Research Article

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Some properties of a class of holomorphic functions associated with tangent function

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Abstract: In this study, we define new class of holomorphic functions associated with tangent function. Furthermore, we examine the differential subordination implementation results related to Janowski and tangent functions. Also, we investigate some extreme point theorem and partial sums results, necessary and sufficient conditions, convex combination, closure theorem, growth and distortion bounds, and radii of close-to-starlikeness and starlikeness for this newly defined functions class of holomorphic functions.

Keywords: holomorphic functions, subordination, tangent function, convolution, Janowski functions

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1 Introduction

We denote by $\mathcal{A}$ the class of all analytic functions h defined in Ω, where

$$\Omega = \{\xi : \xi \in \mathbb{C} \quad \text{and} \quad |\xi| < 1\},$$

and with the usual normalization:

$$h(0) = 0 \quad \text{and} \quad h'(0) = 1.$$

Thus, each $h \in \mathcal{A}$ has the following Taylor series expansion:

$$h(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \quad \xi \in \Omega. \tag{1.1}$$

Also, let $\mathcal{S}$ denotes a subfamily of $\mathcal{A}$, whose members are univalent in the open unit disk Ω.

Moreover, for two functions $h_1, h_2 \in \mathcal{A}$, the expression $h_1 \prec h_2$ denotes that the function h_1 is subordinate to the function h_2 if there exists the holomorphic function μ having properties of the holomorphic function, i.e.,

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$$|\mu(\xi)| \leq |\xi| \quad \text{and} \quad \mu(0) = 0$$

such that

$$h_1(\xi) = h_2(\mu(\xi)), \qquad \forall \xi \in \Omega.$$

Also, if $h_2 \in \mathcal{S}$, then the aforementioned conditions can be expressed as follows:

$$h_1 \prec h_2 \Leftrightarrow h_1(0) = h_2(0) \quad \text{and} \quad h_1(\Omega) \subset h_2(\Omega).$$

In 1992, Ma and Minda defined [1]

$$\mathcal{S}^*(\phi) = \left\{ h \in \mathcal{A} : \frac{\xi h'(\xi)}{h(\xi)} \prec \phi(\xi) \right\}, \tag{1.2}$$

with $\Re(\phi) > 0$ in Ω; additionally, the function ϕ maps Ω onto a star-shaped region and the image domain is symmetric about the real axis and starlike with respect to $\phi(0) = 1$ with $\phi'(0) > 0$. The set $\mathcal{S}^*(\phi)$ generalizes a number of subfamilies of the function class $\mathcal{A}$, including, for instance:

(1) If

$$\phi(\xi) = \frac{1 + L\xi}{1 + M\xi},$$

with $-1 \leq M < L \leq 1$, then

$$\mathcal{S}^*[L, M] \equiv \mathcal{S}^*\left(\frac{1 + L\xi}{1 + M\xi}\right),$$

where $\mathcal{S}^*[L, M]$ is the class of Janowski starlike functions, [2].

(2) For the function

$$\phi(\xi) = 1 + \sinh^{-1} \xi,$$

we obtain the class $\mathcal{S}^*_\rho$, introduced by Kumar and Arora [3].

(3) If we take

$$\phi(\xi) = 1 + \tanh \xi,$$

the class becomes $\mathcal{S}^*_{\tanh}$, which is introduced by Ullah et al. [4].

(4) If $\phi(\xi) = e^\xi$, then the class $\mathcal{S}^*(\phi)$ becomes $\mathcal{S}^*_e$, which is defined and studied by Mendiratta [5].

(5) For $\phi(\xi) = 1 + \sin(\xi)$, the class $\mathcal{S}^*(\phi)$ reduces to the class $\mathcal{S}^*_{\sin}$. The family $\mathcal{S}^*_s$ introduced by Cho et al. [6] is defined as:

$$\mathcal{S}^*_{\sin} = \left\{ h \in \mathcal{A} : \frac{\xi h'(\xi)}{h(\xi)} \prec 1 + \sin \xi, \quad (\xi \in \Omega) \right\}, \tag{1.3}$$

which means that $\frac{\xi h'(\xi)}{h(\xi)}$ lies in an eight-shaped region.

(6) The class $C_{\sin}$ of convex functions related to sine function is introduced by Zulfiqar et al. [7] as:

$$C_{\sin} = \left\{ h \in \mathcal{A} : \frac{(\xi h'(\xi))'}{h'(\xi)} \prec 1 + \sin \xi, \quad \xi \in \Omega \right\}. \tag{1.4}$$

The natural extensions of differential inequalities on the real line in the complex plane are known as differential subordinations. Derivative knowledge is essential for understanding the properties of real-valued functions. Several differential implications may be found in the complex plane, when a function is described by means of differential conditions. The univalency criteria for analytical functions are provided by the Noshiro-Warschawski theorem, an example of such differential implication. The range of the combination of the function's derivatives is frequently used to determine the properties of a function.

Let g be a holomorphic function defined on Ω with $g(0)=1$. Recently, Ali et al. [8] have investigated some differential subordination results. More specifically, they studied the differential subordination:

$$1+\frac{\alpha\xi g'(\xi)}{g^{n}(\xi)}\prec\sqrt{1+\xi},$$

and found

$$g(\xi)\prec\sqrt{1+\xi},\quad n=0,1,2,$$

for some particular range of α. Similar types of results have been investigated by various authors, and one can found the articles contributed by Kumar et al. [9,10], Paprocki et al. [11], Raza et al. [12], Shi et al. [13], and Khan et al. [14].

