DE GRUYTER Demonstratio Mathematica 2024; 57: 20230138

Research Article

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On inverse source term for heat equation with memory term

https://doi.org/10.1515/dema-2023-0138
received July 25, 2023; accepted December 7, 2023

Abstract: In this article, we first study the inverse source problem for parabolic with memory term. We show that our problem is ill-posed in the sense of Hadamard. Then, we construct the convergence result when the parameter tends to zero. We also investigate the regularized solution using the Fourier truncation method. The error estimate between the regularized solution and the exact solution is obtained.

Keywords: inverse source problem, parabolic equation, memory term, regularization method, error estimate

MSC 2020: 35K99, 47J06, 47H10, 35K05

1 Introduction

Let L be a positive constant. In this article, we consider the parabolic equation with memory as follows:

$$\begin{cases} u_t = u_{xx} + \alpha \int_0^t u_{xx}(x,s)\mathrm{d}s + \rho(x), & \text{in } Q_L := (0,L) \times (0,\pi), \\ u(t,0) = u(t,\pi) = 0, & 0 < t < L, \\ u(0,x) = 0, & \text{in } 0 < x < \pi, \end{cases} \tag{1.1}$$

where $0 < \alpha < \frac{1}{4}$. The function ρ is a source term. If the function ρ is known, we can work out the function u using the initial boundary value problem. However, in this article, our main problem is of determining the source term ρ from the addition information, such as

$$u(L,x) = g(x), \quad 0 < x < \pi. \tag{1.2}$$

where g is the terminal data. Our mentioned problem is called inverse space-dependent source problem. It is well-known that the inverse source problem is ill-posed in the sense of Hadamard. In practice, the terminal condition g is unknown, and it is only available as a noisy data g_ε with a noise level ε. When we use the noisy data for our problems (1.1) and (1.2), we will obtain the corresponding source term that has a large deviation from the source function corresponding to g. This shows that the inverse source problem is ill-posed, and they are required approximately by regularization methods.

If $\alpha = 0$, then problem (1.1) is called classical parabolic equation. If $\alpha \neq 0$, then problem (1.1) have a memory term $\alpha\int_0^t u_{xx}(x,s)\mathrm{d}s$, which is called Volterra integrodifferential equations. Volterra integrodifferential

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equations have been studied by many authors [1,2]. These equations have many applications in various fields such as heat conduction in materials with memory, population dynamics, and nuclear reactors, [3]. Yamada [1] considered Volterra diffusion equations with nonlinear terms. They investigated the stability properties of solutions in L^p norms. The local existence results of solutions to a system of partial functional differential equations are also investigated. Gourley and Britton [2] studied the predator-prey system in the form of a coupled system of reaction-diffusion equations. Tao and Gao [3] analyzed the null controllability properties for heat equation with a memory term.

Let us now mention some previous works on the inverse source problem for classical parabolic equation. Savateev [4] examined the existence and uniqueness of solutions to problems of determining the source function in the heat equation. He found sufficient conditions to show the unique solvability of his problem. In [5], Trong et al. have studied the inverse source problem for the heat equation

$$u_t = u_{xx} + \psi(t)\rho(x), \quad 0 < x < 1, \; 0 < t < 1, \tag{1.3}$$

where ψ is the given function. They regularized inverse source problem by using the Fourier truncation method with some new techniques of Fourier transform with a Lebesgue measure. They obtained the error of the exact solution and the regularized solution in logarithmic order. In [6], Yang and Fu have considered the inverse problem of determining a spacewise-dependent heat source in a one-dimensional heat equation. In order to provide a regularized solution, they give a simplified Tikhonov regularization method. They obtained the Hölder-type stability estimate between the regularization solution and the exact solution. Recently, the authors [7] provided quasi-boundary value methods for regularizing inverse source problems where the source function depends only on the space variable x. In [8], the authors applied direct parallel-in-time quasi-boundary value method to regularizing inverse space-dependent source problem. Some of the related issues can be found in the following references [9–31]. They investigated the stability properties of solutions in L^p norms. Local existence results of solutions to a system of partial functional differential equations are also investigated, see in [32–34].

To the best of our knowledge, there are not any results on the inverse source problem for heat equation with memory term. Our current article may be the first study in this direction. Our contribution is described as follows. The first contribution is to investigate the convergence of the source term when $\alpha \to 0$. The second contribution is to regularize the inverse source problem. We apply the Fourier truncation method to provide the regularized solution. We construct the error estimate between the regularized solution and the sought solution under two various cases: the noise error in L^2 and the noise error in L^p.

