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Research Article

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Decay rate of the solutions to the Cauchy problem of the Lord Shulman thermoelastic Timoshenko model with distributed delay

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Abstract: In this study, we address a Cauchy problem within the context of the one-dimensional Timoshenko system, incorporating a distributed delay term. The heat conduction aspect is described by the Lord-Shulman theory. Our demonstration establishes that the dissipation resulting from the coupling of the Timoshenko system with Lord-Shulman's heat conduction is sufficiently robust to stabilize the system, albeit with a gradual decay rate. To support our findings, we convert the system into a first-order form and, utilizing the energy method in Fourier space, and derive point wise estimates for the Fourier transform of the solution. These estimates, in turn, provide evidence for the slow decay of the solution.

Keywords: partial differential equation, decay rate, Lord-Shulman, thermoelasticity, mathematical operators, Fourier transform, distributed delay

MSC 2020: 35B37, 35L55, 74D05, 93D15, 93D20

1 Introduction and preliminaries

Lord Shulman's thermoelasticity has attracted significant attention from scientists in recent years, leading to numerous contributions aimed at elucidating this theory. The foundation of this theory lies in the examination of a set of four hyperbolic equations incorporating heat dissipation. Specifically, in this scenario, the heat equation takes on a noteworthy characteristic it is both equivalent and hyperbolic. This stands in contrast to the formulation based on Fourier's law. For a deeper comprehension of this theory and related ones, you can explore the works of Bazarra et al. and Lord and Shulman [1,2]. Furthermore, Green and Naghdi [3,4] introduced a thermoelasticity model that incorporates thermal displacement gradient and temperature gradient into the constitutive variables, complemented by a suggested heat conduction law. Choucha et al. [5] considered a one-dimensional Cauchy problem in the Timoshenko system, considering thermal effects and damping described by Lord-Shulman's heat conduction theory. It shows that the dissipation from coupling these elements stabilizes the system, albeit with a slow decay rate, using a first-order system transformation and employing the energy method in Fourier space for point wise estimates.

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Several researchers have explored the outcomes arising from coupling the Fourier law of heat conduction with diverse systems. The Timoshenko system has been scrutinized in previous studies [6,7], while the Bresse system, coupled with the Cattaneo law of heat conduction, has been explored in the work presented by Said-Houari and Hamadouche [8]. The investigation of the Bresse system has been undertaken in the range of studies by Said-Houari et al. [9–11]. In addition, the MGT problem has been addressed by Boulaaras et al. [12]. For a more comprehensive understanding, readers are encouraged to explore the recommended papers [13,14]. The foundational evolution equations for one-dimensional Timoshenko thermoelasticity, theories with microtemperature and temperature were initially presented in previous studies [15–18] as follows:

$$\begin{aligned} \rho\mathbf{h}_{tt} &= T_x,\\ I_\rho\mathcal{K}_{tt} &= H_x + G,\\ \rho T_0\zeta_t &= q_x,\\ \rho E_t &= P_x^* + q - Q. \end{aligned} \tag{1.1}$$

In this context, the functions $\mathbf{h}$ and $\mathcal{K}$ represent the elastic material displacement, while $G, T_0, P^*, \zeta, q, T, H$, and E represent equilibrated body force, reference temperature, first heat flux moment, mean heat flux, entropy, heat flux vector, stress, equilibrated stress, and the first moment of energy, respectively. Furthermore, E, I, I_ρ, ρ, and K indicate Young's modulus of elasticity, moment of inertia of a cross section, polar moment of inertia of a cross section, density, and the shear modulus.

For simplification, we set $T_0 = K = \rho = I_\rho = 1$ and $0 < EI = a^2$. In the current study, we consider the natural counterpart to the microtemperatures in the Lord-Shulman theory. Consequently, we need to modify the constitutive equations as follows:

$$\begin{aligned} &T = T_1 + T_2 \quad P^* = -k_2\omega_x,\\ &H = H_1 + H_2 \quad \rho\zeta = \gamma_0\mathbf{h}_x + \gamma_1\mathcal{K} + \beta_1(\kappa\theta_t + \theta),\\ &G = G_1 + G_2 \quad Q = (k_1 - k_3)\omega + (k - k_1)\theta_x,\\ &q = k\theta_x + k_1\omega \quad \rho E = -\beta_2(\kappa\omega_t + \omega) - \gamma_2\mathcal{K}_x, \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} &T_1 = G_1 = K(\mathbf{h}_x - \mathcal{K}) \;\; T_2 = -\gamma_0(\kappa\theta_t + \theta)\\ &H_1 = EI\mathcal{K}_x \;\; G_2 = \gamma_1(\kappa\theta_t + \theta) - \mu_1\mathcal{K}_t - \int_{\tau_1}^{\tau_2}\mu_2(s)\mathcal{K}_t(x, t - s)\mathrm{d}s,\\ &H_2 = -\gamma_2(\kappa\omega_t + \omega). \end{aligned} \tag{1.3}$$

In this context, the function θ indicates the temperature difference, while the microtemperature vector is indicated by ω with $\kappa > 0$ serving as the relaxation parameter. With β_1 and β_2 both being positive, the coefficients γ_1, k, and γ_0, respectively, signify the coupling between the volume fraction and temperature, the thermal conductivity, and the coupling between displacement and temperature.

When accounting for coupling, the coefficients k_1, k_2, k_3, γ_2, and μ_1 are all positive and fulfill the inequality.

$$k_1^2 < kk_3. \tag{1.4}$$

For delay, assume the following **(H1)** $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a limited function considering

$$\int_{\tau_1}^{\tau_2}|\mu_2(s)|\mathrm{d}s < \mu_1. \tag{1.5}$$

This study primarily focuses on thermal effects and distributed delay. We set the heat capacity as $\beta_1 = \beta > 0$, omitting the consideration of microtemperature effects by assuming $\beta_2 = k_1 = k_2 = k_3 = \gamma_2 = 0$.

