Research Article

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Existence of projected solutions for quasi-variational hemivariational inequality

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Abstract: In this short article, we prove the existence of projected solutions to a class of quasi-variational hemivariational inequalities with non-self-constrained mapping, which generalizes the results of Allevi et al. (*Quasi-variational problems with non-self map on Banach spaces: Existence and applications*, Nonlinear Anal. Real World Appl. **67** (2022), 103641, DOI: https://doi.org/10.1016/j.nonrwa.2022.103641.)

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1 Introduction

The theory of variational inequalities was considered and studied by the Kinderlehrer and Stampacchia [1] and Fichera [2]. Since then, it has been greatly developed in various aspects, both from the theoretical and from the applied point of view, because it can provide a powerful tool of investigating various mathematical problems [3,4]. With the deepening of research, a generalized quasi-variational inequality was introduced by Chan and Pang [5]. Recently, Zeng et al. [6] first studied a generalized nonlinear quasi-hemivariational inequality involving a multivalued map in a Banach space, and also, they developed a useful and elegant framework to examine the existence of solution for the considered inequality. The generalized nonlinear quasi-hemivariational inequality introduced in [6] can be a very powerful mathematical model to study various approximations of elastic contact problems with the constitutive law involving convex subdifferential inclusions, the multivalued version of the normal compliance contact condition with frictionless effect, and a frictional contact law with the slip-dependent coefficient of friction.

In general, the solution to the generalized quasi-variational inequality is described as $u \in K(u)$, where $K \coloneqq C \to 2^C$ [7,8]. However, in some applications, the self-constrained mapping condition is not always satisfied and the classical existence theorem is impossible to apply (see [9] and references therein). The concept of projected solutions of generalized quasi-variational inequality with non-self-constrained mapping was introduced in [9]. In [10], the existence of solutions to the generalized quasi-variational inequality where constraint set-valued mapping is not necessarily self-constrained mapping was obtained. The projected solutions of Ky Fan's quasi-equilibrium problem were studied in [11], where the variational inequality becomes a classic

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example. All the results in [9–11] are established in finite dimensional spaces, while [12] provides the existence of projected solutions for the generalized quasi-variational inequality on infinite dimensional Banach spaces.

As we know that the hemivariational inequalities [13–15] can be seen as an important generalization of variational inequalities, which is relatively recent and relates to non-convex energy functionals. If both convex and non-convex functions are involved, they are called variational hemivariational inequalities. Today, variational hemivariational inequalities have been studied (see [16,17] and references therein). Correspondingly, quasi-variational hemivariational inequalities have also been proposed for many application problems (see [18,19] and references therein). This presents an open problem in the study of quasi-variational hemivariational inequalities with non-self-constrained mapping. Hence, the purpose of this short article is to prove the existence of projected solutions to a class of quasi-variational hemivariational inequalities on infinite dimensional Banach space.

Assume that V is a real reflexive Banach space that continuously and compactly embeds into a Banach space X, and C is nonempty, convex, and closed subset of V. Denote by γ the embedding operator from V to X, and denote by $\gamma^* : X^* \to V^*$ its adjoint operator. For simplicity, we write $\gamma(u) \coloneqq \hat{u}$ for all $u \in V$. Let m belong to V^* and $K : C \to 2^V$ and $F : V \to 2^{V^*}$ be the two set-valued maps. We consider the following quasi-variational hemivariational inequality.

Problem 1.1. Find $u \in K(u)$ and $u^* \in F(u)$ such that

$$\langle u^* + m, v - u\rangle + J^0(\hat{u};\hat{v} - \hat{u}) + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in K(u). \tag{1}$$

Let us first introduce the following hypotheses on the data of Problem 3.1.

($\mathbf{H}_K$) K is Mosco continuous such that for all $u \in C$, the set $K(u) \subseteq V$ is nonempty, closed, and convex.
($\mathbf{H}_J$) The function $J : X \to \mathbb{R}$ is locally Lipschitz continuous.
($\mathbf{H}_\varphi$) The function $\varphi : V \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and continuous.
($\mathbf{H}_F$) F has compact values such that
(i) F is upper semicontinuous;
(ii) $F + \gamma^*\partial J\gamma$ is stably φ-pseudomonotone (Definition 2.2) with respect to $\{m\}$;

Remark 1.1. If $J(\cdot,\cdot) \equiv 0, \varphi(\cdot) \equiv 0$, and $m = 0$, then Problem (1) reduces to the following quasi-variational inequality:

$$\text{find } u \in K(u) \quad \text{and} \quad u^* \in F(u) \quad \text{with } \langle u^*, v - u\rangle \geq 0, \quad \forall v \in K(u),$$

where $K : C \to 2^V$ is non-self-constrained mapping and $F : V \to 2^{V^*}$ is a multivalued, which was studied by [12].

