Research Article

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On exact rate of convergence of row sequences of multipoint Hermite-Padé approximants

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Abstract: In this article, we analyze a rate of attraction of poles of an approximated function to poles of incomplete multipoint Padé approximants and use it to derive a sharp bound on the geometric rate of convergence of multipoint Hermite-Padé approximants to a vector of approximated functions in the Montessus de Ballore theorem when a table of interpolation points is Newtonian.

Keywords: Montessus de Ballore's theorem, multipoint Padé approximation, Hermite-Padé approximation, simultaneous approximation, rate of convergence

MSC 2020: 30E10, 41A21, 41A28

1 Introduction

The study of convergence of sequences of rational functions with a fixed number of free poles is a classical problem in approximation theory. Montessus de Ballore's classical theorem is one of the main objects in this problem. To clarify the goal of our work, let us remind the definition of classical Padé approximants:

Let $\mathbb{P}_n$ be the space of all polynomials of degree at most n. Consider a power series

$$f(z) = \sum_{k=0}^{\infty} f_k z^k. \tag{1.1}$$

For given $n, m \in \mathbb{N} \cup \{0\}$, we can find $P \in \mathbb{P}_n$ and $Q \in \mathbb{P}_m$, $Q \not\equiv 0$, such that

$$(Qf - P)(z) = O(z^{n+m+1}), \quad \text{as} \quad z \to 0.$$

The rational function

$$R_{n,m} := \frac{P}{Q} = \frac{P_{n,m}}{Q_{n,m}} \tag{1.2}$$

is called the (n, m) *classical Padé approximant of* f. The polynomials $P_{n,m}$ and $Q_{n,m}$ in equation (1.2) are chosen so that $Q_{n,m}$ is monic and $P_{n,m}$ and $Q_{n,m}$ do not have common zero. It is well known that for any $n, m \in \mathbb{N} \cup \{0\}$, $P_{n,m}$, $Q_{n,m}$, and $R_{n,m}$ are uniquely determined.

For a function f as in equation (1.1), we denote by $R_0(f)$ the radius of the largest open disk at the origin to which f can be extended analytically and by $R_m(f)$ the radius of the largest open disk at the origin to which f can be extended meromorphically with at most m poles counting multiplicities. Set

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$$\mathbb{B}(a, R) := \{z \in \mathbb{C} : |z - a| < R\}.$$

We denote by Q_m^f the monic polynomial whose zeros are the poles of f in $\mathbb{B}(0, R_m(f))$ counting their multiplicities and by $\mathcal{P}_m(f)$ the set of all distinct zeros of Q_m^f. In all what follows, m remains fixed and $\{R_{n,m}\}_{n\geq 0}$ is called the *mth row sequence of classical Padé approximants of f.*

The pioneering result in the study of row sequences of classical Padé approximants is the Montessus de Ballore theorem [1] stated that if $R_0(f) > 0$ and f has exactly m poles in $\mathbb{B}(0, R_m(f))$, then for each compact subset K of $\mathbb{B}(0, R_m(f))\backslash\mathcal{P}_m(f)$,

$$\limsup_{n\to\infty} \|f - R_{n,m}\|_K^{1/n} = \frac{\|z\|_K}{R_m(f)} \tag{1.3}$$

and

$$\limsup_{n\to\infty} \|Q_m^f - Q_{n,m}\|^{1/n} = \frac{\max\{|\lambda| : \lambda \in \mathcal{P}_m(f)\}}{R_m(f)}, \tag{1.4}$$

where $\|\cdot\|_K$ denotes a sup-norm on K and $\|\cdot\|$ denotes the coefficient norm in the space $\mathbb{P}_m$. The reason behind the convergence in equation (1.3) is that as $n \to \infty$, all poles of the approximants $R_{n,m}$ converge to all poles of f in $\mathbb{B}(0, R_m(f))$ as shown in equation (1.4). This allows $R_{n,m}$ to imitate f in $\mathbb{B}(0, R_m(f))$ with the geometric rate of convergence in (1.3). For a precise history, we would like to point out that Montessus de Ballore [1] verified the $\leq$ inequalities in equations (1.3) and (1.4) and the $\geq$ inequalities in equations (1.3) and (1.4) were later obtained by Gonchar [2,3].

In the last several decades, the Montessus de Ballore theorem was generalized in many contexts (see, for example, multipoint Padé approximants [4], orthogonal Padé approximants [5,6], and Hermite-Padé approximants [7,8]). In this study, we will investigate an analogue of the rate of convergence in equation (1.3) for multipoint Hermite-Padé (MHP) approximants defined as follows.

Let E be a bounded continuum subset of the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}}\backslash E$ is simply connected. By $\mathcal{H}(E)$, we denote the space of all functions holomorphic in some neighborhood of E. Set

$$\mathcal{H}(E)^d := \{(f_1, \ldots, f_d) : f_j \in \mathcal{H}(E), j = 1, \ldots, d\}.$$

Let $\alpha \subset E$ be a table of points; more precisely, $\alpha = \{\alpha_{n,k}\}$, $k = 1, \ldots, n$, $n = 1, 2, \ldots$. We say that the table α is *Newtonian* if for each $n \in \mathbb{N}$, $\alpha_{n,k} = \alpha_k$, for all $k = 1, \ldots, n$.

Definition 1. Let $\mathbf{f} \in \mathcal{H}(E)^d$ and $\alpha \subset E$ be a table of points. Fix a multi-index $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$ such that $n \geq |\mathbf{m}|$, where we define

$$|\mathbf{m}| := m_1 + \cdots + m_d.$$

Then, there exist polynomials $Q_{n,\mathbf{m}}$, $P_{n,\mathbf{m},k}$, $k = 1, \ldots, d$ such that
(i) $P_{n,\mathbf{m},k} \in \mathbb{P}_{n-m_k}$, $Q_{n,\mathbf{m}} \in \mathbb{P}_{|\mathbf{m}|}$, $Q_{n,\mathbf{m}} \not\equiv 0$,
(ii) $(Q_{n,\mathbf{m}} f_k - P_{n,\mathbf{m},k})/a_{n+1} \in \mathcal{H}(E)$, where

$$a_n(z) := \prod_{k=1}^{n} (z - \alpha_{n,k}).$$

The vector of rational functions

$$\mathbf{R}_{n,\mathbf{m}} := (R_{n,\mathbf{m},1}, \ldots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},1}, \ldots, P_{n,\mathbf{m},d})/Q_{n,\mathbf{m}} \tag{1.5}$$

is called *a multipoint Hermite-Padé (MHP)* $\mathbf{f}$ *with respect to* $\mathbf{m}$ *and* α.

For given $(n, \mathbf{m})$, the approximant in equation (1.5) always exists but is not uniquely determined. Without loss of generality, we can assume that $Q_{n,\mathbf{m}}$ is a monic polynomial that has no common zero simultaneously with all $P_{n,\mathbf{m},k}$. In all what follows, $\mathbf{m}$ remains fixed and $\{\mathbf{R}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ is called a *row sequence of MHP of* $\mathbf{f}$ *with respect to* $\mathbf{m}$.