Now, by substituting (1.2) and (1.3) into (1.1), we derive:

$$\begin{cases} \mathbf{h}_{tt} - (\mathbf{h}_x - \mathcal{K})_x + \gamma_0(\kappa\theta_t + \theta)_x = 0, \\ \mathcal{K}_{tt} - a^2\mathcal{K}_{xx} - (\mathbf{h}_x - \mathcal{K}) - \gamma_1(\kappa\theta_t + \theta) + \mu_1\mathcal{K}_t + \int\limits_{\tau_1}^{\tau_2} \mu_2(s)\mathcal{K}_t(x, t - s)\mathrm{d}s = 0, \\ \beta(\kappa\theta_t + \theta)_t + \gamma_0\mathbf{h}_{tx} + \gamma_1\mathcal{K}_t - k\theta_{xx} = 0, \end{cases} \tag{1.6}$$

where

$$(x, s, t) \in \mathbb{R} \times (\tau_1, \tau_2) \times \mathbb{R}_+,$$

with initial conditions

$$\begin{aligned} &(\mathbf{h}, \mathbf{h}_t, \mathcal{K}, \mathcal{K}_t, \theta, \theta_t)(x, 0) = (\mathbf{h}_0, \mathbf{h}_1, \mathcal{K}_0, \mathcal{K}_1, \theta_0, \theta_1), \quad x \in \mathbb{R}, \\ &\mathcal{K}_t(x, -t) = g_0(x, t), \quad (x, t) \in \mathbb{R} \times (0, \tau_2). \end{aligned} \tag{1.7}$$

The effect of delay of all kinds always remains very important in any stability of the different systems, so its study is among the priorities. For more information on the effect of delay, the research works by Choucha et al. are recommended [19–22].

Upon delving into the intricacies of distributed delay, several natural questions arise: How does one gauge the intricacies? Is the notion of amortization universally beneficial? Could the incorporation of the distributed delay term have heightened the complexity of solving this type of problem? This study aims to grasp the nuances of the Cauchy problem in the Timoshenko system, where the heat conduction follows Lord-Shulman's theory and involves a distributed delay term, particularly in Fourier space.

We structure this article as follows: In this section, we leverage our initial findings to elucidate our primary decay inference. Following that, in the subsequent section, we formulate the Lyapunov functional and reveal the estimation for the Fourier transform utilizing the energy approach in Fourier space. The concluding segment is dedicated to summarizing our results. Notably, this study represents one of the pioneering investigations addressing this matter in Fourier space, to the best of our knowledge.

We introduce the following variable as in the study by Nicaise and Pignotti [22]:

$$\mathrm{Y}(x, \rho, s, t) = \mathcal{K}_t(x, t - s\rho),$$

thus, the following equation is obtained

$$\begin{cases} s\mathrm{Y}_t(x, \rho, s, t) + \mathrm{Y}_\rho(x, \rho, s, t) = 0, \\ \mathrm{Y}(x, 0, s, t) = \mathcal{K}_t(x, t). \end{cases}$$

Consequently, our problem can be written as follows:

$$\begin{cases} \mathbf{h}_{tt} - (\mathbf{h}_x - \mathcal{K})_x + \gamma_0(\kappa\theta_t + \theta)_x = 0, \\ \mathcal{K}_{tt} - a^2\mathcal{K}_{xx} - (\mathbf{h}_x - \mathcal{K}) - \gamma_1(\kappa\theta_t + \theta) + \mu_1\mathcal{K}_t + \int\limits_{\tau_1}^{\tau_2} \mu_2(s)\mathrm{Y}(x, 1, s, t)\mathrm{d}s = 0, \\ \beta(\kappa\theta_t + \theta)_t + \gamma_0\mathbf{h}_{tx} + \gamma_1\mathcal{K}_t - k\theta_{xx} = 0, \\ s\mathrm{Y}_t(x, \rho, s, t) + \mathrm{Y}_\rho(x, \rho, s, t) = 0, \end{cases} \tag{1.8}$$

where

$$(x, \rho, s, t) \in \mathbb{R} \times (0, 1) \times (\tau_1, \tau_2) \times \mathbb{R}_+,$$

with initial conditions

$$\begin{aligned} &(\mathbf{h}, \mathbf{h}_t, \mathcal{K}, \mathcal{K}_t, \theta, \theta_t)(x, 0) = (\mathbf{h}_0, \mathbf{h}_1, \mathcal{K}_0, \mathcal{K}_1, \theta_0, \theta_1), \quad x \in \mathbb{R}, \\ &\mathrm{Y}(x, \rho, s, 0) = g_0(x, s\rho), \quad (x, \rho, s) \in \mathbb{R} \times (0, 1) \times (0, \tau_2). \end{aligned} \tag{1.9}$$

The following Hausdorff-Young inequality is introduced for the analysis of our work:

Lemma 1.1. [23] *Let* $k, \varrho \geq 0, c > 0$, *then one can find* $C > 0$ *in a manner that* $\forall t \geq 0$ *following satisfies*

$$\int_{|\aleph|\leq 1} |\aleph|^k e^{-c|\aleph|^{\varrho} t} \mathrm{d}\aleph \leq C(1+t)^{-(k+n)/\varrho}, \quad \aleph \in \mathbb{R}^n. \tag{1.10}$$

Theorem 1.2. ([24] Plancherel theorem) *Let* $f(x)$ *be a real line function and its frequency spectrum is given by* $\widehat{f}(\xi)$, *then*

$$\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x = \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2 \mathrm{d}\xi.$$

2 Energy method and decay estimates

In this section, we present the decay estimation for the Fourier transform of the solution to the problem defined by equations (1.8)–(1.9). Through this approach, we can ascertain the rate of decay for the solution within the energy space, utilizing tools such as Plancherel's theorem and integral estimates like Lemma 1.1. We create appropriate Lyapunov functionals and apply the energy method in Fourier space to tackle this issue. The section concludes the main findings of our work.

2.1 The energy method in the Fourier space

For the Lyapunov functional in the Fourier space, we introduce the following new variables

$$f = \mathbf{h}_x - \mathcal{K}, \quad j = \mathbf{h}_t, \quad b = a\mathcal{K}_x, \quad m = \mathcal{K}_t, \quad \zeta = \kappa\theta_t + \theta, \quad \varpi = \theta_x, \tag{2.1}$$

thus, (1.8) can be expressed as follows:

$$\begin{cases} f_t - j_x + m = 0 \\ j_t - f_x + \gamma_0 \zeta_x = 0 \\ b_t - a m_x = 0 \\ m_t - a b_x - f - \gamma_1 \zeta + \mu_1 m + \int_{\tau_1}^{\tau_2} \mu_2(s) \Upsilon(x,1,s,t) \mathrm{d}s = 0 \\ \beta \zeta_t + \gamma_0 j_x + \gamma_1 m - k\varpi_x = 0 \\ \kappa \varpi_t - \zeta_x + \varpi = 0 \\ s\Upsilon_t + \Upsilon_\rho = 0, \end{cases} \tag{2.2}$$

with the initial data

$$(f, j, b, m, \zeta, \varpi, \Upsilon)(x, 0) = (f_0, j_0, b_0, m_0, \zeta_0, \varpi_0, g_0), x \in \mathbb{R}, \tag{2.3}$$

where

$$f_0 = (\mathbf{h}_{0,x} - \mathcal{K}_0), \; j_0 = \mathbf{h}_1, \; b_0 = a\mathcal{K}_{0,x}, \; m_0 = \mathcal{K}_1, \; \zeta_0 = \kappa\theta_1 - \theta_0, \; \varpi_0 = \theta_{0,x}.$$