The article is organized as follows: in Section 2, we introduce preliminaries and the inverse source problem; Section 3 provides the convergence result of the source term; in Section 4, we provide a Fourier truncation method and investigate the error between the regularized solution and the exact solution.

2 Preliminaries and inverse source problem

2.1 Notations and assumptions

We begin this section by introducing some notations and assumptions that are needed for our analysis in the next sections.

Definition 2.1. [19] (Hilbert scale space) Let $\Omega = (0, \pi)$. The Hilbert scale space $\mathbb{H}^p$, $(p > 0)$ is defined by

$$\mathbb{H}^p(\Omega) := \left\{ f \in L^2(\Omega) : \sum_{m=1}^{\infty} \lambda_m^{2p} \langle f, \xi_m(x) \rangle_{L^2(\Omega)}^2 < \infty \right\}, \tag{2.1}$$

is equipped with the norm defined by

$$\|f\|^2_{\mathbb{H}^p(\Omega)} = \sum_{m=1}^{\infty} \lambda_m^{2p} f_m^2, \quad f_m = \langle f, \xi_m(x)\rangle_{L^2(\Omega)}, \quad \xi_m(x) = \sqrt{\frac{2}{\pi}} \sin(mx). \tag{2.2}$$

2.2 The formula of source term for problem (1.1)

The solution to problem (1.1) can be represented in the form of an expansion in the orthogonal series as follows:

$$u(t,x) = \sum_{m=1}^{\infty} u_m(t)\xi_m(x), \quad \text{with } u_m(y) = \langle u(y,x), \xi_m(x)\rangle_{\mathcal{L}^2(\Omega)}. \tag{2.3}$$

By considering that the series (2.3) converges and allows a term by term differentiation (the required number of times), we construct a formal solution to the problem. We obtain the problems

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u_m(t) + m^2 u_m(t) + am^2 \displaystyle\int_0^t u_m(s)\mathrm{d}s = \rho_m, \quad t \in (0,L), \\ u_m(L) = g_m,\ u_m(0) = 0, \end{cases} \tag{2.4}$$

where

$$g(x) = \sum_{m=1}^{\infty} g_m \xi_m(x) \quad \text{and } \rho(x) = \sum_{m=1}^{\infty} \rho_m \xi_m(x).$$

Let us set the following function:

$$w_m(t) = \int_0^t u_m(s)\mathrm{d}s.$$

Then, we obtain $w_m'(t) = u_m(t)$ and $w_m'(0) = u_m(0) = 0$. From the main equation of equation (2.4), we find that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} w_m(t) + m^2 \frac{\mathrm{d}}{\mathrm{d}t} w_m(t) + am^2 w_m(t) = \rho_m. \tag{2.5}$$

By using some simple calculation, we obtain that

$$\begin{aligned} w_m(t) = &\left(A_m + \frac{\rho_m}{\frac{-m^2+\sqrt{m^4-4am^2}}{2}\sqrt{m^4-4am^2}} \right) e^{\left(\frac{-m^2+\sqrt{m^4-4am^2}}{2}\right)t} - \frac{\rho_m}{\frac{-m^2+\sqrt{m^4-4am^2}}{2}\sqrt{m^4-4am^2}} \\ &+ \left(B_m - \frac{\rho_m}{\frac{-m^2-\sqrt{m^4-4am^2}}{2}\sqrt{m^4-4am^2}} \right) e^{\left(\frac{-m^2-\sqrt{m^4-4am^2}}{2}\right)t} + \frac{\rho_m}{\frac{-m^2-\sqrt{m^4-4am^2}}{2}\sqrt{m^4-4am^2}}. \end{aligned} \tag{2.6}$$

Since $w_m(0) = w_m'(0) = 0$, we know that

$$A_m = B_m = 0.$$

Taking the first derivative of w_m, we obtain that

$$u_m(t) = w_m'(t) = \frac{\rho_m}{\sqrt{m^4-4am^2}} \left[e^{\left(\frac{-m^2+\sqrt{m^4-4am^2}}{2}\right)t} - e^{\left(\frac{-m^2-\sqrt{m^4-4am^2}}{2}\right)t} \right]. \tag{2.7}$$