When E is a disk centered at 0 and all interpolation points $\alpha_{n,k}$ are zero, this MHP approximation becomes classical type II Hermite-Padé approximation [9]. Most studies of classical Hermite-Padé approximants concentrated on diagonal or near-diagonal sequences and their applications. There are few studies devoted to row sequences [7,8, 10–12]. In the direction of row sequences, Graves-Morris and Saff [10] proved an analogue of the Montessus de Ballore theorem for classical Hermite-Padé approximants under the concept of polewise independence. Later, such study received a renewed interest by Cacoq et al. [7,8]. In [7], they refined the estimates of convergences of $\{Q_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ and $\{\mathbf{R}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ in [10]. In [8], they proved a reciprocal of the Montessus de Ballore theorem for classical Hermite-Padé approximants. See also further investigations [11,12] involving the relationship between asymptotic properties of zeros of $Q_{n,\mathbf{m}}$ and singularities of $\mathbf{f}$.

Recently, convergence problems of various generalizations of classical Hermite-Padé approximants on row sequences were considered in [13–16]. The results in [16] generalize the ones in [8] to MHP approximants. In particular, Bosuwan et al. [16] computed the exact rate of convergence of $\{Q_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ and estimated the rate of convergence of $\{\mathbf{R}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$. The aim of this study is to show that the rate of convergence of $\{\mathbf{R}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ found in [16] (see (1.11) below) is sharp.

Let Φ be the conformal bijection from $\overline{\mathbb{C}}\backslash E$ to $\overline{\mathbb{C}}\backslash\overline{\mathbb{B}(0,1)}$ with $\Phi(\infty)=\infty$ and $\Phi'(\infty)>0$. It is commonly known that there exist tables of points α satisfying the condition:

$$\lim_{n\to\infty}\frac{a_n(z)}{c^n\Phi^n(z)}=G(z)\neq 0, \tag{1.6}$$

uniformly on compact subsets of $\overline{\mathbb{C}}\backslash E$, where c denotes some positive constant and $G(z)$ is some holomorphic and nonvanishing function on $\overline{\mathbb{C}}\backslash E$, see Chapters 8–9 in [17]. Moreover, it is easy to check that equation (1.6) implies that

$$\lim_{n\to\infty}|a_n(z)|^{1/n}=c|\Phi(z)|, \tag{1.7}$$

uniformly on compact subsets of $\overline{\mathbb{C}}\backslash E$.

The shape of domain of convergence of $\{R_{n,\mathbf{m},k}\}_{n\geq|\mathbf{m}|}$ can be explained using the map Φ. For each $\rho>1$, we define

$$\Gamma_\rho := \{z\in\mathbb{C} : |\Phi(z)|=\rho\} \quad \text{and} \quad D_\rho := E\cup\{z\in\mathbb{C} : |\Phi(z)|<\rho\}$$

as the *level curve of index* ρ and the *canonical domain of index* ρ, respectively. Given $f\in\mathcal{H}(E)$, denote by $\rho_0(f)$ the index of the largest canonical domain D_ρ to which f can be holomorphically extended and by $\rho_m(f)$ the index of the largest canonical domain D_ρ to which f can be meromorphically extended with at most m poles counting multiplicities. Additionally, denote by $D_m(f)$ the canonical domain of the index $\rho_m(f)$.

In the study of row sequences of MHP approximants of $\mathbf{f}$, we consider “a system pole” of $\mathbf{f}$ instead of a pole of $\mathbf{f}$. The concept of system pole was originated by Cacoq et al. [8]. In [16], Bosuwan et al. modified their definition to fit the study of MHP approximants.

Definition 2. Given $\mathbf{f}=(f_1,\ldots,f_d)\in\mathcal{H}(E)^d$ and $\mathbf{m}=(m_1,\ldots,m_d)\in\mathbb{N}^d$, we say that $\lambda\in\mathbb{C}$ is a *system pole of order* τ *of* $(\mathbf{f},\mathbf{m})$ if τ is the largest positive integer such that for each $s=1,\ldots,\tau$, there exists at least one polynomial combination of the form:

$$\sum_{k=1}^{d}p_kf_k,\quad \deg p_k<m_k,\quad k=1,\ldots,d, \tag{1.8}$$

which is analytic in a neighborhood of $\overline{D}_{|\Phi(\lambda)|}$ except for a pole at $z=\lambda$ of exact order s.

Note that a concept of system pole depends on both a vector of functions $\mathbf{f}$ and a multi-index $\mathbf{m}$. However, in this study, we consider a row sequence of MHP of $\mathbf{f}$ with respect to $\mathbf{m}$, so sometimes, we will omit the multi-index $\mathbf{m}$. When $d=1$, the concepts of pole and system pole coincide. However, when $d>1$, the situation is not quite the same. Poles of the individual functions f_k may not be system poles of $\mathbf{f}$ and system poles of $\mathbf{f}$ may not be poles of any of the functions f_k as given in examples in [8]. More importantly, when studying the convergence of $\{Q_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$, instead of considering poles of each individual function f_k, we will consider poles of the polynomial combinations of all components $\mathbf{f}$ in equation (1.8).

Let τ be the order of λ as a system pole of $\mathbf{f}$. For each $s = 1, \ldots, \tau$, let $\rho_{\lambda,s}(\mathbf{f}, \mathbf{m})$ denote the largest of all the numbers $\rho_s(g)$, where g is a polynomial combination of type (1.8) that is holomorphic on a neighborhood of $\overline{D}_{|\Phi(\lambda)|}$ except for a pole at $z = \lambda$ of order s. Then, we define

$$R_{\lambda,s}(\mathbf{f}, \mathbf{m}) := \min_{k=1,\ldots,s} \rho_{\lambda,k}(\mathbf{f}, \mathbf{m})$$

and

$$R_{\lambda}(\mathbf{f}, \mathbf{m}) := R_{\lambda,\tau}(\mathbf{f}, \mathbf{m}) = \min_{k=1,\ldots,\tau} \rho_{\lambda,k}(\mathbf{f}, \mathbf{m}).$$

Fix $k \in \{1, \ldots, d\}$. To explain the domain of convergence of $\{R_{n,\mathbf{m},k}\}_{n\geq|\mathbf{m}|}$, we need to define the following index $R_k^*(\mathbf{f}, \mathbf{m})$ as follows. Let $D_k(\mathbf{f}, \mathbf{m})$ be the largest canonical domain in which all the poles of f_k are system poles of $\mathbf{f}$ with respect to $\mathbf{m}$, their order as poles of f_k does not exceed their order as system poles, and f_k has no other singularity. By $\rho_k(\mathbf{f}, \mathbf{m})$, we denote the index of this canonical domain. Let $\lambda_1, \ldots, \lambda_N$ be the poles of f_k in $D_k(\mathbf{f}, \mathbf{m})$. For each $j = 1, \ldots, N$, let $\hat{\tau}_j$ be the order of λ_j as pole of f_k and τ_j be its order as a system pole. By assumption, $\hat{\tau}_j \leq \tau_j$. Set

$$R_k^*(\mathbf{f}, \mathbf{m}) := \min\left\{\rho_k(\mathbf{f}, \mathbf{m}), \min_{j=1,\ldots,N} R_{\lambda_j,\hat{\tau}_j}(\mathbf{f}, \mathbf{m})\right\},$$

and let $D_k^*(\mathbf{f}, \mathbf{m})$ be the canonical domain with this index.