We are essentially motivated by all of the aforementioned work and the recent research going on in differential subordinations, and we defined a subfamily of holomorphic functions based on the tangent function as follows:

$$\mathcal{S}^{*}_{\tan}=\left\{h\in\mathcal{A}:\frac{\xi h'(\xi)}{h(\xi)}\prec 1+\frac{\tan(\xi)}{2};\quad(\xi\in\Omega)\right\}. \tag{1.5}$$

Geometric interpretation of the fact $f\in\mathcal{S}^{*}_{\tan}$ is that the image of $\frac{\xi h'(\xi)}{h(\xi)}$ under open unit disk lies in the region bounded by domain $1+\frac{\tan(\xi)}{2}$. It is clear that a function $f\in\mathcal{S}^{*}_{\tan}$ if there exists the holomorphic functions q with property that $q(\xi)\prec q_0(\xi)=1-\frac{\tan(\xi)}{2}$, such that

$$f(\xi)=\xi\exp\left(\int_0^{\xi}\frac{q(t)-1}{t}\mathrm{d}t\right).$$

If we take $q_0(\xi)=1+\frac{\tan(\xi)}{2}$, then

$$f(\xi)=\xi\exp\left(\int_0^{\xi}\frac{\frac{\tan(t)}{2}}{t}\mathrm{d}t\right)=\xi+\frac{1}{2}\xi^{2}+\frac{1}{8}\xi^{3}+\frac{11}{144}\xi^{4}+\cdots.$$

The aforementioned function plays the rule of extremal function for many problems.

Our further investigation is organized as follows. In Section 2, we give some necessary and sufficient conditions and the differential subordination implementation results related to Janowski and tangent functions. We also investigate some extreme point for our defined functions classes. The convex combination, closure theorem, growth and distortion bounds, and radii of close-to-starlikeness and starlikeness are also included in this section. In Section 3, we find the partial sum results, for this newly defined functions class of holomorphic functions. At the end, a new section, Conclusion, is included, where we concluded our work and also give a direction for future extension of this work.

We require the following lemma in order to verify our primary findings in the next sections.

Lemma 1.1. [15] *Let μ be holomorphic in Ω with $\mu(0)=0$. If $|\mu(\xi)|$ attains its maximum value on the circle $|\xi|=r$ at a point $\xi_0=re^{i\theta}$, for $\theta\in[-\pi,\pi]$, we can write*

$$\xi_0\mu'(\xi_0)=k\mu(\xi_0),$$

where m is real with $m\geq 1$.

2 Necessary and sufficient conditions

Theorem 2.1. *Let* $-1 \le L_2 < \frac{\sec h^2(1)}{\sec^2(1)} \le L_1 \le 1$ *and suppose*

$$1 + \alpha\xi h'(\xi) \prec \frac{1 + L_1\xi}{1 + L_2\xi}, \quad \xi \in \Omega.$$

If

$$|\alpha| \ge \frac{2(L_1 - L_2)}{\sec h^2(1) - |L_2| \sec^2(1)}, \tag{2.1}$$

then

$$h(\xi) \prec 1 + \frac{1}{2}\tan(\xi), \quad \xi \in \Omega.$$

Proof. Assume $h(\xi) = 1 + \frac{1}{2}\tan(\mu(\xi))$, where μ is the holomorphic function with $\mu(0) = 0$. Let

$$k(\xi) = 1 + \alpha\xi h'(\xi) = 1 + \frac{\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{2}.$$

Now, consider

$$\left|\frac{k(\xi) - 1}{L_1 - L_2 k(\xi)}\right| = \left|\frac{\frac{\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{2}}{L_1 - L_2\left(1 + \frac{\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{2}\right)}\right| = \left|\frac{\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{2(L_1 - L_2) + L_2\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}\right|.$$

Now, to achieve our goal, we have to prove that $|\mu(\xi)| < 1$, $\xi \in \Omega$. In contrast, assume that ξ_0 belongs to Ω in such a way that $\max_{|\xi| \le |\xi_0|} |\mu(\xi)| = |\mu(\xi_0)| = 1$, then by virtue of Lemma 1.1, there exists $k \ge 1$, such that $\xi_0\mu'(\xi_0) = k\mu(\xi_0)$. Let $\mu(\xi_0) = e^{i\theta}$ for $\theta \in [0, \pi]$. Then,

$$\left|\frac{k(\xi_0) - 1}{L_1 - L_2 k(\xi_0)}\right| \ge \frac{k|\alpha||\sec(e^{i\theta})|^2}{2(L_1 - L_2) + |L_2|k|\alpha||\sec(e^{i\theta})|^2}. \tag{2.2}$$

A direct computation gives that

$$|\sec(e^{i\theta})|^2 = \left|\frac{1}{\cos(\cos(\theta))\cosh(\sin(\theta)) - i\sin(\cos(\theta))\sinh(\sin(\theta))}\right|^2 = \frac{1}{\cos^2(\cos(\theta)) + \cosh^2(\sin(\theta)) - 1} = \varphi(\theta).$$

Since $\varphi(-\theta) = \varphi(\theta)$, consider $\theta \in [0, \pi]$. Then,

$$\min\{\varphi(\theta)\} = \varphi\left(\frac{\pi}{2}\right) = \sec h^2(1),$$
$$\max\{\varphi(\theta)\} = \varphi(0) = \varphi(\pi) = \sec^2(1).$$

Therefore,

$$\sec h^2(1) \le |\sec(e^{i\theta})|^2 \le \sec^2(1). \tag{2.3}$$

Substituting (2.3) into (2.2), we obtain

$$\left|\frac{k(\xi_0) - 1}{L_1 - L_2 k(\xi_0)}\right| \ge \frac{k|\alpha|\sec h^2(1)}{2(L_1 - L_2) + |L_2|k|\alpha|\sec^2(1)} = \phi_1(k).$$

Then,

$$\phi_1'(k) = \frac{2|\alpha|\sec h^2(1)(L_1 - L_2)}{(2(L_1 - L_2) + |L_2|k|\alpha|\sec^2(1))^2} > 0,$$

which shows that $\phi_1(k)$ is the increasing function and the maximum value is attained at $k = 1$ and so

$$\left|\frac{k(\xi_0) - 1}{L_1 - L_2 k(\xi_0)}\right| \geq \frac{|\alpha|\sec h^2(1)}{2(L_1 - L_2) + |L_2||\alpha|\sec^2(1)}.$$

Now, by (2.1), we have

$$\left|\frac{k(\xi_0) - 1}{L_1 - L_2 k(\xi_0)}\right| \geq 1,$$

which is a contradiction to the fact that $h(\xi) \prec \frac{1+L_1\xi}{1+L_2\xi}$, and hence, we obtain the required result. □