Under the condition $u_m(L) = g_m$ as in equation (2.4), we know that

$$u_m(L) = \frac{\rho_m}{\sqrt{m^4-4am^2}} \left[e^{\left(\frac{-m^2+\sqrt{m^4-4am^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4am^2}}{2}\right)L} \right]. \tag{2.8}$$

This implies that

$$\rho_m = \frac{\sqrt{m^4 - 4am^2}}{e^{\left(\frac{-m^2+\sqrt{m^4-4am^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4am^2}}{2}\right)L}} g_m. \tag{2.9}$$

Thus, we derive that

$$\rho(x) = \sum_{m=1}^{\infty} \frac{\sqrt{m^4 - 4am^2}}{e^{\left(\frac{-m^2+\sqrt{m^4-4am^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4am^2}}{2}\right)L}} g_m \xi_m(x). \tag{2.10}$$

2.3 Ill-posedness

In the following, we provide an example that shows that the function (2.10) does not depend continuously on the given data g. For $k \in \mathbb{N}^*$, we set

$$\widetilde{g}^k(x) = \frac{1}{\sqrt{k}} \xi_k(x). \tag{2.11}$$

Combining equations (2.10) and (2.11), we obtain that

$$\rho^k(x) = \frac{\sqrt{k^4 - 4\alpha k^2}}{e^{\left(\frac{-k^2+\sqrt{k^4-\alpha k^2}}{2}\right)L} - e^{\left(\frac{-k^2-\sqrt{k^4-\alpha k^2}}{2}\right)L}} \frac{1}{\sqrt{k}} \xi_k(x). \tag{2.12}$$

It is obvious to see that

$$\|\widetilde{g}^k\|_{L^2(\Omega)} = \frac{1}{\sqrt{k}} \to 0, \quad k \to +\infty, \tag{2.13}$$

and

$$\|\rho^k\|_{L^2(\Omega)} = \frac{\sqrt{k^4 - 4\alpha k^2}}{e^{\left(\frac{-k^2+\sqrt{k^4-\alpha k^2}}{2}\right)L} - e^{\left(\frac{-k^2-\sqrt{k^4-\alpha k^2}}{2}\right)L}} \frac{1}{\sqrt{k}}. \tag{2.14}$$

It is easy to check that

$$e^{\left(\frac{-k^2+\sqrt{k^4-\alpha k^2}}{2}\right)L} - e^{\left(\frac{-k^2-\sqrt{k^4-\alpha k^2}}{2}\right)L} \le e^{\left(\frac{-k^2+\sqrt{k^4-\alpha k^2}}{2}\right)L} \le 1. \tag{2.15}$$

Since two latter estimates, we find that

$$\|\rho^k\|_{L^2(\Omega)} \ge \frac{2\sqrt{k^4 - 4\alpha k^2}}{\sqrt{k}} \ge 2\sqrt{k}\sqrt{k^2 - 4\alpha} \ge 2\sqrt{k}\sqrt{1 - 4\alpha}. \tag{2.16}$$

Combining (2.13) and (2.16), we deduce that the inverse source problem is ill-posed in the sense of Hadamard.

3 Convergent of the source term

Theorem 3.1. *Let us assume that $g \in H^2(\Omega)$. Let ρ_α be the source term of the problem* (1.1). *Let ρ be the source term of the following classical parabolic equation*:

$$\begin{cases} u_t = u_{xx} + \rho(x), & \text{in } Q_L := (0, L) \times (0, \pi), \\ u(t, 0) = u(t, \pi) = 0, & 0 < t < L, \\ u(L, x) = g(x), \ u(0, x) = 0, & 0 < x < \pi. \end{cases} \tag{3.1}$$

Then there exist two constants $\mathcal{D}_1$ and $\mathcal{D}_2$ that depend on α and L such that

$$\|\rho_\alpha - \rho\|_{L^2(\Omega)} \le |\mathcal{D}_1(\alpha, L)|\alpha\ \|g\|_{L^2(\Omega)} + |\mathcal{D}_2(\alpha, L)|\alpha\ \|g\|_{\mathbb{H}^2(\Omega)}. \tag{3.2}$$

Proof. Since the source function ρ depends on α, we can denote it by ρ_α. From the formula (2.10), we have

$$\rho_\alpha(x) = \sum_{m=1}^{\infty} \frac{\sqrt{m^4 - 4\alpha m^2}}{e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}} g_m \xi_m(x). \tag{3.3}$$

The source term of the problem (3.1) is given as follows:

$$\rho(x) = \sum_{m=1}^{\infty} \frac{m^2}{1 - e^{-m^2 L}} g_m \xi_m(x). \tag{3.4}$$