By $Q_{\mathbf{m}}^{\mathbf{f}}$, we denote the monic polynomial whose zeros are the system poles of $\mathbf{f}$ with respect to $\mathbf{m}$ taking account of their order. The set of distinct zeros of $Q_{\mathbf{m}}^{\mathbf{f}}$ is denoted by $\mathcal{P}_{\mathbf{m}}^{\mathbf{f}}$. The Montessus de Ballore theorem for MHP is the main result in [16].

Theorem 1.1. *Suppose* (1.6) *takes place. Let* $\mathbf{f} \in \mathcal{H}(E)^d$ *and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *Then, the following two assertions are equivalent*:

(*a*) $\mathbf{f}$ *has exactly* $|\mathbf{m}|$ *system poles with respect to* $\mathbf{m}$ *counting multiplicities.*

(*b*) *For all sufficiently large n, the denominators* $Q_{n,\mathbf{m}}$ *of MHP approximants of* $\mathbf{f}$ *are uniquely determined and there exists a polynomial* $Q_{\mathbf{m}}$ *of degree* $|\mathbf{m}|$ *such that*

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} - Q_{\mathbf{m}}\|^{1/n} = \theta < 1, \tag{1.9}$$

where $\|\cdot\|$ *denotes the coefficient norm in the space* $\mathbb{P}_{|\mathbf{m}|}$.

Moreover, if either (*a*) *or* (*b*) *takes place, then* $Q_{\mathbf{m}} \equiv Q_{\mathbf{m}}^{\mathbf{f}}$,

$$\theta = \max\left\{\frac{|\Phi(\lambda)|}{R_{\lambda}(\mathbf{f}, \mathbf{m})} : \lambda \in \mathcal{P}_{\mathbf{m}}^{\mathbf{f}}\right\}, \tag{1.10}$$

and for any compact subset K of $D_k^*(\mathbf{f}, \mathbf{m}) \setminus \mathcal{P}_{\mathbf{m}}^{\mathbf{f}}$,

$$\limsup_{n\to\infty} \|R_{n,\mathbf{m},k} - f_k\|_K^{1/n} \leq \frac{\|\Phi\|_K}{R_k^*(\mathbf{f}, \mathbf{m})}, \tag{1.11}$$

where $\|\cdot\|_K$ *denotes the sup-norm on K, and if* $K \subset E$, *then* $\|\Phi\|_K$ *is replaced by* 1.

Let us give one concrete example explaining the definition of system pole and Theorem 1.1.

Example 1. Let $\mathbf{m} = (1,1)$, $E = [-1,1]$, $\alpha \subset E$ be a table of interpolation points satisfying (1.6), and $\mathbf{f} = (f_1, f_2)$, where

$$f_1(z) := \frac{1}{z-2} + e^{1/(z-5)} + \frac{1}{z-6} \quad \text{and} \quad f_2(z) := \frac{1}{z-2} + e^{1/(z-5)}.$$

It is well known that $\Phi(z) = z + \sqrt{z^2-1}$, where the branch of the square root is chosen so that $|\Phi(z)| > 1$ outside $[-1,1]$. It is not difficult to see that 2 and 6 are system poles of order 1 of $\mathbf{f}$ with respect to $\mathbf{m}$, which further implies that $\mathbf{f}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$. Notice that 6 is not in the domains of meromorphy of f_1 and f_2 (extended from the set E) because of the essential singularity at 5. However, by (1.9) and (1.10), the zeros of $Q_{n,\mathbf{m}}$ can detect both 2 and 6 with the following rate of convergence:

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} - (z-2)(z-6)\|^{1/n} = \max\left\{\frac{|\Phi(2)|}{R_2(\mathbf{f},\mathbf{m})}, \frac{|\Phi(6)|}{R_6(\mathbf{f},\mathbf{m})}\right\} = \max\left\{\frac{|\Phi(2)|}{|\Phi(5)|}, \frac{|\Phi(6)|}{\infty}\right\} = \frac{|\Phi(2)|}{|\Phi(5)|}.$$

Furthermore, $\rho_1(\mathbf{f},\mathbf{m}) = \rho_2(\mathbf{f},\mathbf{m}) = |\Phi(5)|$ and both $D_1(\mathbf{f},\mathbf{m})$ and $D_2(\mathbf{f},\mathbf{m})$ contain only one pole of order 1 at 2. Consequently, $R_1^*(\mathbf{f},\mathbf{m}) = R_2^*(\mathbf{f},\mathbf{m}) = |\Phi(5)|$. From (1.11), for any compact subset K of $(E \cup \{z \in \mathbb{C} : |\Phi(z)| < |\Phi(5)|\}) \setminus \{2\}$,

$$\limsup_{n\to\infty} \|R_{n,\mathbf{m},k} - f_k\|_K^{1/n} \le \frac{\|\Phi\|_K}{|\Phi(5)|}, \quad k = 1, 2.$$

The purpose of this article is to show that the inequality in equation (1.11) is equality for a certain class of compact sets $K \subset D_k^*(\mathbf{f},\mathbf{m}) \setminus \mathcal{P}_{\mathbf{m}}^{\mathbf{f}}$ (as defined below) and a Newtonian table α.

Let B be a subset of the complex plane $\mathbb{C}$. Denote by $\mathcal{U}(B)$ the class of all coverings of B by at most a countable collection of disks. We define the *1-dimensional Hausdorff content of the set B* as follows:

$$h(B) := \inf\left\{\sum_{j=1}^{\infty} |U_j| : \{U_j\} \in \mathcal{U}(B)\right\},$$

where $|U_j|$ is the radius of the disk U_j.

Definition 3. We say that a compact set $K \subset \mathbb{C}$ is *h-regular* if for each $z_0 \in K$ and for each $\delta > 0$, it holds

$$h\{z \in K : |z - z_0| < \delta\} > 0.$$

Our main result is stated in the following:

Theorem 1.2. *Let* $\mathbf{f} \in \mathcal{H}(E)^d$, $\alpha \subset E$ *be a Newtonian table satisfying* (1.6), *and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *Suppose that either* (*a*) *or* (*b*) *in* Theorem 1.1 *takes place. Then, for any h-regular compact subset K of* $D_k^*(\mathbf{f},\mathbf{m}) \setminus \mathcal{P}_{\mathbf{m}}^{\mathbf{f}}$ *such that* $K \setminus E \neq \emptyset$,

$$\limsup_{n\to\infty} \|R_{n,\mathbf{m},k} - f_k\|_K^{1/n} = \frac{\|\Phi\|_K}{R_k^*(\mathbf{f},\mathbf{m})}. \tag{1.12}$$

An outline of this article is as follows. Section 2 contains a study of incomplete multipoint Padé approximants. We will use the study in Section 2 to prove Theorem 1.2 in Section 3.

2 Incomplete multipoint Padé approximants

An incomplete multipoint Padé approximation defined below plays an important role in proving Theorem 1.2 (see also [7] and [8], where the authors used a concept of incomplete Padé approximation to study a similar problem for classical Hermite-Padé approximation). Our proofs in this section are strongly influenced by the methods employed in [7].