Remark 1.2. If $K : C \to 2^C$ is self-constrained mapping and $F(u) \coloneqq T(u)$ is a single-valued mapping, then Problem (1) is a special case of the following problem: find $u \in K(u)$ such that

$$\langle T(u), v - u\rangle + \varphi(v, u) + J^0(\gamma u; \gamma(v - u)) \geq \langle f, v - u\rangle, \quad \text{for all} \quad v \in K(u),$$

which was considered by [20].

Remark 1.3. If $K(u) \equiv K$, then Problem (1) reduces to the following one: find $u \in K$ and $u^* \in F(u)$ such that

$$\langle u^* + w, v - u\rangle + J^0(iu; iv - iu) + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in K.$$

This problem was studied by [21].

Remark 1.4. If $\varphi(\cdot) \equiv 0$ and $K : C \to 2^C$ is self-constrained mapping, then Problem (1) reduces to the following problem: find $u \in K(u)$ and $u^* \in F(u)$ such that

$$\langle u^*, v - u\rangle + J^0(u, v - u) \geq \langle f, v - u\rangle, \quad \forall v \in K(u).$$

This problem was considered by [18], where the locally Lipschitz function $J : W^{1,p}(\Omega) \to \mathbb{R}$ is of particular interest in applications:

$$J(u) := \int_{\Omega} j(x, u(x))\mathrm{d}x, \quad u \in L^p(\Omega),$$

where $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is the function

$$j(x, t) = \int_0^t \theta(x, \tau)\mathrm{d}\tau, \quad t \in \mathbb{R}.$$

Now, we introduce the concept of projected solutions for quasi-variational hemivariational inequality, Problem (1), on real Banach spaces.

Definition 1.5. A point $u \in C$ is said to be a projected solution of (1) if and only if there exists $\bar{u} \in V$ such that:
(i) u is a projection of $\bar{u}$ on C;
(ii) $\bar{u}$ is a solution to the following problem: find $\bar{u} \in K(u)$ and $u^* \in F(\bar{u})$ such that

$$\langle u^* + m, v - \bar{u}\rangle + J^0(\hat{\bar{u}}; \quad \hat{v} - \hat{\bar{u}}) + \varphi(v) - \varphi(\bar{u}) \geq 0, \quad \forall v \in K(u) \tag{2}$$

where m belongs to V^* and $K : C \to 2^V$ and $F : V \to 2^{V^*}$ are the two set-valued maps.

We recall that if $y \in V$ and $C \subset V$ is a nonempty set, the projection of y on C is defined as $P_C(y)$, i.e.,

$$P_C(y) := \{x \in C : \|y - x\| = d(y, C)\}.$$

Remark 1.6. Clearly, if K is a self-map, then the projected solutions coincide the classical ones.

This article is organized as follows. Section 2 introduces the concepts, definitions, and properties. In Section 3, we prove the existence of projected solutions of quasi-variational hemivariational inequality, Problem (1).

2 Preliminaries

Let X be a real Banach space equipped with the norm $\|\cdot\|_X$, and let $\langle\cdot,\cdot\rangle_X$ be the duality pairing between X and its dual X^*. The strong convergence and weak convergence in X are denoted by symbols $\to$ and $\rightharpoonup$, respectively.

Definition 2.1. Let Y be a real Banach space. A set-valued map $F : X \to 2^Y$ is said to be
(i) Upper semicontinuous at $x_0 \in X$ if for any open set V in Y containing $F(x_0)$, there exists a neighborhood U of x_0 in X such that $F(x) \subseteq V$ for all $x \in U$;
(ii) Lower semicontinuous at $x_0 \in X$ if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, with $x_n \to x_0$, and for any $y \in F(x_0)$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset Y$, with $y_n \in F(x_n)$ for any $n \in \mathbb{N}$, and $y_n \to y$;
(iii) Closed at $x_0 \in X$ if for any sequences $\{x_n\}_{n\in\mathbb{N}} \subset X, \{y_n\}_{n\in\mathbb{N}} \subset Y$, with $x_n \to x_0, y_n \in F(x_n)$ and $y_n \to y$, then $y \in F(x_0)$.