Let us evaluate the difference

$$\mathcal{B}(m, \alpha) = \left| \frac{\sqrt{m^4 - 4\alpha m^2}}{e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}} - \frac{m^2}{1 - e^{-m^2 L}} \right|.$$

Using the triangle inequality, we obtain

$$\mathcal{B}(m, \alpha) \le \mathcal{B}_1(m, \alpha) + \mathcal{B}_2(m, \alpha). \tag{3.5}$$

Here,

$$\mathcal{B}_1(m, \alpha) = \frac{\sqrt{m^4 - 4\alpha m^2} - m^2}{e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}},$$

and

$$\mathcal{B}_2(m, \alpha) = \frac{m^2}{e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}} - \frac{m^2}{1 - e^{-m^2 L}}.$$

It is easy to see that

$$\frac{1}{e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}} = \frac{e^{\frac{m^2 L}{2}}}{e^{\left(\frac{\sqrt{m^4-4\alpha m^2}}{2}\right)L} - e^{\left(\frac{-\sqrt{m^4-4\alpha m^2}}{2}\right)L}} \le \frac{e^{\frac{m^2 L}{2}}}{e^{\left(\frac{\sqrt{m^4-4\alpha m^2}}{2}\right)L} - 1}. \tag{3.6}$$

In addition, we obtain

$$\frac{e^{\frac{m^2 L}{2}}}{e^{\left(\frac{\sqrt{m^4-4\alpha m^2}}{2}\right)L} - 1} = e^{\left(\frac{m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L} \frac{e^{\left(\frac{\sqrt{m^4-4\alpha m^2}}{2}\right)L}}{e^{\left(\frac{\sqrt{m^4-4\alpha m^2}}{2}\right)L} - 1}. \tag{3.7}$$

Since $m^2 - \sqrt{m^4 - 4\alpha m^2} = \frac{4\alpha m^2}{m^2 + \sqrt{m^4 - 4\alpha m^2}} \le 4\alpha$, we find that

$$e^{\left(\frac{m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L} \le e^{2\alpha L}.$$

Noting that $\sqrt{m^4 - 4\alpha m^2} \ge \sqrt{1 - 4\alpha}$ for $m \ge 1$, we derive that

$$\frac{e^{\left(\frac{\sqrt{m^4-4\alpha m^2}}{2}\right)L}}{e^{\left(\frac{\sqrt{m^4-4\alpha m^2}}{2}\right)L} - 1} = \frac{1}{1 - e^{\left(\frac{-\sqrt{m^4-4\alpha m^2}}{2}\right)L}} \le \frac{1}{1 - e^{\left(\frac{-\sqrt{1-4\alpha}}{2}\right)L}}. \tag{3.8}$$

From some above observations, we find that

$$\frac{1}{e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}} \le \frac{e^{2\alpha L}}{1 - e^{\left(\frac{-\sqrt{1-4\alpha}}{2}\right)L}}. \tag{3.9}$$

Using equation (3.9), we obtain

$$|\mathcal{B}_1(m,\alpha)| \le \frac{e^{2\alpha L}}{1 - e^{\left(\frac{-\sqrt{1-4\alpha}}{2}\right)L}}\left(m^2 - \sqrt{m^4 - 4\alpha m^2}\right) \le \frac{4\alpha e^{2\alpha L}}{1 - e^{\left(\frac{-\sqrt{1-4\alpha}}{2}\right)L}} = \mathcal{D}_1(\alpha, L)\alpha. \tag{3.10}$$

Here,

$$\mathcal{D}_1(\alpha, L) = \frac{4e^{2\alpha L}}{1 - e^{\left(\frac{-\sqrt{1-4\alpha}}{2}\right)L}}.$$

Let us now treat the term $\mathcal{B}_2(m,\alpha)$. It is obvious to see that

$$\begin{aligned}
|\mathcal{B}_2(m,\alpha)| &\le \frac{m^2}{\left(e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}\right)(1 - e^{-m^2L})} \\
&\quad \times \left|1 - e^{-m^2L} - e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} + e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}\right| \\
&\le \frac{e^{2\alpha L}m^2}{\left(1 - e^{\left(\frac{-\sqrt{1-4\alpha}}{2}\right)L}\right)(1 - e^{-L})}\left|1 - e^{-m^2L} - e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} + e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}\right|,
\end{aligned} \tag{3.11}$$

where we have used equation (3.9). We give the following bound:

$$\begin{aligned}
&\left|1 - e^{-m^2L} - e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} + e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}\right| \\
&\quad \le \left|1 - e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L}\right| + \left|e^{-m^2L} - e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}\right|.
\end{aligned} \tag{3.12}$$