Definition 4. Let $f \in \mathcal{H}(E)$ and $\alpha \subset E$ be a table of points. Fix integers m and m^* with $m \geq m^* \geq 1$ and an integer $n \geq m$. We say that the rational function $R_{n,m}$ is an *incomplete multipoint Padé approximant of type* (n, m, m^*) *corresponding to* f if $R_{n,m}$ is the quotient of any two polynomials $P_{n,m}$ and $Q_{n,m}$ that verify
(i) $P_{n,m} \in \mathbb{P}_{n-m^*}$, $Q_{n,m} \in \mathbb{P}_m$, $Q_{n,m} \not\equiv 0$,
(ii) $(Q_{n,m}f - P_{n,m})/a_{n+1} \in \mathcal{H}(E)$, where $a_n(z) = \prod_{k=1}^{n}(z - \alpha_{n,k})$.

We normalize $Q_{n,m}$ in the aforementioned definition to be monic. Note that for $k = 1, \ldots, d$, $Q_{n,\mathbf{m}}$ in Definition 1 is a denominator of an incomplete multipoint Padé approximant of type $(n, |\mathbf{m}|, m_k)$ corresponding to f_k. We will make use of this fact in the proof of Theorem 1.2 below.

The concept of convergence in Hausdorff content is defined as follows.

Definition 5. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of functions defined on a domain $D \subset \mathbb{C}$ and g be a function defined on D. We say that $\{g_n\}_{n\in\mathbb{N}}$ *converges in 1-dimensional Hausdorff content to the function g inside D* if for every compact subset K of D and for each $\varepsilon > 0$, we have

$$\lim_{n\to\infty} h\{z \in K : |g_n(z) - g(z)| \geq \varepsilon\} = 0.$$

Such a convergence will be denoted by $h - \lim_{n\to\infty} g_n = g$ in D.

Let $f \in \mathcal{H}(E)$. In order to prove our main lemmas (Lemmas 5 and 6) in this section, we will apply the spherical normalization to $Q_{n,m}$ in Definition 4. For fixed nonnegative integers $m \geq m^* \geq 1$ and for each integer $n \geq m$, let

$$R_{n,m} = \frac{p_{n,m}}{q_{n,m}}$$

be an (n, m, m^*) incomplete multipoint Padé approximant, where $p_{n,m}$ and $q_{n,m}$ are the polynomials such that $\gcd(p_{n,m}, q_{n,m}) = 1$ and $q_{n,m}$ is normalized in terms of its zeros $\lambda_{n,j}$ so that

$$q_{n,m}(z) := \prod_{|\lambda_{n,j}|\leq 1} (z - \lambda_{n,j}) \prod_{|\lambda_{n,j}|>1} \left(1 - \frac{z}{\lambda_{n,j}}\right). \tag{2.1}$$

Now, we discuss some upper and lower estimates on the normalized polynomials $q_{n,m}$ in equation (2.1). Let $\lambda_1, \ldots, \lambda_{m'}$ be the poles of f in $D_{m^*}(f)$. Let $\varepsilon > 0$ be fixed. Suppose that the zeros of $q_{n,m}$ are $\lambda_{n,1}, \ldots, \lambda_{n,\ell_n}$ (they are not necessarily distinct and $\ell_n \leq m$). We cover each λ_j with the open disk centered at λ_j of radius $\varepsilon/(6m)$ and denote by $J_{0,\varepsilon}(f)$ the union of these disks. For each $n \geq m$, we cover each $\lambda_{n,j}$ with the open disk centered at $\lambda_{n,j}$ of radius $\varepsilon/(6mn^2)$ and denote by $J_{n,\varepsilon}(f)$ the union of these disks. Set

$$J_\varepsilon(f) := J_{0,\varepsilon}(f) \cup \left(\bigcup_{n=m}^{\infty} J_{n,\varepsilon}(f)\right).$$

Applying the monotonicity and subadditivity of h, we have

$$h(J_\varepsilon(f)) \leq h(J_{0,\varepsilon}(f)) + \sum_{n=m}^{\infty} h(J_{n,\varepsilon}(f)) \leq \frac{\varepsilon}{6} + \sum_{n=1}^{\infty} \frac{\varepsilon}{6n^2} = \varepsilon\left(\frac{1}{6} + \frac{\pi^2}{6^2}\right) < \varepsilon. \tag{2.2}$$

Note that $J_{\varepsilon_1}(f) \subset J_{\varepsilon_2}(f)$ for $\varepsilon_1 < \varepsilon_2$. For any set $B \subset \mathbb{C}$, we put

$$B(\varepsilon) := B \setminus J_\varepsilon(f).$$

Clearly, if $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly to g on $K(\varepsilon)$ for any compact set $K \subset D_{m^*}(f)$ and $\varepsilon > 0$, then h-$\lim_{n\to\infty} g_n = g$ in $D_{m^*}(f)$.

The normalization of $q_{n,m}$ produces useful upper and lower bounds of $q_{n,m}$, which are given in the following

Lemma 1. *Let $f \in \mathcal{H}(E)$, $\alpha \subset E$ be a table of points, m and m^* be nonnegative integers with $m \geq m^* \geq 1$, $K \subset \mathbb{C}$ be a compact set, and $\varepsilon > 0$ be fixed. Then, there exist constants $c_1 > 0$ and $c_2 > 0$ independent of n such that for all sufficiently large n,*

$$\|q_{n,m}\|_K \leq c_1, \quad \min_{z \in K(\varepsilon)} |q_{n,m}(z)| \geq c_2 n^{-2m}, \tag{2.3}$$

where $\|\cdot\|_K$ is the sup-norm on K and the second inequality is meaningful when $K(\varepsilon)$ is a nonempty set.

Proof. The proof is very standard so we leave this to the reader. □

From now on, we will denote some constants that do not depend on n by $c_1, c_2, c_3, \ldots$.

Lemma 2. *Let $f \in \mathcal{H}(E)$ and $\alpha \subset E$ satisfy equation (1.7). Fix nonnegative integers m and m^* with $m \geq m^* \geq 1$. For each $n \geq m$, let $R_{n,m}$ be an incomplete multipoint Padé approximant of type (n, m, m^*) for f. Then,*

$$h - \lim_{n\to\infty} R_{n,m} = f \quad \text{in } D_{m^*}(f). \tag{2.4}$$

Proof. Let Q_{m^*} be the monic polynomial whose zeros are poles of f in $D_{m^*}(f)$ taking account of their order. Clearly,

$$\frac{Q_{m^*}q_{n,m}f - Q_{m^*}p_{n,m}}{a_{n+1}} \in \mathcal{H}(D_{m^*}(f)).$$

Let $|\Phi(z)| < \rho < \rho_{m^*}(f)$. Applying the Cauchy integral formula to $(Q_{m^*}q_{n,m}f - Q_{m^*}p_{n,m})/a_{n+1}$, we obtain

$$\frac{(Q_{m^*}q_{n,m}f - Q_{m^*}p_{n,m})(z)}{a_{n+1}(z)} = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{(Q_{m^*}q_{n,m}f)(t)}{a_{n+1}(t)}\frac{dt}{t-z} - \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{(Q_{m^*}p_{n,m})(t)}{a_{n+1}(t)}\frac{dt}{t-z} \tag{2.5}$$

$$= \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{(Q_{m^*}q_{n,m}f)(t)}{a_{n+1}(t)}\frac{dt}{t-z}, \tag{2.6}$$

where the second integral in equation (2.5) is zero because $Q_{m^*}p_{n,m}/((t-z)a_{n+1})$ has a zero at ∞ of order at least two.