By setting $L_1 = 1$ and $L_2 = 0$, in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let*

$$1 + \alpha\xi h'(\xi) \prec 1 + \xi, \quad \xi \in \Omega.$$

If

$$|\alpha| \geq \frac{2}{\sec h^2(1)}, \tag{2.4}$$

then

$$h(\xi) \prec 1 + \frac{1}{2}\tan(\xi), \quad \xi \in \Omega.$$

Theorem 2.3. *Let* $-1 \leq L_2 < \frac{\sec h^2(1)}{\sec^2(1)} \leq L_1 \leq 1$ *and suppose*

$$1 + \alpha\frac{\xi h'(\xi)}{h(\xi)} \prec \frac{1 + L_1\xi}{1 + L_2\xi}, \quad \xi \in \Omega. \tag{2.5}$$

If

$$|\alpha| \geq \frac{(2 + \tan(1))(L_1 - L_2)}{\sec h^2(1) - |L_2|\sec^2(1)}, \tag{2.6}$$

then

$$h(\xi) \prec 1 + \frac{1}{2}\tan(\xi), \quad \xi \in \Omega.$$

Proof. Assume $h(\xi) = 1 + \frac{1}{2}\tan(\mu(\xi))$, where μ is the holomorphic function with $\mu(0) = 0$. Let

$$k_1(\xi) = 1 + \alpha\frac{\xi h'(\xi)}{h(\xi)} = 1 + \frac{\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{2 + \tan(\mu(\xi))}.$$

Now, consider

$$\left|\frac{k_1(\xi)-1}{L_1-L_2k_1(\xi)}\right| = \left|\frac{\frac{\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{2+\tan(\mu(\xi))}}{L_1-L_2\left(1+\frac{\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{2+\tan(\mu(\xi))}\right)}\right|$$
$$= \left|\frac{\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{(2+\tan(\mu(\xi)))(L_1-L_2)+L_2\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}\right|.$$

To obtain the required result, we have to prove that $|\mu(\xi)|<1$, $\xi\in\Omega$. On the contrary, assume that ξ_0 belongs to Ω in such a way that $\max_{|\xi|\le|\xi_0|}|\mu(\xi)| = |\mu(\xi_0)| = 1$; then by virtue of Lemma 1.1, there exists $k\ge 1$, such that $\xi_0\mu'(\xi_0) = k\mu(\xi_0)$. Let $\mu(\xi_0) = e^{i\theta}$ for $\theta\in[0,\pi]$. Then,

$$\left|\frac{k_1(\xi_0)-1}{L_1-L_2k_1(\xi_0)}\right| \ge \frac{k|\alpha||\sec(e^{i\theta})|^2}{(2+|\tan(e^{i\theta})|)(L_1-L_2)+|L_2|k|\alpha||\sec(e^{i\theta})|^2}. \tag{2.7}$$

A direct computation gives that

$$|\tan(e^{i\theta})|^2 = \left|\frac{\sin(\cos(x))\cosh(\sin(x))+i\cos(\cos(x))\sinh(\sin(x))}{\cos(\cos(\theta))\cosh(\sin(\theta))-i\sin(\cos(\theta))\sinh(\sin(\theta))}\right|^2$$
$$= \frac{\cosh^2(\sin(x))-\cos^2(\sin(x))}{\cos^2(\cos(\theta))+\cosh^2(\sin(\theta))-1} = \varphi_1(\theta).$$

Since $\varphi_1(-\theta) = \varphi_1(\theta)$, consider $\theta\in[0,\pi]$. Then,

$$\min\{\varphi_1(\theta)\} = \varphi_1\left(\frac{\pi}{2}\right) = \tanh^2(1),$$
$$\max\{\varphi_1(\theta)\} = \varphi_1(0) = \varphi_1(\pi) = \tan^2(1).$$

Therefore,

$$\tanh(1)\le|\tan(e^{i\theta})|\le\tan(1). \tag{2.8}$$

Substituting (2.3) and (2.8) into (2.7), we obtain

$$\left|\frac{k_1(\xi_0)-1}{L_1-L_2k_1(\xi_0)}\right| \ge \frac{k|\alpha|\operatorname{sec}h^2(1)}{(2+\tan(1))(L_1-L_2)+|L_2|k|\alpha|\sec^2(1)} = \phi_2(k).$$

Then,

$$\phi_2'(k) = \frac{|\alpha|\operatorname{sec}h^2(1)(2+\tan(1))(L_1-L_2)}{((2+\tan(1))(L_1-L_2)+|L_2|k|\alpha|\sec^2(1))^2} > 0,$$

which shows that $\phi_2(k)$ is the increasing function and the maximum value is attained at $k=1$ and so

$$\left|\frac{k_1(\xi_0)-1}{L_1-L_2k_1(\xi_0)}\right| \ge \frac{|\alpha|\operatorname{sec}h^2(1)}{(2+\tan(1))(L_1-L_2)+|L_2||\alpha|\sec^2(1)}.$$

Now, by (1.4), we have

$$\left|\frac{k_1(\xi_0)-1}{L_1-L_2k_1(\xi_0)}\right| \ge 1,$$

which contradicts (2.5), and so $|\mu(\xi)|<1$ for $|\xi|<1$, and hence, we obtain the desired result. □

By setting $L_1 = 1$ and $L_2 = 0$, in Theorem 2.3, we obtain the following result.