Hence, problems (2.2) and (2.3) become

$$\begin{cases} F_t + \mathcal{A} F_x + \mathcal{L} F = 0, \\ F(x, 0) = F_0(x), \end{cases} \tag{2.4}$$

where $F = (f, j, b, m, \zeta, \varpi, \Upsilon)^T$, $F_0 = (f_0, j_0, b_0, m_0, \zeta_0, \varpi_0, g_0)$, and

$$\mathcal{A}F = \begin{pmatrix} -j \\ -f + \gamma_0\zeta \\ -am \\ -ab \\ \frac{1}{\beta}(\gamma_0 j - k\varpi) \\ -\frac{1}{\kappa}\zeta \\ 0 \end{pmatrix}, \quad \mathcal{L}F = \begin{pmatrix} m \\ 0 \\ 0 \\ -f - \gamma_1\zeta + \mu_1 m + \int_{\tau_1}^{\tau_2}\mu_2(s)\Upsilon(x,1,s,t)\mathrm{d}s \\ \frac{1}{\beta}(\gamma_1 m) \\ \frac{1}{\kappa}\varpi \\ \frac{1}{s}\Upsilon_\rho \end{pmatrix}.$$

Then the system (2.4) implies

$$\begin{cases} \widehat{F}_t + i\aleph\mathcal{A}\widehat{F} + \mathcal{L}\widehat{F} = 0, \\ \widehat{F}(\aleph, 0) = \widehat{F}_0(\aleph). \end{cases} \tag{2.5}$$

in which $\widehat{F}(\aleph, t) = (\widehat{f}, \widehat{j}, \widehat{b}, \widehat{m}, \widehat{\zeta}, \widehat{\varpi}, \widehat{\Upsilon})^T(\aleph, t)$. Moreover, $(2.5)_1$ becomes

$$\begin{cases} \widehat{f}_t - i\aleph\widehat{j} + \widehat{m} = 0 \\ \widehat{j}_t - i\aleph\widehat{f} + i\aleph\gamma_0\widehat{\zeta} = 0 \\ \widehat{b}_t - ai\aleph\widehat{m} = 0 \\ \widehat{m}_t - ai\aleph\widehat{b} - \widehat{f} - \gamma_1\widehat{\zeta} + \mu_1\widehat{m} + \int_{\tau_1}^{\tau_2}\mu_2(s)\widehat{\Upsilon}(\aleph, 1, s, t)\mathrm{d}s = 0 \\ \beta\widehat{\zeta}_t + i\aleph\gamma_0\widehat{j} + \gamma_1\widehat{m} - i\aleph k\widehat{\varpi} = 0 \\ \kappa\widehat{\varpi}_t - i\aleph\widehat{\zeta} + \widehat{\varpi} = 0 \\ s\widehat{\Upsilon}_t + \widehat{\Upsilon}_\rho = 0. \end{cases} \tag{2.6}$$

Lemma 2.1. *Assume $\widehat{E}(\aleph, t)$ be the energy functional and the solution of* (2.5)*is represented by $\widehat{F}(\aleph, t)$. Then, we have*

$$\begin{aligned} \widehat{E}(\aleph, t) = {} & \frac{1}{2}\{|\widehat{f}|^2 + |\widehat{j}|^2 + |\widehat{b}|^2 + |\widehat{m}|^2 + \beta|\widehat{\zeta}|^2 + k\kappa|\widehat{\varpi}|^2\} \\ & + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)||\widehat{\Upsilon}(\aleph, \rho, s, t)|^2\mathrm{d}s\mathrm{d}\rho, \end{aligned} \tag{2.7}$$

satisfies

$$\frac{\mathrm{d}\widehat{E}(\aleph, t)}{\mathrm{d}t} = -C_1|\widehat{m}|^2 - k|\widehat{\varpi}|^2 \le 0, \tag{2.8}$$

where $C_1 = \left(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|\mathrm{d}s\right) > 0.$

Proof. For the required result, multiply $\overline{\widehat{f}}, \overline{\widehat{j}}, \overline{\widehat{b}}$, and $\overline{\widehat{m}}$ with $(2.6)_{1,2,3,4}$, and multiply $\overline{\widehat{\zeta}}, k\overline{\widehat{\varpi}}$ with $(2.6)_{5,6}$. Take the real part of these equalities and add, we have the following

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[|\widehat{f}|^2+|\widehat{j}|^2+|\widehat{b}|^2+|\widehat{m}|^2+\beta|\widehat{\zeta}|^2+k\kappa|\widehat{\varpi}|^2]\\&\quad+\mu_1|\widehat{m}|^2+k|\widehat{\varpi}|^2+\Re e\left\{\int_{\tau_1}^{\tau_2}\mu_2(s)\overline{\widehat{m}}\,\widehat{\Upsilon}(\aleph,1,s,t)\mathrm{d}s\right\}=0.\end{aligned}\tag{2.9}$$

In next step, multiply $\overline{\widehat{\Upsilon}}|\mu_2(s)|$ with $(2.6)_7$ and integrate over $(0,1)\times(\tau_1,\tau_2)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)||\widehat{\Upsilon}(\aleph,\rho,s,t)|^2\mathrm{d}s\mathrm{d}\rho&=-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\frac{\mathrm{d}}{\mathrm{d}\rho}|\widehat{\Upsilon}(\aleph,\rho,s,t)|^2\mathrm{d}s\mathrm{d}\rho\\&=\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|(|\widehat{\Upsilon}(\aleph,0,s,t)|^2-|\widehat{\Upsilon}(\aleph,1,s,t)|^2)\mathrm{d}s\\&=\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\mathrm{d}s\right)|\widehat{m}|^2-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)||\widehat{\Upsilon}(\aleph,1,s,t)|^2\mathrm{d}s.\end{aligned}\tag{2.10}$$

Here, using the inequality of Young, the following is obtained

$$\Re e\left\{\int_{\tau_1}^{\tau_2}\mu_2(s)\overline{\widehat{m}}\,\widehat{\Upsilon}(\aleph,1,s,t)\mathrm{d}s\right\}\leq\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\mathrm{d}s\right)|\widehat{m}|^2+\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)||\widehat{\Upsilon}(\aleph,1,s,t)|^2\mathrm{d}s,\tag{2.11}$$

and substituting (2.10) and (2.11) into (2.9), one have

$$\frac{\mathrm{d}\widehat{E}(\aleph,t)}{\mathrm{d}t}=-\left(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(s)|\mathrm{d}s\right)|\widehat{m}|^2-k|\widehat{\varpi}|^2.$$

Then, by (1.5), there exits

$$C_1=\left(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(s)|\mathrm{d}s\right)>0,$$

so that

$$\frac{\mathrm{d}\widehat{E}(\xi,t)}{\mathrm{d}t}\leq-C_1|\widehat{m}|^2-k|\widehat{\varpi}|^2\leq 0.\tag{2.12}$$

Therefore, the required (2.7) and (2.8) are obtained. □

In the coming step, we introduce the following lemmas which will be used in the proof of main result.