Let K be a compact subset of $D_{m^*}(f)$, $\varepsilon > 0$, and $\rho \in (1, \rho_{m^*}(f))$ be the index such that D_ρ contains K and all poles of f in $D_{m^*}(f)$. From (2.6), we have for all $z \in K(\varepsilon)$,

$$(Q_{m^*}(f - R_{n,m}))(z) = \frac{1}{2\pi i}\frac{a_{n+1}(z)}{q_{n,m}(z)}\int_{\Gamma_\rho} \frac{(Q_{m^*}q_{n,m}f)(t)}{a_{n+1}(t)}\frac{dt}{t-z}.$$

Then,

$$\|Q_{m^*}(f - R_{n,m})\|_{K(\varepsilon)} \leq c_1 \frac{\|a_{n+1}\|_K \|q_{n,m}\|_{\Gamma_\rho}}{\min_{t\in\Gamma_\rho}|a_{n+1}(t)| \min_{z\in K(\varepsilon)}|q_{n,m}(z)|}.$$

Therefore, by the uniform convergence of (1.7) and Lemma 1,

$$\limsup_{n\to\infty} \|Q_{m^*}(f - R_{n,m})\|_{K(\varepsilon)}^{1/n} \leq \frac{\|\Phi\|_K}{\rho},$$

and letting $\rho \to \rho_{m^*}(f)$,

$$\limsup_{n\to\infty} \|Q_{m^*}(f - R_{n,m})\|_{K(\varepsilon)}^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{m^*}(f)}.$$

Since $\varepsilon > 0$ is arbitrary, $h - \lim_{n\to\infty} Q_{m^*} R_{n,m} = Q_{m^*} f$ in $D_{m^*}(f)$, which further implies that $h - \lim_{n\to\infty} R_{n,m} = f$ in $D_{m^*}(f)$. □

Lemma 3. *Let $f \in \mathcal{H}(E)$ and $\alpha \subset E$ be a Newtonian table. Fix two positive integers m and m^* such that $m \geq m^* \geq 1$. Consider a corresponding sequence of incomplete multipoint Padé approximants. For each $n \geq m$, we have*

$$R_{n+1,m}(z) - R_{n,m}(z) = \frac{A_{n,m} a_{n+1}(z) b_{n,m-m^*}(z)}{q_{n,m}(z) q_{n+1,m}(z)}, \tag{2.7}$$

where $A_{n,m}$ is some constant and $b_{n,m-m^*}(z)$ is a polynomial of degree at most $m - m^*$ normalized as in equation (2.1).

Proof. By (ii) in Definition 4, we know

$$\frac{q_{n,m} f - p_{n,m}}{a_{n+1}}, \frac{q_{n+1,m} f - p_{n+1,m}}{a_{n+2}} \in \mathcal{H}(E). \tag{2.8}$$

By equation (2.8),

$$\frac{q_{n,m} p_{n+1,m} - q_{n+1,m} p_{n,m}}{a_{n+1}} = \left(\frac{q_{n,m} f - p_{n,m}}{a_{n+1}}\right) q_{n+1,m} - \left(\frac{q_{n+1,m} f - p_{n+1,m}}{a_{n+1}}\right) q_{n,m} \in \mathcal{H}(E).$$

Since $(q_{n,m} p_{n+1,m} - q_{n+1,m} p_{n,m}) \in \mathbb{P}_{n+1+m-m^*}$,

$$q_{n,m} p_{n+1,m} - q_{n+1,m} p_{n,m} = A_{n,m} a_{n+1} b_{n,m-m^*},$$

where $b_{n,m-m^*}$ is a polynomial such that it is normalized as in equation (2.1) and $\deg(b_{n,m-m^*}) \leq m - m^*$. Dividing the aforementioned equality by $q_{n,m} q_{n+1,m}$, we arrive at (2.7). □

Let $\varepsilon > 0$. Using a setting similar to $J_{n,\varepsilon}(f)$, for each $n \geq m$, let $J'(f)$ denote the union of all disks centered at all zeros of $b_{n,m-m^*}$ with radii $\varepsilon/(6mn^2)$. Set

$$J'_\varepsilon(f) := \bigcup_{n\geq m} J'(f).$$

For any compact set $B \subset \mathbb{C}$, we put

$$B'(\varepsilon) := B \backslash J'_\varepsilon(f).$$

Note that using the computation similar to equation (2.2), we have $h(J'_\varepsilon(f)) < \varepsilon$ and it is easy to check that the normalization of $b_{n,m-m^*}$ implies that for any compact set K of $\mathbb{C}$ and for any $\varepsilon > 0$, there exist constants $c_1, c_2 > 0$, independent of n, such that

$$\|b_{n,m-m^*}\|_K < c_1, \quad \min_{z\in K'(\varepsilon)} |b_{n,m-m^*}(z)| > c_2 n^{-2m}. \tag{2.9}$$

Define

$$\rho^*_m(f) := \left(c \cdot \limsup_{n\to\infty} |A_{n,m}|^{1/n}\right)^{-1}, \quad D^*_m(f) := \{z \in \mathbb{C} : |\Phi(z)| < \rho^*_m(f)\}. \tag{2.10}$$

Lemma 4. *Let $f \in \mathcal{H}(E)$ and $\alpha \subset E$ be a Newtonian table satisfying equation* (1.7). *Fix two nonnegative integers m and m^* with $m \geq m^* \geq 1$. Let $\{R_{n,m}\}_{n\geq m}$ be a sequence of incomplete multipoint Padé approximants of type (n, m, m^*) for f. Then,*

$$D_{m^*}(f) \subset D^*_m(f) \subset D_m(f)$$

and $D^*_m(f)$ is the largest canonical domain in compact subsets of which $h - \lim_{n\to\infty} R_{n,m} = f$. Moreover, the sequence $\{R_{n,m}\}_{n\geq m}$ is pointwise divergent in $\{z : |\Phi(z)| > \rho^*_m(f)\}$ except possibly on a set of h-content zero.

Proof. From (2.7), the convergence and divergence of $\{R_{n,m}\}_{n\geq m}$ depend on the convergence and divergence of the series

$$\sum_{n\geq m} \frac{A_{n,m}a_{n+1}(z)b_{n,m-m^*}(z)}{q_{n,m}(z)q_{n+1,m}(z)}, \tag{2.11}$$

respectively.

If $|\Phi(z)| > \rho_m^*(f)$ and $z \notin \cap_{\varepsilon>0} J'(f)$, then from equations (1.7), (2.3), (2.9), and (2.10), it follows that

$$\limsup_{n\to\infty} \left| \frac{A_{n,m}a_{n+1}(z)b_{n,m-m^*}(z)}{q_{n,m}(z)q_{n+1,m}(z)} \right|^{1/n} \geq \frac{|\Phi(z)|}{\rho_m^*(f)},$$

and the series diverges. Therefore, the sequence $\{R_{n,m}\}_{n\geq m}$ pointwise diverges in $\{z \in \mathbb{C} : |\Phi(z)| > \rho_m^*(f)\}$ except possibly on a set of h-content zero (namely, $\cap_{\varepsilon>0} J'(f)$). From equation (2.4), we know that $h - \lim_{n\to\infty} R_{n,m} = f$ in $D_{m^*}(f)$. This implies that $\rho_m^*(f) \geq \rho_{m^*}(f) > 1$ and $D_{m^*}(f) \subset D_m^*(f)$.