Using the inequality $1 - e^{-z} \le z$ for any $z > 0$, we find that

$$\begin{aligned}
\left|1 - e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L}\right| &= 1 - e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} \\
&\le \frac{\left(m^2 - \sqrt{m^4 - 4\alpha m^2}\right)}{2}L \\
&\le \frac{1}{2}\frac{4\alpha m^2 L}{m^2 + \sqrt{m^4 - 4\alpha m^2}} \le 2\alpha L.
\end{aligned} \tag{3.13}$$

Using the latter estimate, we continue to obtain

$$\left|e^{-m^2L} - e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}\right| = e^{-m^2L}\left|1 - e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L}\right| \le 4\alpha L. \tag{3.14}$$

This implies that

$$\left|1 - e^{-m^2L} - e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} + e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}\right| \le 4\alpha L. \tag{3.15}$$

It follows from equation (3.11) that

$$|\mathcal{B}_2(m,\alpha)| \le \frac{4\alpha L e^{2\alpha L} m^2}{\left(1 - e^{\left(\frac{-\sqrt{1-4\alpha}}{2}\right)L}\right)(1 - e^{-L})} = \mathcal{D}_2(\alpha, L)\alpha m^2. \tag{3.16}$$

Here,

$$\mathcal{D}_2(\alpha, L) = \frac{4Le^{2\alpha L}}{\left(1 - e^{\left(\frac{-\sqrt{1-4\alpha}}{2}\right)L}\right)(1 - e^{-L})}.$$

Based on some previous evaluations, we obtain that

$$\left| \frac{\sqrt{m^4 - 4\alpha m^2}}{e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}} - \frac{m^2}{1 - e^{-m^2 L}} \right| \le \mathcal{D}_1(\alpha, L)\alpha + \mathcal{D}_2(\alpha, L)\alpha m^2. \tag{3.17}$$

Using Parseval's equality and in view of equations (3.3) and (3.1), we deduce that

$$\begin{aligned}
\|\rho_\alpha - \rho\|^2_{L^2(\Omega)} &= \sum_{m=1}^{\infty} \left| \frac{\sqrt{m^4 - 4\alpha m^2}}{e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}} - \frac{m^2}{1 - e^{-m^2 L}} \right|^2 g_m^2 \\
&\le |\mathcal{D}_1(\alpha, L)|^2\alpha^2\left(\sum_{m=1}^{\infty} g_m^2\right) + |\mathcal{D}_2(\alpha, L)|^2\alpha^2\left(\sum_{m=1}^{\infty} m^4 g_m^2\right) \\
&= |\mathcal{D}_1(\alpha, L)|^2\alpha^2 \,\|g\|^2_{L^2(\Omega)} + |\mathcal{D}_2(\alpha, L)|^2\alpha^2 \,\|g\|^2_{\mathbb{H}^2(\Omega)}.
\end{aligned} \tag{3.18}$$

The latter estimate implies that the desired result (3.2).

4 Regularization by Fourier truncation method

Let us assume that g^ε be a measured data that satisfies

$$\|g^\varepsilon - g\|_{L^2(\Omega)} \le \varepsilon. \tag{4.1}$$

Let us define Fourier truncation solution as follows:

$$\rho^\varepsilon(x) = \sum_{m \le M_\varepsilon} \frac{\sqrt{m^4 - 4\alpha m^2}}{e^{\left(\frac{-m^2+\sqrt{m^4-4\alpha m^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4\alpha m^2}}{2}\right)L}} g_m^\varepsilon \xi_m(x), \tag{4.2}$$

where $M_\varepsilon > 0$ depends on ε. We will choose it for later theorem.