Corollary 2.4. *Let*

$$1 + \alpha\frac{\xi h'(\xi)}{h(\xi)} \prec 1 + \xi, \quad \xi \in \Omega. \tag{2.9}$$

If

$$|\alpha| \geq \frac{(2 + \tan(1))}{\sec h^2(1)}, \tag{2.10}$$

then

$$h(\xi) \prec 1 + \frac{1}{2}\tan(\xi), \quad \xi \in \Omega.$$

Theorem 2.5. *Let* $-1 \leq L_2 < \frac{\sec h^2(1)}{\sec^2(1)} \leq L_1 \leq 1$ *and suppose*

$$1 + \alpha\frac{\xi h'(\xi)}{(h(\xi))^2} \prec \frac{1 + L_1\xi}{1 + L_2\xi}, \quad \xi \in \Omega. \tag{2.11}$$

If

$$|\alpha| \geq \frac{(2 + \tan(1))^2(L_1 - L_2)}{2[\sec h^2(1) - |L_2|\sec^2(1)]}, \tag{2.12}$$

then

$$h(\xi) \prec 1 + \frac{1}{2}\tan(\xi), \quad \xi \in \Omega.$$

Proof. Assume $h(\xi) = 1 + \frac{1}{2}\tan(\mu(\xi))$, where μ is the holomorphic function with $\mu(0) = 0$. Let

$$k_2(\xi) = 1 + \alpha\frac{\xi h'(\xi)}{(h(\xi))^2} = 1 + \frac{2\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{(2 + \tan(\mu(\xi)))^2}.$$

Now, consider

$$\begin{aligned}\left|\frac{k_2(\xi) - 1}{L_1 - L_2k_2(\xi)}\right| &= \left|\frac{\frac{2\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{(2+\tan(\mu(\xi)))^2}}{L_1 - L_2\left(1 + \frac{2\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{(2+\tan(\mu(\xi)))^2}\right)}\right| \\ &= \left|\frac{2\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{(2 + \tan(\mu(\xi)))^2(L_1 - L_2) + 2L_2\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}\right|.\end{aligned}$$

To prove the required result, we have to prove that $|\mu(\xi)| < 1, \xi \in \Omega$. In contrast, assume that ξ_0 belongs to Ω in such a way that $\max_{|\xi|\leq|\xi_0|}|\mu(\xi)| = |\mu(\xi_0)| = 1$; then, by virtue of Lemma 1.1, there exists $k \geq 1$, such that $\xi_0\mu'(\xi_0) = k\mu(\xi_0)$. Let $\mu(\xi_0) = e^{i\theta}$ for $\theta \in [0, \pi]$. Then,

$$\left|\frac{k_2(\xi_0) - 1}{L_1 - L_2k_2(\xi_0)}\right| \geq \frac{2k|\alpha||\sec(e^{i\theta})|^2}{(2 + |\tan(e^{i\theta})|)^2(L_1 - L_2) + 2|L_2|k|\alpha||\sec(e^{i\theta})|^2}. \tag{2.13}$$

Substituting (2.3) and (2.8) into (2.13), we obtain

$$\left|\frac{k_2(\xi_0) - 1}{L_1 - L_2k_2(\xi_0)}\right| \geq \frac{2k|\alpha|\sec h^2(1)}{(2 + \tan(1))^2(L_1 - L_2) + 2|L_2|k|\alpha|\sec^2(1)} = \phi_3(k).$$

Then,

$$\phi_3'(k) = \frac{2|\alpha|\sec h^2(1)(2 + \tan(1))^2(L_1 - L_2)}{((2 + \tan(1))^2(L_1 - L_2) + 2|L_2|k|\alpha|\sec^2(1))^2} > 0,$$

which shows that $\phi_3(k)$ is the increasing function and the maximum value is attained at $k = 1$ and so

$$\left|\frac{k_2(\xi_0) - 1}{L_1 - L_2 k_2(\xi_0)}\right| \geq \frac{2|\alpha| \operatorname{sec}h^2(1)}{(2 + \tan(1))^2 (L_1 - L_2) + 2|L_2||\alpha| \sec^2(1)}.$$

Now, by (2.12), we have

$$\left|\frac{k_2(\xi_0) - 1}{L_1 - L_2 k_2(\xi_0)}\right| \geq 1,$$

which contradicts (2.11), and so $|\mu(\xi)| < 1$ for $|\xi| < 1$, and hence, we obtain the desired result. □

By setting $L_1 = 1$ and $L_2 = 0$, in Theorem 2.5, we obtain the following result.

Corollary 2.6. *Let*

$$1 + \alpha \frac{\xi h'(\xi)}{(h(\xi))^2} \prec 1 + \xi, \quad \xi \in \Omega. \tag{2.14}$$

If

$$|\alpha| \geq \frac{(2 + \tan(1))^2}{2 \operatorname{sec}h^2(1)}, \tag{2.15}$$

then

$$h(\xi) \prec 1 + \frac{1}{2}\tan(\xi), \quad \xi \in \Omega.$$

Theorem 2.7. *Let* $-1 \leq L_2 < \frac{\operatorname{sec}h^2(1)}{\sec^2(1)} \leq L_1 \leq 1$ *and suppose*

$$1 + \alpha \frac{\xi h'(\xi)}{(h(\xi))3} \prec \frac{1 + L_1\xi}{1 + L_2\xi}, \quad \xi \in \Omega. \tag{2.16}$$

If

$$|\alpha| \geq \frac{(2 + \tan(1))^3 (L_1 - L_2)}{4[\operatorname{sec}h^2(1) - |L_2| \sec^2(1)]}, \tag{2.17}$$

then

$$h(\xi) \prec 1 + \frac{1}{2}\tan(\xi), \quad \xi \in \Omega.$$

Proof. Assume that $h(\xi) = 1 + \frac{1}{2}\tan(\mu(\xi))$, where μ is the holomorphic function with $\mu(0) = 0$. Let

$$k_3(\xi) = 1 + \alpha \frac{\xi h'(\xi)}{(h(\xi))^3} = 1 + \frac{4\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{(2 + \tan(\mu(\xi)))^3}.$$

Now, consider

$$\left|\frac{k_3(\xi) - 1}{L_1 - L_2 k_3(\xi)}\right| = \left|\frac{\frac{4\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{(2 + \tan(\mu(\xi)))^3}.}{L_1 - L_2\left(1 + \frac{4\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{(2 + \tan(\mu(\xi)))^3}.\right)}\right|$$

$$= \left|\frac{4\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}{(2 + \tan(\mu(\xi)))^3 (L_1 - L_2) + 4L_2\alpha\xi\mu'(\xi)\sec^2(\mu(\xi))}\right|.$$