Let $K \subset D_m^*(f)$. Fix $\varepsilon > 0$. Using equations (1.7), (2.3), (2.9), and (2.10), we have

$$\limsup_{n\to\infty} \left\| \frac{A_{n,m}a_{n+1}(z)b_{n,m-m^*}(z)}{q_{n,m}(z)q_{n+1,m}(z)} \right\|_{K(\varepsilon)}^{1/n} \leq \frac{\|\Phi\|_K}{\rho_m^*(f)} < 1. \tag{2.12}$$

From this, the series in equation (2.11) converges uniformly on $K(\varepsilon)$ for any $K \subset D_m^*(f)$ and $\varepsilon > 0$. Thus, $h - \lim_{n\to\infty} R_{n,m} = \varphi$ in $D_m^*(f)$ for some function φ. By Gonchar's lemma (see Lemma 1 in [3]), φ is (except on a set of h-content zero) a meromorphic function with at most m poles in $D_m^*(f)$. By the uniqueness of the limit (up to h-content zero), $\varphi = f$ in $D_m^*(f)$. Then, f has poles at most m in $D_m^*(f)$. Hence, $D_m^*(f) \subset D_m(f)$. □

In [2], Gonchar defined two indicators used to quantify a rate of attraction of a point to poles of classical Padé approximants. In what follows, we use the same indicators to quantify such rate of attraction of a pole of $f \in \mathcal{H}(E)$ to poles of incomplete multipoint Padé approximants. Let $f \in \mathcal{H}(E)$ and m, m^* be fixed nonnegative integers with $m \geq m^* \geq 1$. Let

$$\mathcal{P}_{n,m} \coloneqq \{\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,\ell_n}\}, \quad \ell_n \leq m, \quad n \geq m,$$

denote the collection of zeros of $q_{n,m}$ (repeated according to their multiplicity). Define

$$|z - w|_1 \coloneqq \min\{1, |z - w|\}, \quad z, w \in \mathbb{C}.$$

Let $\lambda \in \mathbb{C}$. The first indicator is defined by:

$$\Delta(\lambda) \coloneqq \limsup_{n\to\infty} \prod_{j=1}^{\ell_n} |\lambda_{n,j} - \lambda|_1^{1/n} = \limsup_{n\to\infty} \prod_{|\lambda_{n,j}-\lambda|<1} |\lambda_{n,j} - \lambda|^{1/n}.$$

Clearly, $0 \leq \Delta(\lambda) \leq 1$ (when $\ell_n = 0$ or $|\lambda_{n,j} - \lambda| \geq 1$ for all $j = 1, 2, \ldots, \ell_n$, the product is taken to be 1). To define the second indicator, we suppose that for each n, the points in

$$\mathcal{P}_{n,m} = \{\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,\ell_n}\}$$

are enumerated in nondecreasing distance to the point λ. We set

$$\delta_j(\lambda) \coloneqq \limsup_{n\to\infty} |\lambda_{n,j} - \lambda|_1^{1/n}, \quad j = 1, 2, \ldots, m', \tag{2.13}$$

where $m' \coloneqq \liminf_{n\to\infty} \ell_n$ and for $j = m' + 1, \ldots, m$, we define $\delta_j(\lambda) = 1$. Obviously, we have $0 \leq \delta_j(\lambda) \leq 1$. The second indicator, a nonnegative integer $\mu(\lambda)$, is defined as follows. If $\Delta(\lambda) = 1$ (in that case, all $\delta_j(\lambda) = 1$), then $\mu(\lambda) = 0$. If $\Delta(\lambda) < 1$, then for some τ, $1 \leq \tau \leq m$, we have that $\delta_1(\lambda) \leq \ldots \leq \delta_\tau(\lambda) < 1$ and $\delta_{\tau+1}(\lambda) = 1$ or $\tau = m$; in this case, we take $\mu(\lambda) = \tau$.

Lemma 5. *Let $f \in \mathcal{H}(E)$ and $\alpha \subset E$ be a Newtonian table satisfying equation* (1.7). *Fix m and m^* as nonnegative integers, $m \geq m^* \geq 1$. For each $n \geq m$, let $R_{n,m}$ be an incomplete multipoint Padé approximant of type* (n, m, m^*) *for f. Let λ be a pole of f in $D_m^*(f)$ of order τ. Then,*

$$\Delta(\lambda) \le \frac{|\Phi(\lambda)|}{\rho_m^*(f)} \quad \textit{and} \quad \mu(\lambda) \ge \tau.$$

Proof. Let $\varepsilon > 0$ and λ be a pole of f in $D_m^*(f)$ of order τ. Let $r > 0$ be sufficiently small so that $\mathbb{B}(\lambda, r)$ contains no other pole of f and

$$\partial\mathbb{B}(\lambda, r) := \{z \in \mathbb{C} : |z - \lambda| = r\} \subset D_m^*(f) \setminus J_\varepsilon(f).$$

By Gonchar's lemma (see Lemma 1 in [3]), since $h - \lim_{n\to\infty} R_{n,m} = f$ in $D_m^*(f)$, $q_{n,m}$ has more than τ zeros in $\mathbb{B}(\lambda, r)$, so we name these zeros $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,\mu_n}$ indexed in nondecreasing distance from λ, i.e.,

$$|\lambda - \lambda_{n,1}| \le |\lambda - \lambda_{n,2}| \le \dots \le |\lambda - \lambda_{n,\mu_n}|.$$

Set

$$Q_{n,\lambda}(z) := \prod_{j=1}^{\mu_n} (z - \lambda_{n,j}).$$

For any ρ with $\|\Phi\|_{\partial\mathbb{B}(\lambda,r)} < \rho < \rho_m^*(f)$, it follows from (2.7), (2.11), and (2.12) that

$$\|f - R_{n,m}\|_{\partial\mathbb{B}(\lambda,r)} \le c_1 q^n, \quad q := \frac{\|\Phi\|_{\partial\mathbb{B}(\lambda,r)}}{\rho} < 1, \tag{2.14}$$

for some constant c_1 and for sufficiently large n.

Let $p(z)/(z - \lambda)^\tau$ be the principal part of the function f at the point λ and $p_n/Q_{n,\lambda}$ be the sum of the principal parts of $R_{n,m}$ corresponding to its poles in $\mathbb{B}(\lambda, r)$. We have $\deg p < \tau$, $p(\lambda) \ne 0$, and $\deg p_n < \mu_n$. It is known that the norm of the holomorphic component of a meromorphic function may be bounded in terms of the norm of the function and the number of poles (see Theorem 1 in [18]). Thus, using this and (2.14), we obtain

$$\left\| \frac{p(z)}{(z - \lambda)^\tau} - \frac{p_n(z)}{Q_{n,\lambda}(z)} \right\|_{\partial\mathbb{B}(\lambda,r)} \le c_2 q^n, \tag{2.15}$$

for some constant c_2 and for sufficiently large n. Therefore, from (2.15) and the maximal modulus principle, we have

$$\|p(z)Q_{n,\lambda}(z) - (z - \lambda)^\tau p_n(z)\|_{\overline{\mathbb{B}(\lambda,r)}} \le c_3 q^n, \tag{2.16}$$

for some constant c_3 and for sufficiently large n. Since $p(\lambda) \ne 0$, replacing z by λ in equation (2.16) and taking the limit supremum of the n-root from both sides as $n \to \infty$, we obtain

$$\Delta(\lambda) = \limsup_{n\to\infty} \prod_{j=1}^{\mu_n} |\lambda_{n,j} - \lambda|_1^{1/n} = \limsup_{n\to\infty} |Q_{n,\lambda}(\lambda)|^{1/n} \le q.$$

Because of the arbitrariness of r, ε, and ρ, we obtain $\Delta(\lambda) \le |\Phi(\lambda)|/\rho_m^*(f) < 1$.