Lemma 2.2. *The functional*

$$\mathcal{D}_1(\aleph,t)\coloneqq\Re e\{i\aleph(\widehat{f}\overline{\widehat{j}}+\widehat{m}\overline{\widehat{b}})\}\tag{2.13}$$

fulfills

$$\begin{aligned}\frac{\mathrm{d}\mathcal{D}_1(\aleph,t)}{\mathrm{d}t}&\leq-\frac{1}{2}\aleph^2|\widehat{j}|^2-\frac{a}{2}\aleph^2|\widehat{b}|^2+c(1+\aleph^2)|\widehat{f}|^2+c(1+\aleph^2)|\widehat{m}|^2\\&\quad+c\aleph^2|\widehat{\zeta}|^2+c\int_{\tau_1}^{\tau_2}|\mu_2(s)||\widehat{\Upsilon}(\aleph,1,s,t)|^2\mathrm{d}s.\end{aligned}\tag{2.14}$$

Proof. Differentiating $\mathcal{D}_1$ and applying (2.6), one have

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{D}_1(\aleph,t)}{\mathrm{d}t} &= \mathfrak{Re}\{i\aleph\hat{f}_t\bar{\hat{j}} - i\aleph\hat{j}_t\bar{\hat{f}} + i\aleph\hat{m}_t\bar{\hat{b}} - i\aleph\hat{b}_t\bar{\hat{m}}\} \\
&= -\aleph^2|\hat{j}|^2 - a\aleph^2|\hat{b}|^2 + \aleph^2|\hat{f}|^2 + a\aleph^2|\hat{m}|^2 - \mathfrak{Re}\{i\aleph\hat{m}\bar{\hat{j}}\} + \mathfrak{Re}\{i\aleph\hat{f}\bar{\hat{b}}\} \\
&\quad + \mathfrak{Re}\{i\gamma_1\aleph\hat{\zeta}\bar{\hat{b}}\} - \mathfrak{Re}\{i\mu_1\aleph\hat{m}\bar{\hat{b}}\} - \mathfrak{Re}\{\gamma_0\aleph^2\hat{\zeta}\bar{\hat{f}}\} \\
&\quad - \mathfrak{Re}\left\{i\xi\int_{\tau_1}^{\tau_2}\mu_2(s)\bar{\hat{b}}\,\hat{\Upsilon}(\aleph,1,s,t)\mathrm{d}s\right\}.
\end{aligned}
\tag{2.15}
$$

From (2.15) and the inequality of Young, there exists $\delta_1, \delta_2 > 0$ such that

$$
\begin{aligned}
-\mathfrak{Re}\{i\aleph\hat{m}\bar{\hat{j}}\} &\le \delta_1\aleph^2|\hat{j}|^2 + c(\delta_1)|\hat{m}|^2, \\
+\mathfrak{Re}\{i\aleph\hat{f}\bar{\hat{b}}\} &\le \delta_2\aleph^2|\hat{b}|^2 + c(\delta_2)|\hat{f}|^2, \\
+\mathfrak{Re}\{i\gamma_1\aleph\hat{\zeta}\bar{\hat{b}}\} &\le \delta_2\aleph^2|\hat{b}|^2 + c(\delta_2)|\hat{\zeta}|^2, \\
-\mathfrak{Re}\{i\mu_1\aleph\hat{m}\bar{\hat{b}}\} &\le \delta_2\aleph^2|\hat{b}|^2 + c(\delta_2)|\hat{m}|^2, \\
-\mathfrak{Re}\left\{i\xi\int_{\tau_1}^{\tau_2}\mu_2(s)\bar{\hat{b}}\,\hat{\Upsilon}(\aleph,1,s,t)\mathrm{d}s\right\} &\le \delta_2\aleph^2|\hat{b}|^2 + c(\delta_2)\int_{\tau_1}^{\tau_2}|\mu_2(s)||\hat{\Upsilon}(\aleph,1,s,t)|^2\mathrm{d}s, \\
-\mathfrak{Re}\{\gamma_0\aleph^2\hat{\zeta}\bar{\hat{f}}\} &\le c\aleph^2|\hat{\zeta}|^2 + c\aleph^2|\hat{f}|^2.
\end{aligned}
\tag{2.16}
$$

Substituting (2.16) into (2.15) and take $\delta_1 = \frac{1}{2}$, $\delta_2 = \frac{a}{8}$, we obtained (2.14). □

Lemma 2.3. *The functional*

$$
\mathcal{D}_2(\aleph,t) := \mathfrak{Re}\{i\aleph(\kappa\beta\hat{\varpi}\bar{\hat{\zeta}})\}
\tag{2.17}
$$

holds, for any $\varepsilon_1 > 0$,

$$
\frac{\mathrm{d}\mathcal{D}_2(\aleph,t)}{\mathrm{d}t} \le -\frac{\beta}{2}\aleph^2|\hat{\zeta}|^2 + \varepsilon_1\frac{\aleph^4}{(1+\aleph^2)^2}|\hat{j}|^2 + c\aleph^2|\hat{m}|^2 + c(\varepsilon_1)(1+\aleph^2)^2|\hat{\varpi}|^2.
\tag{2.18}
$$

Proof. For the required proof, differentiate $\mathcal{D}_2$ and using (2.6), we have

$$
\begin{aligned}
\frac{\mathcal{D}_2(\aleph,t)}{\mathrm{d}t} &= \mathfrak{Re}\{i\aleph(\kappa\beta\hat{\varpi}_t\bar{\hat{\zeta}} + \kappa\beta\hat{\varpi}\bar{\hat{\zeta}}_t)\} \\
&= -\beta\aleph^2|\hat{\zeta}|^2 + k\kappa\aleph^2|\hat{\varpi}|^2 - \mathfrak{Re}\{i\aleph\beta\hat{\varpi}\bar{\hat{\zeta}}\} + \mathfrak{Re}\{i\gamma_1\kappa\aleph\hat{m}\bar{\hat{\varpi}}\} - \mathfrak{Re}\{\gamma_0\kappa\aleph^2\hat{j}\bar{\hat{\varpi}}\}.
\end{aligned}
\tag{2.19}
$$