Theorem 4.1. *Let us assume that $\rho \in \mathbb{H}^s(\Omega)$ for $s > 0$. Let us choose $M_\varepsilon > 0$ such that*

$$\lim_{\varepsilon\to 0} M_\varepsilon^2 \varepsilon = \lim_{\varepsilon\to 0} (M_\varepsilon)^{-1} = 0. \tag{4.3}$$

Then, we obtain

$$\|\rho - \rho^\varepsilon\|_{L^2(\Omega)} \le \frac{e^{L\alpha}}{1 - e^{-L\sqrt{1-\alpha}}} M_\varepsilon^2 \varepsilon + (M_\varepsilon)^{-s}\|\rho\|_{\mathbb{H}^s(\Omega)}. \tag{4.4}$$

Remark 4.1. Let us choose $M_\varepsilon = \varepsilon^{\frac{\mu-1}{2}}$ for any $0 < \mu < 1$. Then, the error $\|\rho - \rho^\varepsilon\|_{L^2(\Omega)}$ is of order

$$\max\left(\varepsilon^\mu, \varepsilon^{\frac{(1-\mu)s}{2}}\right).$$

Proof. We set the following function:

$$\theta^\varepsilon(x) = \sum_{m\le M_\varepsilon} \frac{\sqrt{m^4-4am^2}}{e^{\left(\frac{-m^2+\sqrt{m^4-4am^2}}{2}\right)L} - e^{\left(\frac{-m^2-\sqrt{m^4-4am^2}}{2}\right)L}} g_m \xi_m(x). \tag{4.5}$$

Using Parseval's equality, we derive that

$$\|\rho^\varepsilon - \theta^\varepsilon\|^2_{L^2(\Omega)} = \sum_{m\le M_\varepsilon}\left(\frac{2\sqrt{m^4-am^2}}{e^{\left(\frac{-m^2+\sqrt{m^4-am^2}}{2}\right)L} - e^{\frac{(-m^2-\sqrt{m^4-am^2})}{2}L}}\right)^2 |g^\varepsilon_m - g_m|^2. \tag{4.6}$$

It is clear to see that

$$\begin{aligned}\frac{\sqrt{m^4-am^2}}{e^{\frac{(-m^2+\sqrt{m^4-am^2})}{2}L} - e^{\frac{(-m^2-\sqrt{m^4-am^2})}{2}L}} &= \frac{\sqrt{m^4-am^2}\, e^{\frac{m^2}{2}L}}{e^{\frac{\sqrt{m^4-am^2}}{2}L}-1}\\ &= \sqrt{m^4-am^2}\, e^{\frac{(m^2-\sqrt{m^4-am^2})}{2}L}\frac{e^{\frac{\sqrt{m^4-am^2}}{2}L}}{e^{\frac{\sqrt{m^4-am^2}}{2}L}-1}.\end{aligned} \tag{4.7}$$

Since

$$\frac{m^2-\sqrt{m^4-am^2}}{2} = \frac{1}{2}\frac{am^2}{m^2+\sqrt{m^4-am^2}} \le \frac{a}{2},$$

we know that

$$e^{\frac{(m^2-\sqrt{m^4-am^2})}{2}L} \le e^{\frac{aL}{2}}.$$

In addition, we derive that

$$\frac{e^{\frac{\sqrt{m^4-am^2}}{2}L}}{e^{\frac{\sqrt{m^4-am^2}}{2}L}-1} = \frac{1}{1-e^{-\frac{\sqrt{m^4-am^2}}{2}L}} \le \frac{1}{1-e^{-L\frac{\sqrt{1-a}}{2}}}\ . \tag{4.8}$$

From some above observations, we deduce that

$$\frac{\sqrt{m^4-am^2}}{e^{\frac{(-m^2+\sqrt{m^4-am^2})}{2}L} - e^{\frac{(-m^2-\sqrt{m^4-am^2})}{2}L}} \le \frac{e^{\frac{La}{2}}}{1-e^{-L\frac{\sqrt{1-a}}{2}}}\sqrt{m^4-am^2} \le \frac{e^{\frac{La}{2}}}{1-e^{-L\frac{\sqrt{1-a}}{2}}} m^2. \tag{4.9}$$

It follows from equation (4.6) that

$$\|\rho^\varepsilon - \theta^\varepsilon\|^2_{L^2(\Omega)} = \left(\frac{e^{\frac{La}{2}}}{1-e^{-L\frac{\sqrt{1-a}}{2}}}\right)^2 M_\varepsilon^4 \sum_{m\le M_\varepsilon} |g^\varepsilon_m - g_m|^2. \tag{4.10}$$

Hence, we have immediately that

$$\|\rho^\varepsilon - \theta^\varepsilon\|_{L^2(\Omega)} \le \left(\frac{e^{\frac{La}{2}}}{1-e^{-L\frac{\sqrt{1-a}}{2}}}\right) M_\varepsilon^2\ \|g^\varepsilon - g\|_{L^2(\Omega)} \le \left(\frac{e^{\frac{La}{2}}}{1-e^{-L\frac{\sqrt{1-a}}{2}}}\right) M_\varepsilon^2 \varepsilon. \tag{4.11}$$