Definition 2.2. (See [21, Definition 3.1]) Let X be a real Banach space. A set-valued map $F : X \to 2^{X^*}$ is said to be
(i) ϕ-pesudomonotone if whenever $u, v \in X$ and $u^* \in F(u)$ it holds

$$\langle u^*, v - u\rangle + \phi(v) - \phi(u) \geq 0 \Rightarrow \langle v^*, v - u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v^* \in F(v);$$

(ii) Stably ϕ-pesudomonotone with respect to a set $W \subset X^*$ if F and its translations $F + y$ are ϕ-pesudomonotone for every $y \in W$.

Definition 2.3. Assume that $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous. The subdifferential of φ at $u \in D(\varphi)$ is defined to be

$$\partial_c\varphi(u) := \{w \in X^* : \varphi(v) - \varphi(u) \geq \langle w, v - u\rangle_X, \quad \text{for all } v \in X\}.$$

An element $w \in \partial_c\varphi(u)$ is called a subgradient of φ at $u \in X$.

Definition 2.4. Assume that $f : X \to \mathbb{R}$ is locally Lipschitz continuous. The Clarke generalized directional derivative of f at the point $u \in X$ in the direction $v \in X$ is defined to be

$$f^0(u; v) = \limsup_{y \to u, t \downarrow 0} \frac{f(y + tv) - f(y)}{t}.$$

The Clarke generalized gradient (subdifferential) of f at u is a subset of the dual space X^* given by $\partial f(u) = \{\zeta \in X^* | f^0(u; v) \geq \langle \zeta, v\rangle_X$ for all $v \in X\}$.

Definition 2.5. (See [12, Definition 3]) Let $M \subset X$ be a nonempty and convex set. A set-valued map $\Gamma : M \to 2^M$ is said to be Kakutani factorizable if $\Gamma = \Gamma_N \circ \Gamma_{N-1} \circ \ldots \circ \Gamma_0$, i.e., if there is a diagram

$$\Gamma : M_0 \xrightarrow{\Gamma_0} 2^{M_1} \xrightarrow{\Gamma_1} 2^{M_2} \longrightarrow \cdots \xrightarrow{\Gamma_N} 2^{M_{N+1}}, \quad \text{with } M_0 = M_{N+1} = M,$$

where, for each $i = 0, \ldots, N$, $\Gamma_i : M_i \to M_{i+1}$ is upper semicontinuous with nonempty, convex, and compact values and $M_i \subseteq X$ being convex set.

Lemma 2.6. (See [22, Theorem 1]) *Let $M \subset X$ be a nonempty and convex set. Suppose that $\Gamma : M \to 2^M$ is a Kakutani factorizable set-valued map such that $\Gamma(M)$ is relatively compact. Then, Γ has a fixed point.*

Lemma 2.7. (See [23, Lemma 1.10]) *Let X and Y be the real Banach spaces and let $T : X \to 2^Y$ be a set-valued map such that $T(x)$ is nonempty and compact for all $x \in X$. Then, T is upper semicontinuous at $x \in X$ if and only if for any sequences $\{x_n\}_{n \in \mathbb{N}} \subset X$ with $x_n \to x$ and $\{y_n\}_{n \in \mathbb{N}} \subset Y$ with $y_n \in T(x_n)$, there exists a subsequence $\{y_\nu\}_{\nu \in \mathbb{N}} \subset \{y_n\}_{n \in \mathbb{N}}$ such that $y_\nu \to y$ for some $y \in T(x)$.*

Proposition 2.8. *Let $J : X \to \mathbb{R}$ be locally Lipschitz of rank $L_u > 0$ near $u \in E_1$. Then, there hold:*

(*a*) *The function $v \mapsto J^\circ(u; v)$ is positively homogeneous, subadditive and satisfies $|J^\circ(u; v)| \leq L_u \|v\|_X$ for all $v \in X$;*

(*b*) *$J^\circ(u; v)$ is upper semicontinuous as a function of (u, v);*

(*c*) *$\partial J(u)$ is a nonempty, convex, weakly* compact subset of X^* with $\|\xi\|_{X^*} \leq L_u$ for all $\xi \in \partial J(u)$;*

(*d*) *For each $v \in X$, we have $J^\circ(u; v) = \max\{\langle \xi, v\rangle : \xi \in \partial J(u)\}$.*

3 Main result

In this section, we prove the existence of projected solutions to a class of quasi-variational hemivariational inequalities with non-self-constrained mapping on a real Banach space. To this end, we need to use Lemma 2.6. Therefore, we prove that all conditions of Lemma 2.6 are available. First, we give some notations.