Now, let us show that $\mu(\lambda) \ge \tau$. To do that, we will show that $\delta_\tau(\lambda) < 1$ by induction. Because $\Delta(\lambda) < 1$, we have $\delta_1(\lambda) < 1$. Let $\delta_1(\lambda) \le \delta_2(\lambda) \le \dots \le \delta_k(\lambda) < 1$ and $k < \tau$. We differentiate the polynomial inside the norm in equation (2.16) k times. Notice that $\deg(pQ_{n,\lambda} - (z - \lambda)^\tau p_n) < 2m$. Using (2.16) and Cauchy's integral formula, we obtain that its kth derivative satisfies an inequality like (2.16). If we put $z = \lambda$ in the corresponding inequality, we have

$$\left| \left(p(z) \prod_{j=1}^{\mu_n} (z - \lambda_{n,j}) \right)^{(k)} (\lambda) \right| \le c_4 q^n. \tag{2.17}$$

It is not difficult to see that $(p(z)\prod_{j=1}^{\mu_n}(z - \lambda_{n,j}))^{(k)}(\lambda)$ differs from $p(\lambda)\prod_{j=k+1}^{\mu_n}(\lambda - \lambda_{n,j})$ by a group of terms, each of which has one of the terms $(\lambda - \lambda_{n,j})$, $j \in \{1, 2, \dots, k\}$, as a factor (the moduli of the products of the other factors are bounded at λ). Since we assumed that $\delta_j(\lambda) < 1$ for $j = 1, 2, \dots, k$, we obtain

$$\limsup_{n\to\infty} \prod_{j=k+1}^{\mu_n} |\lambda_{n,j} - \lambda|^{1/n} < 1,$$

which in turn implies $\delta_{k+1}(\lambda) = \limsup_{n\to\infty} |\lambda - \lambda_{n,k+1}|^{1/n} < 1$. Therefore, $\mu(\lambda) \geq \tau$ holds. This completes the proof. □

Let us continue using the notation in the proof of Lemma 5 in the following lemma and its proof.

Lemma 6. *Let $f \in \mathcal{H}(E)$, $\alpha \subset E$ be a Newtonian table satisfying equation* (1.7) *and λ be a pole of f in $D_m^*(f)$ of order τ. Assume that* $\liminf_{n\to\infty}|\lambda - \lambda_{n,\tau+1}| > 0$. *Then,*

$$\delta_1(\lambda) \leq \cdots \leq \delta_\tau(\lambda) \leq \left(\frac{|\Phi(\lambda)|}{\rho_m^*(f)}\right)^{1/\tau}. \tag{2.18}$$

In particular, $\delta_1(\lambda) = \cdots = \delta_\tau(\lambda) = (|\Phi(\lambda)|/\rho_m^*(f))^{1/\tau}$ *if and only if* $\Delta(\lambda) = |\Phi(\lambda)|/\rho_m^*(f)$.

Proof. We assume that

$$Q_{n,\lambda}(z) = \prod_{j=1}^{\tau}(z - \lambda_{n,j}).$$

Recall that $p(\lambda) \neq 0$. Replacing $z = \lambda$ in equation (2.16), we obtain $|Q_{n,\lambda}(\lambda)| \leq c_1 q^n$, for sufficiently large n. By the Leibniz formula,

$$(pQ_{n,\lambda})^{(k)}(\lambda) = p(\lambda)Q_{n,\lambda}^{(k)}(\lambda) + \sum_{j=0}^{k-1}\binom{k}{j}p^{(k-j)}(\lambda)Q_{n,\lambda}^{(j)}(\lambda).$$

Using the aforementioned equation, Inequality (2.17), and the mathematical induction on k, we obtain

$$|Q_{n,\lambda}^{(k)}(\lambda)| \leq c_2 q^n, \quad k = 0, 1, 2, \ldots, \tau - 1, \tag{2.19}$$

for some constant c_2 and for sufficiently large n. Inequality (2.19) and the expression

$$Q_{n,\lambda}(z) = (z - \lambda)^\tau + \sum_{k=0}^{\tau-1}\frac{Q_{n,\lambda}^{(k)}(\lambda)}{k!}(z - \lambda)^k$$

imply that there exists $N \in \mathbb{N}$ such that $\|(z - \lambda)^\tau - Q_{n,\lambda}(z)\|_{\overline{\mathbb{B}(\lambda,r)}} \leq c_3 q^n$, for all $n \geq N$. Replacing $z = \lambda_{n,\tau}$, we obtain

$$|\lambda_{n,\tau} - \lambda|^\tau \leq c_3 q^n, \quad n \geq N,$$

which means $(\delta_\tau(\lambda))^\tau \leq q$. As $q = \|\Phi\|_{\partial\mathbb{B}(\lambda,r)}/\rho, r > 0$, and $\rho < \rho_m^*(f)$ are arbitrary, we have

$$\delta_\tau(\lambda) \leq \left(\frac{|\Phi(\lambda)|}{\rho_m^*(f)}\right)^{1/\tau}.$$

Combining this and the fact that $\delta_1(\lambda) \leq \delta_2(\lambda) \ldots \leq \delta_\tau(\lambda)$, we obtain equation (2.18).

Finally, let us show the last statement. Assume that $\delta_1(\lambda) = \cdots = \delta_\tau(\lambda) = (|\Phi(\lambda)|/\rho_m^*(f))^{1/\tau}$. Then,

$$\frac{|\Phi(\lambda)|}{\rho_m^*(f)} = \delta_1(f)^\tau \leq \Delta(\lambda) \leq \prod_{j=1}^{\tau}\delta_j(\lambda) = \frac{|\Phi(\lambda)|}{\rho_m^*(f)},$$

which implies $\Delta(\lambda) = |\Phi(\lambda)|/\rho_m^*(f)$.

Suppose that $\Delta(\lambda) = |\Phi(\lambda)|/\rho_m^*(f)$. Then, from equation (2.18), $\delta_\tau(\lambda) \leq (\Delta(\lambda))^{1/\tau}$, which further implies that

$$(\delta_\tau(\lambda))^\tau \leq \Delta(\lambda) \leq \prod_{j=1}^{\tau}\delta_j(\lambda).$$

Therefore, $\delta_1(\lambda) = \cdots = \delta_\tau(\lambda) = (|\Phi(\lambda)|/\rho_m^*(f))^{1/\tau}$. □

Under the condition of Lemma 6, as a consequence of equation (2.19), since r and $\rho < \rho_m^*(f)$ are arbitrary, we note that

$$\limsup_{n\to\infty} |Q_{n,\lambda}^{(k)}(\lambda)|^{1/n} \leq \frac{|\Phi(\lambda)|}{\rho_m^*(f)}, \quad k = 0, 1, 2, \ldots, \tau - 1. \tag{2.20}$$

We will make use of this in the proof of Theorem 1.2.