Next, estimate of the term $\|\rho - \theta^\varepsilon\|_{L^2(\Omega)}$. Indeed, we obtain the following bound:

$$\|\rho - \theta^\varepsilon\|^2_{L^2(\Omega)} = \sum_{m\le M_\varepsilon}\rho_m^2 = \sum_{m\le M_\varepsilon} m^{-2s}m^{2s}\rho_m^2 \le (M_\varepsilon)^{-2s}\|\rho\|^2_{\mathbb{H}^s(\Omega)}. \tag{4.12}$$

Hence, we obtain the following bound:

$$\|\rho - \theta^\varepsilon\|_{L^2(\Omega)} \le (M_\varepsilon)^{-s}\|\rho\|_{\mathbb{H}^s(\Omega)}. \tag{4.13}$$

By collecting equations (4.11) and (4.13), we derive that

$$\begin{aligned}\|\rho - \rho^\varepsilon\|_{L^2(\Omega)} &\leq \|\rho - \theta^\varepsilon\|_{L^2(\Omega)} + \|\rho^\varepsilon - \theta^\varepsilon\|_{L^2(\Omega)} \\ &\leq \frac{e^{\frac{La}{2}}}{1 - e^{-L\frac{\sqrt{1-a}}{2}}} M_\varepsilon^2 \varepsilon + (M_\varepsilon)^{-s} \|\rho\|_{\mathbb{H}^s(\Omega)}.\end{aligned} \tag{4.14}$$

□

Theorem 4.2. *Let us assume that g^ε is the noisy data for g that satisfies*

$$\|g^\varepsilon - g\|_{L^p(\Omega)} \leq \varepsilon \tag{4.15}$$

for $\varepsilon > 0$ and $1 \leq p < 2$. Let $\rho \in \mathbb{H}^{s+b}(\Omega)$ for $b \geq 0$ and $s \geq 0$. Then, we obtain

$$\|\rho - \rho^\varepsilon\|_{\mathbb{H}^b(\Omega)} \leq C_{p,k} \left(\frac{e^{\frac{La}{2}}}{1 - e^{-L\frac{\sqrt{1-a}}{2}}} \right) (M_\varepsilon)^{b-k+2} \varepsilon + (M_\varepsilon)^{-s} \|\rho\|_{\mathbb{H}^{s+b}(\Omega)}. \tag{4.16}$$

Here,

$$-\frac{1}{4} < k \leq \frac{p-2}{4p},$$

and M_ε is chosen such that

$$\lim_{\varepsilon \to 0} (M_\varepsilon)^{b-k+2} \varepsilon = (M_\varepsilon)^{-1} = 0. \tag{4.17}$$

Remark 4.2. Let $M_\varepsilon = \varepsilon^{\frac{\mu-1}{b-k+2}}$ for any $0 < \mu < 1$. Then, the error $\|\rho - \rho^\varepsilon\|_{\mathbb{H}^b(\Omega)}$ is of order

$$\max\left(\varepsilon^\mu, \varepsilon^{\frac{(1-\mu)s}{b-k+2}} \right).$$

Proof. Let us recall the function θ^ε that is defined in equation (4.5). Let us take a positive constant b. We need to give the upper bound for the error

$$\|\rho - \rho^\varepsilon\|_{\mathbb{H}^b(\Omega)}. \tag{4.18}$$

It is obvious to obtain that Sobolev embedding $L^p(\Omega) \hookrightarrow H^k(\Omega)$ for (p, k), satisfying the constraints $\frac{-1}{4} \leq k < 0$ and $p \geq \frac{2}{1-4k}$. Hence, we can find a constant $C_{p,k} > 0$, which depends on p and k such that

$$\|g^\varepsilon - g\|_{\mathbb{H}^k(\Omega)} \leq C_{p,k} \|g^\varepsilon - g\|_{L^p(\Omega)} \leq \varepsilon C_{p,k}. \tag{4.19}$$