Fixed $u \in C$, let us introduce the following parametric quasi-variational hemivariational inequality: find $\bar{u} \in K(u)$ and $\exists u^* \in F(\bar{u})$ such that

$$\langle u^* + m, v - \bar{u}\rangle + J^0(\hat{\bar{u}}; \hat{v} - \hat{\bar{u}}) + \varphi(v) - \varphi(\bar{u}) \geq 0, \quad \forall v \in K(u). \tag{3}$$

Let $S : C \to 2^{K(C)} \subset V$ be the solution mapping of Problem (3), namely,

$$S(u) := \{\bar{u} \in K(u) : \bar{u} \text{ is a solution to (3)}\}.$$

The following lemma is needed for our proof.

Lemma 3.1. *Assume that* (H_K), (H_J), (H_φ), *and* (H_F) *hold. Then,*
(*i*) *The solution set of Problem* (3) *is nonempty*;
(*ii*) $\bar{u}$ *is a solution to* (3) *if and only if it solves the following inequality problem: find* $\bar{u} \in K(u)$ *such that*

$$\langle v^* + m, v - \bar{u}\rangle + J^0(\hat{v}; \hat{v} - \hat{\bar{u}}) + \varphi(v) - \varphi(\bar{u}) \geq 0, \quad \forall v^* \in F(v), \quad \forall v \in K(u). \tag{4}$$

Proof. It is a direct consequence of [21, Lemma 3.3]. □

In what follows, we will prove that S satisfies all conditions of Lemma 2.6.

Proposition 3.2. *Let* $K(C)$ *be relatively compact. Under hypotheses* (H_K), (H_J), (H_φ), *and* (H_F), S *is upper semicontinuous with nonempty, convex, and compact values.*

Proof. We split the proof in some parts.

Step 1. S has nonempty values.

By hypotheses (H_K), $K(u)$ is nonempty, closed, and convex. Let $K(u) \coloneqq K$, the imposed conditions and [21, Theorem 3.4] imply that for all $u \in C$, the S is well defined.

Step 2. S is closed.

Let $\{u_n\}_{n\in\mathbb{N}} \subseteq C$ and $\{\bar{u}_n\}_{n\in\mathbb{N}} \subseteq K(C) \subset V$ be such that $u_n \to u$, $\bar{u}_n \to \bar{u}$, and $\bar{u}_n \in S(u_n)$ for all $n \in \mathbb{N}$. We prove that $\bar{u} \in S(u)$. Since $\bar{u}_n \in S(u_n)$ for all $n \in \mathbb{N}$, there exists $u_n^* \in F(\bar{u}_n)$ such that

$$\langle u_n^* + m, v_n - \bar{u}_n\rangle + J^0(\widehat{\bar{u}}_n;\widehat{v}_n - \widehat{\bar{u}}_n) + \varphi(v_n) - \varphi(\bar{u}_n) \geq 0, \quad \forall v_n \in K(u_n).$$

Since F is upper semicontinuous, Lemma 2.7 ensures that there exists a subset of $\{u_n^*\}$, which we may assume without loss of generality to be $\{u_n^*\}$ itself, converging to some point of $u^* \in F(\bar{u})$. On the other hand, by (H_K), we know that K is closed and lower semicontinuous; thus, for any $w \in K(u)$, there exists $\{w_n\}_{n\in\mathbb{N}}$ such that $\{w_n\} \in K(u_n)$ and $w_n \to w$. Hence, according to Proposition 2.8(b) and (H_φ), we have

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty}[\langle u_n^* + m, w_n - \bar{u}_n\rangle + J^0(\widehat{\bar{u}}_n;\widehat{w}_n - \widehat{\bar{u}}_n) + \varphi(w_n) - \varphi(\bar{u}_n)] \\
&\leq \limsup_{n\to\infty}\langle u_n^* + m, w_n - \bar{u}_n\rangle + \limsup_{n\to\infty} J^0(\widehat{\bar{u}}_n;\widehat{w}_n - \widehat{\bar{u}}_n) + \limsup_{n\to\infty}\varphi(w_n) - \liminf_{n\to\infty}\varphi(\bar{u}_n) \\
&\leq \langle u^* + m, w - \bar{u}\rangle + J^0(\hat{\bar{u}};\widehat{w} - \hat{\bar{u}}) + \varphi(w) - \varphi(\bar{u}).
\end{aligned}$$

Since $w \in K(u)$ is arbitrary, so S is closed.