3 Proof of Theorem 1.2

Proof. Let $k \in \{1, 2, \ldots, d\}$ be fixed. From (1.11), we recall that

$$\limsup_{n\to\infty} \|R_{n,\mathbf{m},k} - f_k\|_K^{1/n} \leq \frac{\|\Phi\|_K}{R_k^*(\mathbf{f}, \mathbf{m})}, \tag{3.1}$$

for any compact subset K of $D_k^*(\mathbf{f}, \mathbf{m}) \setminus \mathcal{P}_{\mathbf{m}}^{\mathbf{f}}$. This implies that $h - \lim_{n\to\infty} R_{n,\mathbf{m},k} = f_k$ in $D_k^*(\mathbf{f}, \mathbf{m})$. Because $\rho_{|\mathbf{m}|}^*(f_k)$ is the largest index of canonical domain inside of which such h-convergence is valid, $R_k^*(\mathbf{f}, \mathbf{m}) \leq \rho_{|\mathbf{m}|}^*(f_k)$.

Next, we want to show that $R_k^*(\mathbf{f}, \mathbf{m}) = \rho_{|\mathbf{m}|}^*(f_k)$. To the contrary, we assume that

$$R_k^*(\mathbf{f}, \mathbf{m}) < \rho_{|\mathbf{m}|}^*(f_k). \tag{3.2}$$

Note that the h-convergence of $\{R_{n,\mathbf{m},k}\}_{n \geq |\mathbf{m}|}$ inside $D_{|\mathbf{m}|}^*(f_k)$ implies that all singularities of f_k inside $D_{|\mathbf{m}|}^*(f_k)$ are zeroes of $Q_{\mathbf{m}}^{\mathbf{f}}$ which are system poles $\mathbf{f}$ by Theorem 1.1. Therefore, the boundary of $D_k^*(\mathbf{f}, \mathbf{m})$ cannot contain "a singularity of f_k, which is not a system pole of $\mathbf{f}$" or "a system pole of $\mathbf{f}$ with its order as a pole of f_k greater than its order as a system pole." Consequently, $\rho_k(\mathbf{f}, \mathbf{m}) > R_k^*(\mathbf{f}, \mathbf{m})$. This additionally implies that

$$\rho_{|\mathbf{m}|}^*(f_k) > R_k^*(\mathbf{f}, \mathbf{m}) = \min_{j=1,\ldots,N} R_{\lambda_j,\hat{\tau}_j}(\mathbf{f}, \mathbf{m}), \tag{3.3}$$

where we recall that $\lambda_1, \ldots, \lambda_N$ are all poles of f_k in $D_k(\mathbf{f}, \mathbf{m})$ and $\hat{\tau}_j$ is the order of λ_j as a pole of f_k. Since $D_k(\mathbf{f}, \mathbf{m}) \subset D_{|\mathbf{m}|}^*(f_k)$ (by Lemma 2) and $Q_{n,\mathbf{m}}$ is a denominator of an incomplete multipoint Padé approximant of type $(n, |\mathbf{m}|, m_k)$, from equations (1.9) and (2.20), for each pole λ of order $\hat{\tau}$ of f_k inside $D_k(\mathbf{f}, \mathbf{m})$,

$$\limsup_{n\to\infty} |Q_{n,\mathbf{m}}^{(j)}(\lambda)|^{1/n} \leq \frac{|\Phi(\lambda)|}{\rho_{|\mathbf{m}|}^*(f_k)}, \quad j = 0, 1, \ldots, \hat{\tau} - 1. \tag{3.4}$$

Because such λ is also a system pole of order at least $\hat{\tau}$, from Corollary 2.2 in [16], we have

$$\max_{j=0,1,\ldots,\hat{\tau}-1} \limsup_{n\to\infty} |Q_{n,\mathbf{m}}^{(j)}(\lambda)|^{1/n} = \frac{|\Phi(\lambda)|}{R_{\lambda,\hat{\tau}}(\mathbf{f}, \mathbf{m})}. \tag{3.5}$$

Combining equations (3.4) and (3.5), we obtain $\rho_{|\mathbf{m}|}^*(f_k) \leq R_{\lambda,\hat{\tau}}(\mathbf{f}, \mathbf{m})$ for all poles λ of f_k in $D_k(\mathbf{f}, \mathbf{m})$. This contradicts (3.3). Therefore, $R_k^*(\mathbf{f}, \mathbf{m}) = \rho_{|\mathbf{m}|}^*(f_k)$.

Suppose that $K \subset D_k^*(\mathbf{f}, \mathbf{m}) \setminus \mathcal{P}_{\mathbf{m}}^{\mathbf{f}}$ is a h-regular compact set and $K \setminus E \neq \emptyset$. Let us consider the constants $A_{n,\mathbf{m},k}$ and the polynomials $b_{n,m-m^*,k}$ defined according to Lemma 3 for the incomplete multipoint Padé approximant $R_{n,\mathbf{m},k}$, where $m = |\mathbf{m}|$ and $m^* = m_k$. Put $J_0' := \cap_{\varepsilon>0} J_\varepsilon'(f_k)$ and take $z_0 \in K$ such that $\|\Phi\|_K = |\Phi(z_0)| > 1$. As J' is a set of h-content zero and the compact set K is h-regular, there exists a sequence $\{z_j\}_{j\in\mathbb{N}} \subset K \setminus J_0'$ verifying $\lim_{j\to\infty} z_j = z_0$. We may assume that $|\Phi(z_j)| > 1$ for all $j \in \mathbb{N}$.

From Lemma 3, it follows that

$$|A_{n,\mathbf{m},k}| = \frac{|(q_{n+1,\mathbf{m},k} q_{n,\mathbf{m},k})(z_j)||R_{n+1,\mathbf{m},k}(z_j) - R_{n,\mathbf{m},k}(z_j)|}{|a_{n+1}(z_j)||b_{n,m-m^*,k}(z_j)|},$$

where we denote the denominator of $R_{n,\mathbf{m},k}$ by $q_{n,\mathbf{m},k}$ normalized as in equation (2.1). We may write

$$|R_{n+1,\mathbf{m},k}(z_j) - R_{n,\mathbf{m},k}(z_j)| \leq \|R_{n+1,\mathbf{m},k} - f_k\|_K + \|f_k - R_{n,\mathbf{m},k}\|_K.$$

So, applying (2.3), (2.9), and (2.10), we obtain

$$\frac{1}{c\rho^*_{|\mathbf{m}|}(f_k)} = \limsup_{n\to\infty} |A_{n,\mathbf{m},k}|^{1/n} \le \frac{1}{c|\Phi(z_j)|}\limsup_{n\to\infty}\|f_k - R_{n,\mathbf{m},k}\|_K^{1/n}.$$

Taking the limit as $j \to \infty$ in the aforementioned expression, we verify that Inequality (3.1) is actually an equality when K is a h-regular compact set with $K \setminus E \neq \emptyset$, as we wanted to prove. □

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