To obtain the required conditions, we have to prove that $|\mu(\xi)| < 1, \xi \in \Omega$. Contrary, assume that ξ_0 belongs to Ω in such a way that $\max_{|\xi|\leq|\xi_0|}|\mu(\xi)| = |\mu(\xi_0)| = 1$, then by virtue of Lemma 1.1, there exists $k \geq 1$, such that $\xi_0\mu'(\xi_0) = k\mu(\xi_0)$. Let $\mu(\xi_0) = e^{i\theta}$ for $\theta \in [0, \pi]$. Then, we have

$$\left|\frac{k_3(\xi_0) - 1}{L_1 - L_2k_3(\xi_0)}\right| \geq \frac{4k|\alpha||\sec(e^{i\theta})|^2}{(2 + |\tan(e^{i\theta})|)^3(L_1 - L_2) + 4|L_2|k|\alpha||\sec(e^{i\theta})|^2}. \tag{2.18}$$

Adjusting (2.3) and (2.8) into (2.18), we obtain

$$\left|\frac{k_3(\xi_0) - 1}{L_1 - L_2k_3(\xi_0)}\right| \geq \frac{4k|\alpha|\sec h^2(1)}{(2 + \tan(1))^3(L_1 - L_2) + 4|L_2|k|\alpha|\sec^2(1)} = \phi_4(k).$$

Then,

$$\phi_4'(k) = \frac{4|\alpha|\sec h^2(1)(2 + \tan(1))^3(L_1 - L_2)}{((2 + \tan(1))^3(L_1 - L_2) + 4|L_2|k|\alpha|\sec^2(1))^2} > 0,$$

which shows that $\phi_4(k)$ is the increasing function and the maximum value is attained at $k = 1$ and so

$$\left|\frac{k_3(\xi_0) - 1}{L_1 - L_2k_3(\xi_0)}\right| \geq \frac{4|\alpha|\sec h^2(1)}{(2 + \tan(1))^3(L_1 - L_2) + 4|L_2||\alpha|\sec^2(1)}.$$

Now, by (2.12), we have

$$\left|\frac{k_3(\xi_0) - 1}{L_1 - L_2k_3(\xi_0)}\right| \geq 1,$$

which contradicts (2.16), and so $|\mu(\xi)| < 1$ for $|\xi| < 1$, and hence, we obtain the desired result. □

By setting $L_1 = 1$ and $L_2 = 0$, in Theorem 2.7, we obtain the following result.

Corollary 2.8. *Let*

$$1 + \alpha\frac{\xi h'(\xi)}{(h(\xi))3} \prec 1 + \xi, \quad \xi \in \Omega. \tag{2.19}$$

If

$$|\alpha| \geq \frac{(2 + \tan(1))^3}{\sec h^2(1) - \sec^2(1)}, \tag{2.20}$$

then

$$h(\xi) \prec 1 + \frac{1}{2}\tan(\xi), \quad \xi \in \Omega.$$

Theorem 2.9. *Suppose that $h \in \mathcal{A}$ and of the form given in* (1.1) *then $h \in \mathcal{S}^*_{\tan}$ if and only if*

$$\frac{1}{\xi}\left[h(\xi) * \frac{\xi - H\xi^2}{(1 - \xi)(1 - q\xi)}\right] \neq 0, \tag{2.21}$$

where

$$H = \frac{2 + \tan(e^{i\theta})}{4 + \tan(e^{i\theta})} \quad \textit{and also for } H = 1 \tag{2.22}$$

Proof. For any function $h \in \mathcal{S}^*_{\tan}$ holomorphic in Ω, follows that $\frac{1}{\xi}h(\xi) \neq 0$ for all ξ in $\Omega^* = \Omega - \{0\}$. Thus, we can have equation (2.21) if $H = 1$. Now, by using (1.5) along with the principal of subordination, a Schwarz function μ exists with $\mu(0) = 0$ and $|\mu(\xi)| < 1$, so

$$\frac{\xi h'(\xi)}{h(\xi)} = 1 + \frac{\tan(\mu(\xi))}{2}.$$

Here we consider $\mu(\xi) = e^{i\theta}$, where $0 \le \theta \le 2\pi$ then above expression becomes

$$\frac{\xi h'(\xi)}{h(\xi)} \neq 1 + \frac{\tan(e^{i\theta})}{2}. \tag{2.23}$$

$$\xi h'(\xi) - \left(\frac{2 + \tan(e^{i\theta})}{2}\right) h(\xi) \neq 0. \tag{2.24}$$

Now, by using the relation

$$h(\xi) = h(\xi) * \frac{\xi}{1 - \xi} \quad \text{and}\, \xi h'(\xi) = h(\xi) * \frac{\xi}{(1 - \xi)^2},$$

then equation (2.24), becomes

$$h(\xi) * \frac{\xi}{(1 - \xi)^2} - \left(\frac{2 + \tan(e^{i\theta})}{2}\right)\left(h(\xi) * \frac{\xi}{1 - \xi}\right) \neq 0.$$

After some simple calculations, we obtain

$$h(\xi) * \left(\frac{\left(\frac{4 + \tan(e^{i\theta})}{2}\right)\xi - \left(\frac{2 + \tan(e^{i\theta})}{2}\right)\xi^2}{(1 - \xi)^2}\right) \neq 0.$$

$$h(\xi) * \left(\frac{\xi - H\xi^2}{(1 - \xi)^2}\right) \neq 0,$$

where $H = \frac{2 + \tan(e^{i\theta})}{4 + \tan(e^{i\theta})}$; thus, the necessary condition

$$\frac{1}{\xi}\left[h(\xi) * \left(\frac{\xi - H\xi^2}{(1 - \xi)^2}\right)\right] \neq 0$$

holds.