Step 3. S is compact.

We know that S is closed by **Step 2**, so it has closed values. Furthermore, $S(u) \subseteq \overline{K(C)}$, with $\overline{K(C)}$ being a compact set, which means that $S(u)$ is compact set for all $u \in C$. Moreover, $\overline{S(C)}$ is compact; hence, the set-valued map S is compact.

Step 4. S has convex values.

It can be obtained by using the same arguments as in [21, Theorem 3.4].

Step 5. S is upper semicontinuous.

Let $\{u_n\}_{n\in\mathbb{N}} \subseteq C$ and $\{\bar{u}_n\}_{n\in\mathbb{N}} \subseteq K(C) \subset V$ be such that $u_n \to u$ and $\bar{u}_n \in S(u_n)$ for all $n \in \mathbb{N}$. We prove that there exists $\{\bar{u}_k\} \subset \{\bar{u}_n\}$, such that $\bar{u}_k \to \bar{u}$ for some $\bar{u} \in S(u)$. Since S is compact, we can get that there exists $\{\bar{u}_k\} \subset \{\bar{u}_n\}$, such that $\bar{u}_k \to \bar{u}$. Since S is closed, we have $\bar{u} \in S(u)$. Thus, we obtain S that is upper semicontinuous by Lemma 2.7. □

Finally, we give the main result of this article.

Theorem 3.3. *Let* $K(C)$ *be relatively compact. Under hypotheses* (H_K), (H_J), (H_φ), *and* (H_F), *Problem* (1) *admits a projected solution.*

Proof. We define the set-valued map $\Gamma \coloneqq \Gamma_1 \circ \Gamma_0 : C \to 2^C$, where

$$\Gamma_0 : C \to 2^V \quad \text{such that } u \to \Gamma_0(u) := S(u),$$
$$\Gamma_1 : V \to 2^C \quad \text{such that } \bar{u} \to \Gamma_1(\bar{u}) := P_C(\bar{u}) = \{u \in C : \|\bar{u} - u\| = d(\bar{u}, C)\}.$$

One has:
(i) From Proposition 3.2, Γ_0 is upper semicontinuous with nonempty, compact, and convex values;
(ii) Since C is nonempty and convex, closed subset, it is easy to see that Γ_1 is upper semicontinuous with nonempty, compact, and convex values.

Therefore, Γ is a Kakutani factorizable set-valued map. Then, applying [12, Theorem 4], we have Γ that is relatively compact. From Lemma 2.6, we know that there exists $\bar{x}$, which is the fixed point of $\Gamma : \bar{x} \in \Gamma(\bar{x})$; i.e., there exists $\bar{y} \in V$ such that

$$\bar{y} \in \Gamma_0(\bar{x}) \quad \text{and} \quad \bar{x} \in \Gamma_1(\bar{y}).$$

Hence, $\bar{x}$ is a projected solution to (1). □

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References

[1] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, SIAM, New York, 2000.
[2] G. Fichera, *Problemi Elastostatici con Vincoli Unilaterali: il Problema di Signorini con Ambigue Condizioni al Contorno*, Memorie DellaAccademia Nazionale Dei Lincei, 1964.
[3] S. D. Zeng and S. Migórski, *Noncoercive hyperbolic variational inequalities with applications to contact mechanics*, J. Math. Anal. Appl. **455** (2017), no. 1, 619–637, DOI: https://doi.org/10.1016/j.jmaa.2017.05.072.
[4] R. T. Rockafellar and J. Sun, *Solving monotone stochastic variational inequalities and complementarity problems by progressive hedging*, Math. Program. **174** (2019), no. 1–2, 453–471, DOI: https://doi.org/10.1007/s10107-018-1251-y.
[5] D. Chan and J. S. Pang, *The generalized Quasi variational inequality problem*, Math. Oper. Res. **7** (1982), no. 2, 211–222, DOI: https://doi.org/10.1287/moor.7.2.211.
[6] S. D. Zeng, S. Migórski, and A. A. Khan, *Nonlinear quasi-hemivariational inequalities: Existence and optimal control*, SIAM J. Control Optim. **59** (2021), 1246–1274, DOI: https://doi.org/10.1137/19M1282210.
[7] D. Aussel and J. Cotrina, *Quasimonotone quasivariational inequalities: Existence results and applications*, J. Optim. Theory Appl. **158** (2013), no. 3, 637–652, DOI: https://doi.org/10.1007/s10957-013-0270-3.
[8] N. X. Tan, *Quasi-variational inequality in topological linear locally convex Hausdorff spaces*, Math. Nachr. **122** (1985), no. 1, 231–245, DOI: https://doi.org/10.1002/mana.19851220123.