In the same way, from (2.19) and the inequality of Young, there exits $\varepsilon_1, \delta_3 > 0$ such that

$$
\begin{aligned}
-\mathfrak{Re}\{i\aleph\beta\hat{\varpi}\bar{\hat{\zeta}}\} &\le \delta_3\aleph^2|\hat{\zeta}|^2 + c(\delta_3)|\hat{\varpi}|^2, \\
+\mathfrak{Re}\{i\gamma_1\kappa\aleph\hat{m}\bar{\hat{\varpi}}\} &\le c\aleph^2|\hat{m}|^2 + c\,|\hat{\varpi}|^2, \\
-\mathfrak{Re}\{\gamma_0\kappa\aleph^2\hat{j}\bar{\hat{\varpi}}\} &\le \varepsilon_1\frac{\aleph^4}{(1+\aleph^2)^2}|\hat{j}|^2 + c(\varepsilon_1)(1+\aleph^2)^2|\hat{\varpi}|^2.
\end{aligned}
\tag{2.20}
$$

Here, substituting (2.20) into (2.19) and assuming $\delta_3 = \frac{\beta}{2}$, the required (2.18) is obtained. □

Lemma 2.4. *The functional*

$$
\mathcal{D}_3(\aleph,t) := -\mathfrak{Re}\{\hat{f}\bar{\hat{m}} + a\hat{j}\bar{\hat{b}}\}
\tag{2.21}
$$

fulfills

- *If* $a = 1$. *For any* $\varepsilon_2 > 0$, *we have*

$$\frac{\mathrm{d}\mathcal{D}_3(\aleph,t)}{\mathrm{d}t} \le -\frac{1}{2}|\hat{f}|^2 + \varepsilon_2\frac{\aleph^2}{1+\aleph^2}|\hat{b}|^2 + c|\hat{m}|^2 + c(\varepsilon_2)(1+\aleph^2)|\hat{\zeta}|^2 + c\int_{\tau_1}^{\tau_2}|\mu_2(s)||\hat{\Upsilon}(\aleph,1,s,t)|^2\mathrm{d}s. \tag{2.22}$$

- *For* $a \neq 1$. *Then, for any* $\varepsilon_2, \varepsilon_3 > 0$

$$\frac{\mathrm{d}\mathcal{D}_3(\aleph,t)}{\mathrm{d}t} \le -\frac{1}{2}|\hat{f}|^2 + \varepsilon_2\frac{\aleph^2}{1+\aleph^2}|\hat{b}|^2 + \varepsilon_3\frac{\aleph^2}{1+\aleph^2}|\hat{j}|^2 + c(\varepsilon_3)(1+\aleph^2)|\hat{m}|^2 + c(\varepsilon_2)(1+\aleph^2)|\hat{\zeta}|^2 + c\int_{\tau_1}^{\tau_2}|\mu_2(s)||\hat{\Upsilon}(\aleph,1,s,t)|^2\mathrm{d}s. \tag{2.23}$$

Proof. For the proof, differentiate $\mathcal{D}_3$ and utilize (2.6), we have the following

$$\frac{\mathrm{d}\mathcal{D}_3(\aleph,t)}{\mathrm{d}t} = -|\hat{f}|^2 + |\hat{m}|^2 - \Re e\{\gamma_1\hat{\zeta}\overline{\hat{f}}\} + \Re e\{\mu_1\hat{m}\overline{\hat{f}}\} + \Re e\{ia\gamma_0\aleph\hat{\zeta}\overline{\hat{b}}\} - \Re e\{i(1-a^2)\aleph\hat{j}\,\overline{\hat{m}}\} + \Re e\left\{\int_{\tau_1}^{\tau_2}\mu_2(s)\overline{\hat{f}}\,\hat{\Upsilon}(\aleph,1,s,t)\mathrm{d}s\right\}. \tag{2.24}$$

Here, we have two cases:

Case 1. ($a = 1$).

Using the inequality of Young and (2.24), there exists $\varepsilon_2, \delta_4 > 0$ such that

$$\begin{aligned}
-\Re e\{\gamma_1\hat{\zeta}\overline{\hat{f}}\} &\le \delta_4|\hat{f}|^2 + c(\delta_4)|\hat{\zeta}|^2,\\
\Re e\{\mu_1\hat{m}\overline{\hat{f}}\} &\le \delta_4|\hat{f}|^2 + c(\delta_4)|\hat{m}|^2,\\
\Re e\left\{\int_{\tau_1}^{\tau_2}\mu_2(s)\overline{\hat{f}}\,\hat{\Upsilon}(\aleph,1,s,t)\mathrm{d}s\right\} &\le \delta_4|\hat{f}|^2 + c(\delta_4)\int_{\tau_1}^{\tau_2}|\mu_2(s)||\hat{\Upsilon}(\aleph,1,s,t)|^2\mathrm{d}s,\\
\Re e\{ia\gamma_0\aleph\hat{\zeta}\overline{\hat{b}}\} &\le \varepsilon_2\frac{\aleph^2}{1+\aleph^2}|\hat{b}|^2 + c(\varepsilon_2)(1+\aleph^2)|\hat{\zeta}|^2.
\end{aligned} \tag{2.25}$$

By substituting (2.25) in (2.24) and assuming $\delta_4 = \frac{1}{6}$, the required (2.22) is obtained.

Case 2. ($a \neq 1$).

Applying the inequality of Young and (2.24), there exists $\varepsilon_3 > 0$ such that

$$(a^2-1)\Re e\{i\aleph\hat{j}\,\overline{\hat{m}}\} \le \varepsilon_3\frac{\aleph^2}{1+\aleph^2}|\hat{j}|^2 + c(\varepsilon_3)(1+\aleph^2)|\hat{m}|^2. \tag{2.26}$$

By substituting (2.26) and (2.25) in (2.24), the required (2.23) is obtained, which completes the proof.

Lemma 2.5. *The functional*

$$\mathcal{D}_4(\aleph,t) := \int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)||\hat{\Upsilon}(\aleph,\rho,s,t)|^2\mathrm{d}s\mathrm{d}\rho$$

satisfies

$$\frac{\mathrm{d}\mathcal{D}_4(\aleph, t)}{\mathrm{d}t} \leq -\eta_1 \int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)||\widehat{\Upsilon}(\aleph, \rho, s, t)|^2 \mathrm{d}s\mathrm{d}\rho + \mu_1 |\widehat{m}|^2 - \eta_1 \int_{\tau_1}^{\tau_2} |\mu_2(s)||\widehat{\Upsilon}(\aleph, 1, s, t)|^2 \mathrm{d}s, \tag{2.27}$$

where $\eta_1 > 0$.