Conversely, let the condition in (2.21) holds, then $\frac{1}{\xi} h(\xi) \neq 0$ for all $\xi \in \Omega$. Let $\xi(\xi) = \frac{\xi h'(\xi)}{h(\xi)}$, which is holomorphic in Ω and $\xi(0) = 1$. Furthermore, suppose that $h(\xi) = 1 + \frac{\tan(\xi)}{2}$ and then from equation (2.23), it is clear that $h(\partial\Omega) \cap \xi(\xi) = \Phi$. Therefore, the connected component $\mathbb{C} \backslash h(\partial\Omega)$ contains the domain $\xi(\xi)$, which is also connected. Now, by the univalence of "ξ" together with the assumption $h(0) = \xi(0) = 1$, it is clear that $\xi \prec h$ so $h(\xi) \in \mathcal{S}^*_{\tan}$. □

Theorem 2.10. *Suppose that $h \in \mathcal{A}$ and of the form given in* (1.1), *then $h(\xi) \in \mathcal{S}^*_{\tan}$, if and only if*

$$1 - \sum_{n=2}^{\infty} \frac{-2n - 2 - \tan(e^{i\theta})}{4 + \tan(e^{i\theta})} a_n \xi^{n-1} \neq 0. \tag{2.25}$$

Proof. From Theorem (3.1), a function $h \in \mathcal{S}^*_{\tan}$, then (2.21) holds true:

$$
\begin{aligned}
0 &\neq \frac{1}{\xi}\left[h(\xi) * \left(\frac{\xi - H\xi^2}{(1-\xi)^2}\right)\right] \\
&= \frac{1}{\xi}\left[h(\xi) * \frac{\xi}{(1-\xi)^2} - h(\xi) * \frac{H\xi^2}{(1-\xi)^2}\right] \\
&= \frac{1}{\xi}\left[h(\xi) * \frac{\xi}{(1-\xi)^2} - H\left(h(\xi) * \frac{\xi^2}{(1-\xi)^2} - h(\xi) * \frac{\xi}{(1-\xi)^2}\right)\right] \\
&= (1-H)h'(\xi) + H\frac{h(\xi)}{\xi} \\
&= 1 - \sum_{n=2}^{\infty}(n(H-1) - H)a_n\xi^{n-1} \\
&= 1 - \sum_{n=2}^{\infty}\frac{-2n - 2 - \tan(e^{i\theta})}{4 + \tan(e^{i\theta})}a_n\xi^{n-1}.
\end{aligned}
$$

□

Theorem 2.11. *If $h \in \mathcal{A}$ is defined in* (1.1) *and satisfies the condition*

$$\sum_{n=2}^{\infty}\left|\frac{2n + 2 + \tan(e^{i\theta})}{4 + \tan(e^{i\theta})}\right||a_n| < 1, \tag{2.26}$$

then $h \in \mathcal{S}^*_{\tan}$.

Proof. To prove the required result, we consider (2.25):

$$\left|1 - \sum_{n=2}^{\infty}\frac{-2n - 2 - \tan(e^{i\theta})}{4 + \tan(e^{i\theta})}a_n\xi^{n-1}\right| > 1 - \sum_{n=2}^{\infty}\left|\frac{2n + 2 + \tan(e^{i\theta})}{4 + \tan(e^{i\theta})}\right||a_n||\xi^{n-1}|.$$

By using the given condition in (2.26), we have

$$1 - \sum_{n=2}^{\infty}\left|\frac{2n + 2 + \tan(e^{i\theta})}{4 + \tan(e^{i\theta})}\right||a_n| > 0. \tag{2.27}$$

So by Theorems 2.1 and 2.2, we see that $h \in \mathcal{S}^*_{\tan}$. □

Theorem 2.12. *Let $h \in \mathcal{S}^*_{\tan}$. and $|\xi| = r$. Then,*

$$r - \left|\frac{4 + \tan(e^{i\theta})}{6 + \tan(e^{i\theta})}\right| r^2 \leq |h(\xi)| \leq r + \left|\frac{4 + \tan(e^{i\theta})}{6 + \tan(e^{i\theta})}\right| r^2. \tag{2.28}$$

Proof. Consider

$$|h(\xi)| = \left|\xi + \sum_{n=2}^{\infty}a_n\xi^n\right| \leq r + \sum_{n=2}^{\infty}|a_n|r^n.$$

Since $r^n \leq r^2$ for $n \geq 2$ and $r < 1$, we have

$$|h(\xi)| \leq r + r^2\sum_{n=2}^{\infty}|a_n|. \tag{2.29}$$

Similarly,

$$|h(\xi)| \geq r - r^2\sum_{n=2}^{\infty}|a_n|. \tag{2.30}$$

Now, from (2.26), it implies that

$$\sum_{n=2}^{\infty}\left|\frac{2n+2+\tan(e^{i\theta})}{4+\tan(e^{i\theta})}\right||a_n|<1.$$

Since

$$\left|\frac{6+\tan(e^{i\theta})}{4+\tan(e^{i\theta})}\right|\sum_{n=2}^{\infty}|a_n|\le\sum_{n=2}^{\infty}\left|\frac{2n+2+\tan(e^{i\theta})}{4+\tan(e^{i\theta})}\right||a_n|,$$

from this, we obtain

$$\left|\frac{6+\tan(e^{i\theta})}{4+\tan(e^{i\theta})}\right|\sum_{n=2}^{\infty}|a_n|<1.$$

One can easily write this as:

$$\sum_{n=2}^{\infty}|a_n|<\left|\frac{4+\tan(e^{i\theta})}{6+\tan(e^{i\theta})}\right|.$$

Now, substituting the aforementioned value into (2.29) and (2.30), we obtain the desired inequality. □

Theorem 2.13. *Let $h\in\mathcal{S}^*_{\tan}$ and $|\xi|=r$. Then,*

$$1-2\left|\frac{4+\tan(e^{i\theta})}{6+\tan(e^{i\theta})}\right|r\le|h'(\xi)|\le1+2\left|\frac{4+\tan(e^{i\theta})}{6+\tan(e^{i\theta})}\right|r. \tag{2.31}$$

Proof. Consider

$$|h'(\xi)|=\left|1+\sum_{n=2}^{\infty}na_n\xi^n\right|\le1+\sum_{n=2}^{\infty}|a_n|r^{n-1}.$$

Since $r^{n-1}\le r$ for $n\ge2$ and $r<1$, we have

$$|h'(\xi)|\le1+2r\sum_{n=2}^{\infty}|a_n|. \tag{2.32}$$

Similarly,

$$|h'(\xi)|\ge1-2r\sum_{n=2}^{\infty}|a_n|. \tag{2.33}$$

Now, from (2.26), it implies that

$$\sum_{n=2}^{\infty}\left|\frac{2n+2+\tan(e^{i\theta})}{4+\tan(e^{i\theta})}\right||a_n|<1.$$

