx}\cdot(\sigma_x\times\sigma_y) + 2x'y'\sigma_{xy}\cdot(\sigma_x\times\sigma_y)\\
&\quad + y'^2\sigma_{yy}\cdot(\sigma_x\times\sigma_y)\},\\
&= \frac{\mu_2(r)}{\kappa(r)}\{x'^2\sigma_{xx}\cdot(\sigma_x\times\sigma_y) + 2x'y'\phi_{xy}\cdot(\phi_x\times\phi_y) + y'^2\sigma_{yy}\cdot(\sigma_x\times\sigma_y)\},\\
&= \frac{\mu_2(r)}{\kappa(r)}\{x'^2X + 2x'y'Y + y'^2Z\},\\
&= \frac{\mu_2(r)}{\kappa(r)}(\kappa_n),
\end{aligned}
$$

where κ_n represents the normal curvature of $\delta(r)$, and X, Y, and Z denote the magnitudes of the second fundamental form [10].

Now, considering the normal component of the curve $\delta(r)$ along the surface normal $\mathbb{N}$, we obtain

$$
\begin{aligned}
\tilde{\delta}\cdot\tilde{\mathbb{N}} - \zeta^4\delta\cdot\mathbb{N} &= \frac{\mu_2(\tilde{r})}{\kappa(\tilde{r})}(\tilde{\kappa}_n) - \zeta^4\frac{\mu_2(r)}{\kappa(r)}(\kappa_n),\\
&= \frac{\mu_2(r)}{\kappa(s)}(\tilde{\kappa}_n) - \zeta^4\frac{\mu_2(r)}{\kappa(r)}(\kappa_n),\\
&= \frac{\mu_2(r)}{\kappa(r)}(\tilde{\kappa}_n - \zeta^4\kappa_n).
\end{aligned}
$$

This proves equation (15). □

Corollary 3. *If Q and $\tilde{Q}$ are two homothetic smooth and regular surfaces in the Euclidean space $\mathbb{R}^3$, and let $\delta(r)$ be an osculating curve on Q, then the normal component of the curve $\delta(r)$ along the surface normal $\mathbb{N}$ satisfying the following condition*:

$$\tilde{\delta}\cdot\tilde{\mathbb{N}} - c^4\delta\cdot\mathbb{N} = \frac{\mu_2(r)}{\kappa(r)}(\tilde{\kappa}_n - c^4\kappa_n), \tag{16}$$

where $c \neq \{0,1\}$ is a constant and κ_n is the normal curvature of $\delta(r)$.

Furthermore, the normal component of $\delta(r)$ exhibits conformal invariance when either the position vector of $\delta(r)$ aligns with the tangent vector or the normal curvature is conformally invariant.

Proof. Since for homothetic conformal transformation, we have $\zeta(x,y) = c$, where c is a constant and $c \neq \{0,1\}$. If we substitute $\zeta(x,y) = c$ into equation (15), we obtained the desired result.

Also, in view of equation (16), the curve $\delta(r)$ is conformally invariant if and only if $\mu_2(r) = 0$, i.e.,

$$\delta(r) = \mu_1(r)\vec{t}(r) \quad \text{or} \quad \tilde{\kappa}_n = c^4\kappa_n.$$ □

Corollary 4. *If Q and $\tilde{Q}$ are two isometric smooth and regular surfaces in the Euclidean space $\mathbb{R}^3$ and let $\delta(r)$ be an osculating curve on Q, then the component of the curve $\delta(r)$ along the surface normal $\mathbb{N}$ is given by*

$$\tilde{\delta}\cdot\tilde{\mathbb{N}} - \delta\cdot\mathbb{N} = \frac{\mu_2(r)}{\kappa(r)}(\tilde{\kappa}_n - \kappa_n),$$

where κ_n is the normal curvature of the curve $\delta(r)$.

Moreover, in cases where the position vector of $\delta(r)$ aligns with the tangent vector or the normal curvature remains unchanged under isometry, then the component of $\delta(r)$ along the surface normal $\mathbb{N}$ remains unchanged with respect to such isometry.

Proof. Since a conformal transformation is made up of a dilation function and an isometry [10]. To prove this corollary, we substitute $\zeta(x, y) = 1$ into equation (15), we obtained the desired result. □

Theorem 3. *Consider Q and $\tilde{Q}$ as two smooth and regular conformal surfaces in the Euclidean space $\mathbb{R}^3$, and let $\delta(r)$ be an osculating curve on Q. Then, for the tangential component $\mathbb{T}$ of the curve $\delta(r)$, we have*

$$\tilde{\delta}\cdot\tilde{\mathbb{T}} - \zeta^2\delta\cdot\mathbb{T} = ah_1 + bh_2, \tag{17}$$

where h_1 and h_2 are defined by equations (18) and (19), respectively, and "a" and "b" are any real numbers and ζ is the dilation function.

Proof. Suppose that $\tilde{Q}$ is the conformal image of Q. Consider that $\sigma(x, y)$ and $\tilde{\sigma}(x, y)$ are the coordinate maps of the surfaces Q and $\tilde{Q}$, respectively, such that $\tilde{\sigma}(x, y) = \mathcal{G}\circ\sigma(x, y)$.