Proof. For the proof, differentiate $\mathcal{D}_4$ and utilize $(2.6)_7$, we have

$$\frac{\mathrm{d}\mathcal{D}_4(\aleph, t)}{\mathrm{d}t} = -\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)||\widehat{\Upsilon}(\aleph, \rho, s, t)|^2 \mathrm{d}s\mathrm{d}\rho - \int_{\tau_1}^{\tau_2} |\mu_2(s)|[e^{-s}\,|\widehat{\Upsilon}(\aleph, 1, s, t)|^2 - |\widehat{\Upsilon}(\aleph, 0, s, t)|^2]\mathrm{d}s.$$

Utilizing $\Upsilon(\aleph, 0, s, t) = \mathcal{K}_t(\aleph, t) = m$, and $e^{-s} \leq e^{-s\rho} \leq 1$, for all $0 < \rho < 1$, one have

$$\frac{\mathrm{d}\mathcal{D}_4(\aleph, t)}{\mathrm{d}t} \leq -\int_0^1\int_{\tau_1}^{\tau_2} se^{-s}|\mu_2(s)||\widehat{\Upsilon}(\aleph, \rho, s, t)|^2 \mathrm{d}s\mathrm{d}\rho - \int_{\tau_1}^{\tau_2} e^{-s}|\mu_2(s)||\widehat{\Upsilon}(\aleph, 1, s, t)|^2 \mathrm{d}s + \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\mathrm{d}s\right) |\widehat{m}|^2.$$ □

We have $-e^{-s} \leq -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$. Because $-e^{-s}$ is a monotonically increasing function, we can finally derive (2.27) by setting $\eta_1 = e^{-\tau_2}$ and recalling (1.5). □

For the two cases, we introduce the Lyapunov functional as follows:

$$\mathcal{J}(\aleph, t) \coloneqq N(1 + \aleph^2)^2\widehat{E}(\aleph, t) + \frac{\aleph^2}{1 + \aleph^2}\left\{\frac{1}{1 + \aleph^2}N_1\mathcal{D}_1(\aleph, t) + N_3\mathcal{D}_3(\aleph, t)\right\} + N_2\mathcal{D}_2(\aleph, t) + N_4(1 + \aleph^2)^2\mathcal{D}_4(\aleph, t), \tag{2.28}$$

in which the positive constants N, N_i, $i = 1, 2, 3, 4$ will be selected in a later stage.

Lemma 2.6. *There exist* $\mu_3, \mu_4, \mu_5 > 0$ *in a manner that the functional* $\mathcal{J}(\aleph, t)$ *given in* (2.28)*fulfills*

$$\begin{cases} \mu_3(1 + \aleph^2)^2\widehat{E}(\aleph, t) \leq \mathcal{J}(\aleph, t) \leq \mu_4(1 + \aleph^2)^2\widehat{E}(\aleph, t), \\ \mathcal{J}'(\aleph, t) \leq -\mu_5\rho(\aleph)\mathcal{J}(\aleph, t), \quad \forall t > 0. \end{cases} \tag{2.29}$$

where

$$\rho(\aleph) = \frac{\aleph^4}{(1 + \aleph^2)^4}. \tag{2.30}$$

Proof. If $a = 1$, differentiate (2.28) and utilize (2.8), (2.14), (2.18), (2.22), and (2.27) with $\frac{\aleph^2}{1+\aleph^2} \leq \min\{1, \aleph^2\}$ and $\frac{1}{1+\aleph^2} \leq 1$, one have

$$\begin{aligned}\mathcal{J}'(\aleph, t) \leq &-\frac{\aleph^4}{(1+\aleph^2)^2}\left\{\left[\frac{1}{2}N_1 - \varepsilon_1 N_2\right]|\hat{j}|^2 + \left[\frac{a}{2}N_1 - \varepsilon_2 N_3\right]|\hat{b}|^2\right\} \\ &- (1+\aleph^2)^2[C_1 N - cN_1 - cN_2 - cN_3 - cN_4]|\widehat{m}|^2 \\ &- (1+\aleph^2)^2[kN - c(\varepsilon_1)N_2]|\widehat{\varpi}|^2 - \frac{\aleph^2}{1+\aleph^2}\left[\frac{1}{2}N_3 - cN_1\right]|\hat{f}|^2 \\ &- \aleph^2\left[\frac{\beta}{2}N_2 - cN_1 - c(\varepsilon_2)N_3\right]|\hat{\zeta}|^2 \\ &- (1+\aleph^2)^2[\eta_1 N_4 - cN_1 - cN_3]\int_{\tau_1}^{\tau_2}|\mu_2(s)||\widehat{\Upsilon}(\aleph, 1, s, t)|^2 ds \\ &- (1+\aleph^2)^2\eta_1 N_4\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)||\widehat{\Upsilon}(\aleph, \rho, s, t)|^2 ds d\rho.\end{aligned} \tag{2.31}$$

By setting

$$\varepsilon_1 = \frac{N_1}{4N_2}, \qquad \varepsilon_2 = \frac{aN_1}{4N_3}.$$

we obtain

$$\begin{aligned}\mathcal{J}'(\aleph, t) \leq &-\frac{\aleph^4}{(1+\aleph^2)^2}N_1\left\{\frac{1}{4}|\hat{j}|^2 + \frac{a}{4}|\hat{b}|^2\right\} \\ &- (1+\aleph^2)^2[C_1 N - cN_1 - cN_2 - cN_3 - cN_4]|\widehat{m}|^2 \\ &- (1+\aleph^2)^2[kN - c(N_1, N_2)N_2]|\widehat{\varpi}|^2 - \frac{\aleph^2}{1+\aleph^2}\left[\frac{1}{2}N_3 - cN_1\right]|\hat{f}|^2 \\ &- \aleph^2\left[\frac{\beta}{2}N_2 - cN_1 - c(N_1, N_3)N_3\right]|\hat{\zeta}|^2 \\ &- (1+\aleph^2)^2[\eta_1 N_4 - cN_1 - cN_3]\int_{\tau_1}^{\tau_2}|\mu_2(s)||\widehat{\Upsilon}(\aleph, 1, s, t)|^2 ds \\ &- (1+\aleph^2)^2\eta_1 N_4\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)||\widehat{\Upsilon}(\aleph, \rho, s, t)|^2 ds d\rho.\end{aligned} \tag{2.32}$$