Since

$$\left|\frac{6+\tan(e^{i\theta})}{4+\tan(e^{i\theta})}\right|\sum_{n=2}^{\infty}|a_n|\le\sum_{n=2}^{\infty}\left|\frac{2n+2+\tan(e^{i\theta})}{4+\tan(e^{i\theta})}\right||a_n|,$$

from this, we obtain

$$\left|\frac{6+\tan(e^{i\theta})}{4+\tan(e^{i\theta})}\right|\sum_{n=2}^{\infty}|a_n|<1.$$

One can easily write this as:

$$\sum_{n=2}^{\infty}|a_n| < \left|\frac{4+\tan(e^{i\theta})}{6+\tan(e^{i\theta})}\right|.$$

Now, inserting this value into (2.32) and (2.33), we obtain the desired inequality. □

Theorem 2.14. *Let $h_k \in \mathcal{S}^*_{\tan}$, $k = 1, 2, \ldots,$ such that*

$$h_k(\xi) = \xi + \sum_{n=2}^{\infty} a_{n,k}\xi^n.$$

Then, $H(\xi) = \sum_{k=1}^{\infty}\eta_k h_k(\xi)$, where $\sum_{n=1}^{\infty}\eta_k = 1$ is in class $\mathcal{S}^*_{\tan}$.

Proof. We have

$$H(\xi) = \xi + \sum_{k=1}^{\infty}\sum_{n=2}^{\infty}\eta_k a_{n,k}\xi^n.$$

Due to Theorem 2.ii, it is sufficient to prove

$$\begin{aligned}\sum_{n=2}^{\infty}&\left|\frac{2n+2+\tan(e^{i\theta})}{4+\tan(e^{i\theta})}\right|\eta_k|a_{n,k}| \\ &= \sum_{k=1}^{\infty}\eta_k\left(\sum_{n=2}^{\infty}\left|\frac{2n+2+\tan(e^{i\theta})}{4+\tan(e^{i\theta})}\right||a_{n,k}|\right) \\ &< \sum_{k=1}^{\infty}\eta_k = 1.\end{aligned}$$

Hence, $H(\xi) \in \mathcal{S}^*_{\tan}$. □

Theorem 2.15. *The class $\mathcal{S}^*_{\tan}$ is closed under convex combination.*

Proof. Let h_1 and h_2 be any functions in set $\mathcal{S}^*_{\tan}$ with the following series representation:

$$h_1(\xi) = \xi + \sum_{n=2}^{\infty}a_n\xi^n \quad \text{and} \quad h_2(\xi) = \xi + \sum_{n=2}^{\infty}b_n\xi^n.$$

We have to show that $h(\xi) = \lambda h_1(\xi) + (1-\lambda)h_2(\xi)$, with $0 \le \lambda \le 1$, is in the class $\mathcal{S}^*_{\tan}$. Since

$$h(\xi) = \xi + \sum_{n=2}^{\infty}[\lambda a_n + (1-\lambda)b_n]\xi^n.$$

Consider

$$\begin{aligned}\sum_{n=2}^{\infty}&\left|\frac{2n+2+\tan(e^{i\theta})}{4+\tan(e^{i\theta})}\right||\lambda a_n + (1-\lambda)b_n| \\ &\le \lambda\sum_{n=2}^{\infty}\left|\frac{2n+2+\tan(e^{i\theta})}{4+\tan(e^{i\theta})}\right||a_n| + (1-\lambda)\sum_{n=2}^{\infty}\left|\frac{2n+2+\tan(e^{i\theta})}{4+\tan(e^{i\theta})}\right||b_n| \\ &< 1.\end{aligned}$$

Hence, $h \in \mathcal{S}^*_{\tan}$. □

Theorem 2.16. *Let $h \in \mathcal{S}^*_{\tan}$. Then, for $|\xi| = r_1$, the function is the starlike of order α, where*

$$r_1 = \inf\left\{\frac{(1-\alpha)(2n+2+\tan(e^{i\theta}))}{(n-\alpha)(4+\tan(e^{i\theta}))}\right\}^{\frac{1}{n-1}} \quad \textit{for } n \in \mathbb{N}\backslash\{1\}.$$

Proof. To show h is the starlike function of order α, it is enough to show that

$$\left|\frac{\xi h'(\xi) - h(\xi)}{\xi h'(\xi) - (2\alpha - 1)h(\xi)}\right| < 1.$$

After simple computation, we obtain

$$\left|\frac{\xi h'(\xi) - h(\xi)}{\xi h'(\xi) - (2\alpha - 1)h(\xi)}\right| \leq \sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}\right)|a_n||\xi|^{n-1}. \tag{2.34}$$

Also from (2.26), we have

$$\sum_{n=2}^{\infty}\left|\frac{2n + 2 + \tan(e^{i\theta})}{4 + \tan(e^{i\theta})}\right||a_n| < 1. \tag{2.35}$$

From (2.34), the function on the left-hand side is bounded by 1 if

$$\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}\right)|\xi|^{n-1} < \sum_{n=2}^{\infty}\left|\frac{2n + 2 + \tan(e^{i\theta})}{4 + \tan(e^{i\theta})}\right|,$$

which implies that

$$|\xi| < \left(\frac{(1-\alpha)(2n + 2 + \tan(e^{i\theta}))}{(n-\alpha)(4 + \tan(e^{i\theta}))}\right)^{\frac{1}{n-1}} = r_1.$$

Hence, the proof is completed. □

3 Partial sum results

Silverman [16] was the first who found the sharp bounds for the ratio of a function to its partial sums, considering the subclasses of regular functions. Then, many researchers took inspiration from the work of Silverman and have studied the partial sum problems for various q-subclasses. In recent past, Jabeen et al. [17] investigated the partial sums results for q-convex functions defined by the q-Ruscheweyh differential operator in conic regions. In this portion, we discuss some lower bounds results of the function defined in (1.1) to its partial sum, which is defined as:

$$h_m(\xi) = \xi + \sum_{n=2}^{m} a_n\xi^n, \quad (r \in \Omega).$$

Theorem 3.1. *For $h \in \mathcal{S}^*_{\tan}$, the coefficients of h are so small that satisfies Condition* (2.26), *then*

$$\Re\left(\frac{h(\xi)}{h_m(\xi)}\right) \geq 1 - \frac{1}{\varepsilon_{m+1}}, \quad (r \in \Omega) \tag{3.1}$$

and

$$\Re\left(\frac{h_m(\xi)}{h(\xi)}\right) \geq \frac{\varepsilon_{m+1}}{1 + \varepsilon_{m+1}} \quad (r \in \Omega), \tag{3.2}$$

where

$$\varepsilon_n = \left|\frac{2n + 2 + \tan(e^{i\theta})}{4 + \tan(e^{i\theta})}\right| \tag{3.3}$$

and

$$\varepsilon_n \geq \begin{cases} 1, & for\ \ m = 2, 3, 4 \ldots, n \\ \varepsilon_{m+1}, & for\ \ m = n + 1. \end{cases}$$