[9] D. Aussel, A. Sultana, and V. Vetrivel, *On the existence of projected solutions of quasi-variational inequalities and generalized Nash equilibrium problems*, J. Optim. Theory Appl. **170** (2016), no. 3, 818–837, DOI: https://doi.org/10.1007/s10957-016-0951-9.

[10] P. Bhattacharyya and V. Vetrivel, *An existence theorem on generalized quasi-variational inequality problem*, J. Math. Anal. Appl. **188** (1994), no. 2, 610–615, DOI: https://doi.org/10.1006/jmaa.1994.1448.

[11] J. Cotrina and J. Zuniga, *Quasi-equilibrium problems with non-self constraint map*, J. Global Optim. **75** (2019), no. 1, 177–197, DOI: https://doi.org/10.1007/s10898-019-00762-5.

[12] E. Allevi, M. E. D. Giuli, M. Milasi, and D. Scopelliti, *Quasi-variational problems with non-self map on Banach spaces: Existence and applications*, Nonlinear Anal. Real World Appl. **67** (2022), 103641, DOI: https://doi.org/10.1016/j.nonrwa.2022.103641.

[13] P. D. Panagiotopoulos, *Nonconvex energy functions, hemivariational inequalities and substationary principles*, Acta Mech., **42** (1983), no. 3–4, 160–183, DOI: https://doi.org/10.1007/BF01170410.

[14] P. D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston, 1985.

[15] P. D. Panagiotopoulos, *Hemivariational Inequalities, Applications in Mechanics and Engineering*, Springer-Verlag, Berlin, 1993.

[16] W. Han, M. Sofonea, and D. Danan, *Numerical analysis of stationary variational-hemivariational inequalities*, Numer. Math. **139** (2018), 563–592, DOI: https://doi.org/10.1007/s00211-018-0951-9.

[17] S. Migórski, A. Ochal, and M. Sofonea, *History-dependent variational-hemivariational inequalities in contact mechanics*, Nonlinear Anal. Real World Appl. **22** (2015), 604–618, DOI: https://doi.org/10.1016/j.nonrwa.2014.09.021.

[18] Z. H. Liu, *Generalized quasi-variational hemi-variational inequalities*, Appl. Math. Lett. **17** (2004), no. 6, 741–745, DOI: https://doi.org/10.1016/S0893-9659(04)90115-2.

[19] G. J. Tang, X. Wang, and Z. B. Wang, *Existence of variational quasi-hemivariational inequalities involving a set-valued operator and a nonlinear term*, Optim. Lett. **9** (2015), 75–90, DOI: https://doi.org/10.1007/s11590-014-0739-5.

[20] S. Migórski, A. A. Khan, and S. D. Zeng, *Inverse problems for nonlinear quasi-hemivariational inequalities with application to mixed boundary value problems*, Inverse Problems **36** (2020), no. 2, 024006, DOI: https://doi.org/10.1088/1361-6420/ab44d7.

[21] Z. H. Liu, D. Motreanu, and S. D. Zeng, *Nonlinear evolutionary systems driven by quasi-hemivariational inequalities*, Math. Methods Appl. Sci. **41** (2017), no. 3, 409–421, DOI: https://doi.org/10.1002/mma.4660.

[22] M. Lassonde, *Fixed points for Kakutani factorizable multifunctions*, J. Math. Anal. Appl. **152** (1990), no. 1, 46–60, DOI: https://doi.org/10.1016/0022-247X(90)90092-T.

[23] Q. H. Ansari, E. Kobis, and J.-C. Yao, *Vector Variational Inequalities and Vector Optimization*, Springer, New York, 2018.