In the same way, if $a \neq 1$, differentiate (2.28) and utilize (2.8), (2.14), (2.18),(2.23), and (2.27) with $\frac{\aleph^2}{1+\aleph^2} \leq \min\{1, \aleph^2\}$ and $\frac{1}{1+\aleph^2} \leq 1$, one have

$$\begin{aligned}\mathcal{J}'(\aleph, t) \leq &-\frac{\aleph^4}{(1+\aleph^2)^2}\left\{\left[\frac{1}{2}N_1 - \varepsilon_1 N_2 - \varepsilon_3 N_3\right]|\hat{j}|^2 + \left[\frac{a}{2}N_1 - \varepsilon_2 N_3\right]|\hat{b}|^2\right\} \\ &- (1+\aleph^2)^2[C_1 N - cN_1 - cN_2 - c(\varepsilon_3)N_3 - cN_4]|\widehat{m}|^2 \\ &- (1+\aleph^2)^2[kN - c(\varepsilon_1)N_2]|\widehat{\varpi}|^2 - \frac{\aleph^2}{1+\aleph^2}\left[\frac{1}{2}N_3 - cN_1\right]|\hat{f}|^2 \\ &- \aleph^2\left[\frac{\beta}{2}N_2 - cN_1 - c(\varepsilon_2)N_3\right]|\hat{\zeta}|^2 \\ &- (1+\aleph^2)^2[\eta_1 N_4 - cN_1 - cN_3]\int_{\tau_1}^{\tau_2}|\mu_2(s)||\widehat{\Upsilon}(\aleph, 1, s, t)|^2 ds \\ &- (1+\aleph^2)^2\eta_1 N_4\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)||\widehat{\Upsilon}(\aleph, \rho, s, t)|^2 ds d\rho.\end{aligned} \tag{2.33}$$

By setting

$$\varepsilon_1 = \frac{N_1}{8N_2}, \qquad \varepsilon_2 = \frac{aN_1}{4N_3}, \qquad \varepsilon_3 = \frac{N_1}{8N_3}.$$

we obtain

$$\begin{aligned}\mathcal{J}'(\varkappa, t) \le &-\frac{\varkappa^4}{(1+\varkappa^2)^2} N_1\left\{\frac{1}{4}|\hat{j}|^2 + \frac{a}{4}|\hat{b}|^2\right\} \\ &- (1+\varkappa^2)^2[C_1 N - cN_1 - cN_2 - c(N_1, N_3)N_3 - cN_4]|\widehat{m}|^2 \\ &- (1+\varkappa^2)^2[kN - c(N_1, N_2)N_2]|\widehat{\varpi}|^2 - \frac{\varkappa^2}{1+\varkappa^2}\left[\frac{1}{2}N_3 - cN_1\right]|\hat{f}|^2 \\ &- \varkappa^2\left[\frac{\beta}{2}N_2 - cN_1 - c(N_1, N_3)N_3\right]|\hat{\zeta}|^2 \\ &- (1+\varkappa^2)^2[\eta_1 N_4 - cN_1 - cN_3]\int_{\tau_1}^{\tau_2}|\mu_2(s)||\hat{\Upsilon}(\varkappa, 1, s, t)|^2 \mathrm{d}s \\ &- (1+\varkappa^2)^2\eta_1 N_4\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)||\hat{\Upsilon}(\varkappa, \rho, s, t)|^2 \mathrm{d}s\mathrm{d}\rho.\end{aligned} \tag{2.34}$$

Next, for the two cases (2.32) and (2.34), we fixed N_1 and choosing N_3 large in a manner that

$$\frac{1}{2}N_3 - cN_1 > 0,$$

next we choose N_4 large enough such that

$$\eta_1 N_4 - cN_1 - cN_3 > 0,$$

then we select N_2 large enough such as

$$\frac{\beta}{2}N_2 - cN_1 - c(N_1, N_3)N_3 > 0.$$

Hence, for the two cases, we arrive at

$$\begin{aligned}\mathcal{J}'(\varkappa, t) \le &-\frac{\varkappa^4}{(1+\varkappa^2)^2}(\alpha_0|\hat{j}|^2 + \alpha_1|\hat{b}|^2) - (1+\varkappa^2)^2[C_1 N - c]|\widehat{m}|^2 \\ &- \frac{\varkappa^2}{1+\varkappa^2}\alpha_2|\hat{f}|^2 - \varkappa^2\alpha_3|\hat{\zeta}|^2 - (1+\varkappa^2)^2[kN - c]|\widehat{\varpi}|^2 \\ &- (1+\varkappa^2)^2\alpha_4\int_{\tau_1}^{\tau_2}|\mu_2(s)||\hat{\Upsilon}(\varkappa, 1, s, t)|^2 \mathrm{d}s \\ &- (1+\varkappa^2)^2\alpha_5\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)||\hat{\Upsilon}(\varkappa, \rho, s, t)|^2 \mathrm{d}s\mathrm{d}\rho.\end{aligned} \tag{2.35}$$

In addition, we have

$$\begin{aligned}|\mathcal{J}(\varkappa, t) - N(1+\varkappa^2)^2\hat{E}(\varkappa, t)| = &N_1\frac{\varkappa^2}{(1+\varkappa^2)^2}|\mathcal{D}_1(\varkappa, t)| + N_3\frac{\varkappa^2}{1+\varkappa^2}|\mathcal{D}_3(\varkappa, t)| \\ &+ N_2|\mathcal{D}_2(\varkappa, t)| + (1+\varkappa^2)^2 N_4|\mathcal{D}_4(\varkappa, t)|.\end{aligned}$$

Using the inequality of Young and the inequality $\frac{\varkappa^2}{1+\varkappa^2} \le \min\{1, \varkappa^2\}$ and $\frac{1}{1+\varkappa^2} \le 1$, one obtained

$$|\mathcal{J}(\varkappa, t) - N(1+\varkappa^2)^2\hat{E}(\varkappa, t)| \le c(1+\varkappa^2)^2\hat{E}(\varkappa, t).$$

Thus, we have

$$(N - c)(1+\varkappa^2)^2\hat{E}(\varkappa, t) \le \mathcal{J}(\varkappa, t) \le (N + c)(1+\varkappa^2)^2\hat{E}(\varkappa, t). \tag{2.36}$$