Now,

$$\begin{aligned}\delta(r)\cdot\ \sigma_x &= \{\mu_1(r)(\sigma_x x' + \sigma_y y') + \frac{\mu_2(r)}{\kappa(r)}(x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})\}\cdot\ \sigma_x,\\ &= \mu_1(r)(\sigma_x\cdot\sigma_x x' + \sigma_y\cdot\sigma_x y') + \frac{\mu_2(r)}{\kappa(r)}(x''\sigma_x\cdot\sigma_x + y''\sigma_y\cdot\sigma_x + x'^2\sigma_{xx}\cdot\sigma_x + 2x'y'\sigma_{xy}\cdot\sigma_x + y'^2\sigma_{yy}\cdot\sigma_x),\\ &= \mu_1(r)(x'E + y'F) + \frac{\mu_2(r)}{\kappa(r)}\left\{x''E + y''F + x'^2\frac{E_x}{2} + 2x'y'\frac{E_y}{2} + y'^2\left(F_y - \frac{G_x}{2}\right)\right\},\\ &= \mu_1(r)(x'E + y'F) + \frac{\mu_2(r)}{2\kappa(r)}\{2x''E + 2y''F + x'^2E_x + 2x'y'E_y + y'^2(2F_y - G_x)\}.\end{aligned}$$

In the same manner, we can write

$$\delta(r)\cdot\sigma_y = \mu_1(r)(x'F + y'G) + \frac{\mu_2(r)}{2\kappa(r)}\{2x''F + 2y''G + 2x'^2F_x - x'^2E_y + 2x'y'G_x + y'^2G_y\}.$$

Now, for the conformal image $\tilde{\delta(r)}$ of $\delta(r)$ on $\tilde{Q}$, we can write

$$\begin{aligned}\tilde{\delta}(r)\cdot\tilde{\sigma}_x &= \tilde{\mu}_1(r)(x'\tilde{E} + y'\tilde{F}) + \frac{\tilde{\mu}_2(r)}{2\tilde{\kappa}(r)}\{2x''\tilde{E} + 2y''\tilde{F} + x'^2\tilde{E}_x + 2x'y'\tilde{E}_y + y'^2(2\tilde{F}_y - \tilde{G}_x)\},\\ &= \mu_1(r)(x'\zeta^2E + y'\zeta^2F) + \frac{\mu_2(r)}{2\kappa(r)}\Bigg\{2x''\zeta^2E + 2y''\zeta^2F + x'^2(2\zeta\zeta_xE + \zeta^2E_x) + 2x'y'(2\zeta\zeta_yE + \zeta^2E_y)\\ &\quad + 2y'^2\left(2\zeta\zeta_yF + \zeta^2F_y - \frac{2\zeta\zeta_xG + \zeta^2G_x}{2}\right)\Bigg\}.\end{aligned}$$

Similarly, we can find $\tilde{\delta}(r)\cdot\ \tilde{\sigma}_y$.

Now,

$$\begin{aligned}\tilde{\delta}(r)\cdot\ \tilde{\sigma}_x - \zeta^2(\delta(r)\cdot\ \sigma_x) &= \frac{\mu_2(r)}{2\kappa(r)}\{2x'^2\zeta\zeta_xE + 2x'y'(2\zeta\zeta_yE) + 2y'^2(2\zeta\zeta_yF - \zeta\zeta_xG)\},\\ &= \frac{\mu_2(r)}{\kappa(r)}\{x'^2\zeta\zeta_xE + 2x'y'\zeta\zeta_yE + y'^2(2\zeta\zeta_yF - \zeta\zeta_xG)\},\\ &= h_1(E, F, G, \zeta).\end{aligned}\tag{18}$$

Similarly, we can find

$$\tilde{\delta}(r)\cdot\ \tilde{\sigma}_y - \zeta^2(\delta(r)\cdot\ \sigma_y) = h_2(E, F, G, \zeta). \tag{19}$$

Now,

$$\begin{aligned}\tilde{\delta}\cdot\tilde{\mathbb{T}}-\zeta^2(\delta\cdot\mathbb{T}) &= \tilde{\delta}\cdot(a\tilde{\sigma}_x+b\tilde{\sigma}_y)-\zeta^2(\delta\cdot(a\sigma_x+b\sigma_y)),\\ &= a(\tilde{\delta}\cdot\tilde{\sigma}_x-\zeta^2(\delta\cdot\sigma_x))+b(\tilde{\delta}\cdot\tilde{\sigma}_y-\zeta^2(\delta\cdot\sigma_y)),\\ &= ah_1(E,F,G,\zeta)+bh_2(E,F,G,\zeta).\end{aligned}$$

This gives the required result. □

Corollary 5. *If $\mathcal{G}:Q\rightarrow\tilde{Q}$ is an isometry and $\delta(r)$ is an osculating curve, then the tangential components of the curve $\delta(r)$ are invariant under $\mathcal{G}$, i.e., $\tilde{\delta}\cdot\tilde{\mathbb{T}}=\delta\cdot\mathbb{T}$.*

Proof. In the context of isometry, the conformal dilation factor is $\zeta(x,y)=1$. Therefore, from Theorem 3, we can conclude that under such an isometry, both h_1 and h_2 become zero. So, by substituting both $h_1=0$ and $h_2=0$ into Theorem 3, we demonstrated that in an osculating curve, the tangential components remain invariant when the surfaces are isometric. □

4 Conclusion

In this article, we explored the adequate conditions for the invariants of the conformal image of an osculating curve under various transformations. We obtained expressions for the tangential and normal components of the osculating curves by applying these conditions and investigated their invariance under isometric transformations.

In future research, it is possible to build upon the findings of Shaikh and Chen regarding such curves and introduces additional results about the normal and geodesic curvatures. One could explore the conformal and isometric invariance of these curves under conformal and isometric transformations using the Darboux frame instead of the Frenet frame in Euclidean spaces.

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