Proof. To show the inequality in (3.1), we set

$$\mu(\xi) = \varepsilon_{m+1}\left[\frac{h(\xi)}{h_m(\xi)} - \left(1 - \frac{1}{\varepsilon_{m+1}}\right)\right].$$

After some simple calculations, we have

$$\mu(\xi) = \left[\frac{\sum_{n=2}^{m} a_n\xi^{n-1} + \varepsilon_{m+1}\sum_{n=m+1}^{\infty} a_n\xi^{n-1} + 1}{\sum_{n=2}^{m} a_n\xi^{n-1}}\right].$$

Now, consider that

$$\left|\frac{\mu(\xi) - 1}{\mu(\xi) + 1}\right| = \left|\frac{\varepsilon_{m+1}\sum_{n=m+1}^{\infty} a_n\xi^{n-1}}{2 + 2\sum_{n=2}^{m} a_n\xi^{n-1} + \varepsilon_{m+1}\sum_{n=m+1}^{\infty} a_n\xi^{n-1}}\right|.$$

Using the triangular inequality, we obtain

$$\left|\frac{\mu(\xi) - 1}{\mu(\xi) + 1}\right| \leq \frac{\varepsilon_{m+1}\sum_{n=m+1}^{\infty}|a_m|}{2 - 2\sum_{n=2}^{m}|a_n| - \varepsilon_{m+1}\sum_{n=m+1}^{\infty}|a_n|}.$$

Now, $\left|\frac{\mu(r) - 1}{\mu(r) + 1}\right| \leq 1$, if and only if

$$\sum_{n=2}^{m}|a_n| + \varepsilon_{m+1}\sum_{n=m+1}^{\infty}|a_n| \leq 1. \tag{3.4}$$

For the desired result, it would be enough to show that the left-hand side of (3.4) is bounded above by $\sum_{n=2}^{\infty}\varepsilon_n|a_n|$, i.e.,

$$\sum_{n=2}^{m}|a_n| + \varepsilon_{m+1}\sum_{n=m+1}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}\varepsilon_n|a_n|.$$

On simple calculations, it gives

$$\sum_{n=2}^{m}(\varepsilon_n - 1)|a_n| + \sum_{n=m+1}^{\infty}(\varepsilon_n - \varepsilon_{n+1})|a_n| \geq 0,$$

which gives the proof of (3.1).

For (3.2), we set

$$\mu(\xi) = (1 + \varepsilon_{m+1})\left(\frac{h(\xi)}{h_m(\xi)} - \frac{\varepsilon_{m+1}}{1 + m_{x+1}}\right).$$

After some simple calculations, we have

$$\mu(\xi) = \frac{1 + \sum^{m}{}_{n=2} a_n\xi^{n-1} - \varepsilon_{m+1}\sum_{n=m+1}^{\infty} a_n\xi^{n-1}}{1 + \sum^{m}{}_{n=2} a_n\xi^{n-1}},$$

which gives

$$\left|\frac{\mu(\xi) - 1}{\mu(\xi) + 1}\right| \leq \frac{(1 + \varepsilon_{m+1})\sum_{n=m+1}^{\infty}|a_n|}{2 - 2\sum_{n=2}^{m}|a_n| - (\varepsilon_{m+1} - 1)\sum_{n=m+1}^{\infty}|a_n|}.$$

Now, $\left|\frac{\mu(\xi) - 1}{\mu(\xi) + 1}\right| \leq 1$, where

$$\sum_{n=2}^{m}|a_n| + \varepsilon_{m+1}\sum_{n=m+1}^{\infty}|a_n| \leq 1. \tag{3.5}$$

Finally, we can see that the left-hand side of the inequality in (3.5) is bounded above by:

$$\sum_{n=2}^{\infty}\varepsilon_n|a_n|,$$

so we have completed the proof of the second assertion. And so, we have completed the proof of our theorem. □

Theorem 3.2. *If* $h \in \mathcal{S}^{*}_{\tan}$, *of the form given in* (1.1), *then*

$$\Re\left(\frac{h'(\xi)}{h'_m(\xi)}\right) \geq 1 - \frac{m+1}{\varepsilon_{m+1}}, \quad (\forall r \in \Omega) \tag{3.6}$$

and

$$\Re\left(\frac{h'_m(\xi)}{h'(\xi)}\right) \geq \frac{\varepsilon_{m+1}}{m+1+\varepsilon_{m+1}}, \quad (\forall r \in \Omega), \tag{3.7}$$

where ε_m *is given in* (3.3).

Proof. The relevant details are skipped here because the proof is identical to that of Theorem 3.8. □

4 Conclusions

In this article, we have first introduced a new family of holomorphic tangent functions. Then, we examined some useful numbers of geometric properties of the holomorphic functions connected with trigonometric tangent function, which is symmetric about the real axis. These results comprising the implementation of subordination, necessary and sufficient conditions based on the use of the convolution operator, growth and distortion bounds, the closure theorem, convex combination, radii of starlikeness and close-to-starlikeness, the extreme point theorem, and partial sums results for newly defined classes have also been covered.

Furthermore, some other results such as coefficient estimates, logarithmic coefficient estimates, Hankel determinant, and many more results can be debated in future work. Moreover, this concept can be expanded to include meromorphic, multivalent, and quantum calculus functions. Also, one can apply the idea of basic (or q-) calculus and obtain the required results (see for details [18–26]).

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