Next, we pick N in a manner that

$$N - c > 0, \quad C_1 N - c > 0, \quad kN - c > 0,$$

and exploiting (2.7), estimates (2.35) and (2.36), one can find a positive constant $\alpha > 0$, $\forall t > 0$ and $\forall \aleph \in \mathbb{R}$ in a manner that

$$\mu_3(1 + \aleph^2)^2 \hat{E}(\aleph, t) \le \mathcal{J}(\aleph, t) \le \mu_4(1 + \aleph^2)^2 \hat{E}(\aleph, t) \tag{2.37}$$

and

$$\mathcal{J}'(\aleph, t) \le -\alpha \frac{\aleph^4}{(1 + \aleph^2)^2}(|\hat{j}|^2 + |\hat{b}|^2 + |\hat{\varpi}|^2 + |\hat{f}|^2 + |\hat{m}|^2 + |\hat{\zeta}|^2 + h(|\hat{\Upsilon}|^2)), \tag{2.38}$$

where

$$h(|\hat{\Upsilon}|^2) = \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)||\hat{\Upsilon}(\aleph, \rho, s, t)|^2 \mathrm{d}s\mathrm{d}\rho.$$

Then

$$\mathcal{J}'(\aleph, t) \le -\lambda_1 \rho(\aleph) \hat{E}(\aleph, t), \quad \forall t \ge 0. \tag{2.39}$$

As a result, for some positive constant $\mu_5 = \frac{\lambda_1}{\mu_4} > 0$, we have

$$\mathcal{J}'(\aleph, t) \le -\mu_5 \rho(\aleph) \mathcal{J}(\aleph, t), \quad \forall t \ge 0, \tag{2.40}$$

in which $\rho(\aleph) = \frac{\aleph^4}{(1+\aleph^2)^4}$, for some $\lambda_1, \mu_i > 0$, $i = 3, 4, 5$. Hence, the required proof is completed. □

2.2 Decay estimates

Theorem 2.7. *Assume s and $F_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Then, the solution F of* (2.2)*and* (2.3)*satisfies the below decay estimates*

$$\|\partial_x^k F(t)\|_2 \le C\|F_0\|_1(1 + t)^{-\frac{1}{8}-\frac{k}{4}} + C(1 + t)^{-\frac{\ell}{4}}\|\partial_x^{k+\ell} F_0\|_2. \tag{2.41}$$

Here, ℓ, s, and k represent nonnegative integers with the condition that $k + \ell \le s$, and $C > 0$ stands for a positive constant.

Proof. To prove the result, $|\hat{F}(\aleph, t)|^2 \sim \hat{E}(\aleph, t)$ can be concluded easily, and then by using the theorem of Plancherel, we obtain

$$\begin{aligned} \|\partial_x^k F(t)\|_2^2 &= \int_{\mathbb{R}} |\aleph|^{2k} |\hat{F}(\aleph, t)|^2 \mathrm{d}\aleph \\ &\le c \int_{\mathbb{R}} |\aleph|^{2k} e^{-\mu_5 \rho(\aleph) t} |\hat{F}(\aleph, 0)|^2 \mathrm{d}\aleph \\ &\le \underbrace{c \int_{|\aleph| \le 1} |\aleph|^{2k} e^{-\mu_5 \rho(\aleph) t} |\hat{F}(\aleph, 0)|^2 \mathrm{d}\aleph}_{R_1} \\ &\quad + \underbrace{c \int_{|\aleph| \ge 1} |\aleph|^{2k} e^{-\mu_5 \rho(\aleph) t} |\hat{F}(\aleph, 0)|^2 \mathrm{d}\aleph}_{R_2}. \end{aligned} \tag{2.42}$$

Next, we estimate R_1 and R_2, the low-frequency part $|\aleph| \le 1$ and the high-frequency part $|\aleph| \ge 1$, respectively.

Here, we have $\rho(\aleph) \geq \frac{1}{16}\aleph^4$, for $|\aleph| \leq 1$. Then

$$\begin{aligned} R_1 &\leq c \int\limits_{|\aleph|\leq 1} |\aleph|^{2k} e^{-\frac{\mu_5}{16}|\aleph|^4 t} |\widehat{F}(\aleph, 0)|^2 \mathrm{d}\aleph \\ &\leq c \sup_{|\aleph|\leq 1}\{|\widehat{F}(\aleph, 0)|^2\} \int\limits_{|\aleph|\leq 1} |\aleph|^{2k} e^{-\frac{\mu_5}{16}|\aleph|^4 t} \mathrm{d}\aleph, \end{aligned} \tag{2.43}$$

and by using Lemma 1.1, one have

$$R_1 \leq c \sup_{|\aleph|\leq 1}\{|\widehat{F}(\aleph, 0)|^2\}(1+t)^{-\frac{k}{2}-\frac{1}{4}} \leq c\|F_0\|_1^2(1+t)^{-\frac{k}{2}-\frac{1}{4}}. \tag{2.44}$$

After that, we have $\rho(\aleph) \geq \frac{1}{16}\aleph^{-4}$, for $|\aleph| \geq 1$. Then

$$R_2 \leq c \int\limits_{|\aleph|\geq 1} |\aleph|^{2k} e^{-\frac{\mu_5}{16}|\aleph|^{-4} t} |\widehat{F}(\aleph, 0)|^2 \mathrm{d}\aleph, \quad \forall t \geq 0. \tag{2.45}$$

Exploiting the inequality

$$\sup_{|\aleph|\geq 1}\{|\aleph|^{-2\ell} e^{-c|\aleph|^{-2} t}\} \leq C(1+t)^{-\ell}, \tag{2.46}$$

we obtain that

$$\begin{aligned} R_2 &\leq c \sup_{|\aleph|\geq 1}\left\{|\aleph|^{-2\ell} e^{-\frac{\mu_5}{16}|\aleph|^{-4} t}\right\} \int\limits_{|\aleph|\geq 1} |\aleph|^{2(k+\ell)}|\widehat{F}(\aleph, 0)|^2 \mathrm{d}\aleph \\ &\leq c(1+t)^{-\frac{\ell}{2}}\|\partial_x^{k+\ell} F(x, 0)\|_2^2, \quad \forall t \geq 0. \end{aligned} \tag{2.47}$$

By substituting (2.44) and (2.47) into (2.42), we find (2.41). □

3 Conclusion

In this study, we explored the general decay behavior of solutions in a one-dimensional Lord-Shulman Timoshenko system with thermal effects and a distributed delay term. We established optimal decay outcomes for the L^2-norm of the solution, specifically demonstrating that the decay rate follows the form $(1 + t)^{-1/8}$. For the proof of our results, the energy method in the Fourier space is used to build some very delicate Lyapunov functionals. Furthermore, the inclusion of mechanical damping $\mu_1\mathcal{K}_t$ appears to be essential for our approach in our system (1.8). It is an interesting problem to prove the same result for the equality $\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\mathrm{d}s = \mu_1\right)$ in the hypothesis (1.5), and we will attempt to utilize the same methodology for prove this result.

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