Using Parseval's equality, we have

$$\begin{aligned}\|\rho^\varepsilon - \theta^\varepsilon\|^2_{\mathbb{H}^b(\Omega)} &= \sum_{m \leq M_\varepsilon} m^{2b} \left(\frac{2\sqrt{m^4 - am^2}}{e^{\frac{(-m^2 + \sqrt{m^4 - am^2})}{2} L} - e^{\frac{(-m^2 - \sqrt{m^4 - am^2})}{2} L}} \right)^2 |g^\varepsilon_m - g_m|^2 \\ &= \sum_{m \leq M_\varepsilon} m^{2b-2k} \left(\frac{2\sqrt{m^4 - am^2}}{e^{\frac{(-m^2 + \sqrt{m^4 - am^2})}{2} L} - e^{\frac{(-m^2 - \sqrt{m^4 - am^2})}{2} L}} \right)^2 m^{2k} |g^\varepsilon_m - g_m|^2 \\ &\leq \left(\frac{e^{\frac{La}{2}}}{1 - e^{-L\frac{\sqrt{1-a}}{2}}} \right)^2 \sum_{m \leq M_\varepsilon} m^{2b-2k+4} m^{2k} |g^\varepsilon_m - g_m|^2,\end{aligned} \tag{4.20}$$

where we have used the fact that (4.10). This follows from equation (4.19) that

$$\begin{aligned}\|\rho^\varepsilon - \theta^\varepsilon\|^2_{\mathbb{H}^b(\Omega)} &\leq \left(\frac{e^{\frac{La}{2}}}{1 - e^{-L\frac{\sqrt{1-a}}{2}}} \right)^2 (M_\varepsilon)^{2b-2k+4} \|g^\varepsilon - g\|^2_{\mathbb{H}^k(\Omega)} \\ &\leq C^2_{p,k} \left(\frac{e^{\frac{La}{2}}}{1 - e^{-L\frac{\sqrt{1-a}}{2}}} \right)^2 (M_\varepsilon)^{2b-2k+4} \varepsilon^2.\end{aligned} \tag{4.21}$$

Hence, we deduce that

$$\|\rho^{\varepsilon} - \theta^{\varepsilon}\|_{\mathbb{H}^{b}(\Omega)} \leq C_{p,k}\left(\frac{e^{\frac{L\alpha}{2}}}{1 - e^{-L\frac{\sqrt{1-\alpha}}{2}}}\right)(M_{\varepsilon})^{b-k+2}\varepsilon. \tag{4.22}$$

Let us now to treat the term $\|\rho - \theta^{\varepsilon}\|_{\mathbb{H}^{b}(\Omega)}$. Indeed, we obtain the following bound:

$$\|\rho - \theta^{\varepsilon}\|^{2}_{\mathbb{H}^{b}(\Omega)} = \sum_{m \leq M_{\varepsilon}} \rho_{m}^{2} = \sum_{m \leq M_{\varepsilon}} m^{-2s} m^{2s+2b} \rho_{m}^{2} \leq (M_{\varepsilon})^{-2s} \|\rho\|^{2}_{\mathbb{H}^{s+b}(\Omega)}. \tag{4.23}$$

Thus, we find that

$$\|\rho - \theta^{\varepsilon}\|_{\mathbb{H}^{b}(\Omega)} \leq (M_{\varepsilon})^{-s} \|\rho\|_{\mathbb{H}^{s+b}(\Omega)}. \tag{4.24}$$

Combining equations (4.22) and (4.24), we obtain

$$\begin{aligned} \|\rho - \rho^{\varepsilon}\|_{\mathbb{H}^{b}(\Omega)} &\leq \|\rho - \theta^{\varepsilon}\|_{\mathbb{H}^{b}(\Omega)} + \|\rho^{\varepsilon} - \theta^{\varepsilon}\|_{\mathbb{H}^{b}(\Omega)} \\ &\leq C_{p,k}\left(\frac{e^{\frac{L\alpha}{2}}}{1 - e^{-L\frac{\sqrt{1-\alpha}}{2}}}\right)(M_{\varepsilon})^{b-k+2}\varepsilon + (M_{\varepsilon})^{-s}\|\rho\|_{\mathbb{H}^{s+b}(\Omega)}. \end{aligned} \tag{4.25}$$

The proof is completed. □

5 Conclusion

In this work, we investigate model (1.1) with memory term. We show the exact solution and present an example of the problem to prove the ill-posed of equation (2.10), and the regularized solution is built based on the Fourier transform formula. We prove that solution (2.10) converges to solution (3.4) in Theorem 3.1. Next, Theorems 4.1 and 4.2 investigate the convergence results between the exact solution and the regularized solution with data belong to $L^{2}(\Omega)$ and $L^{p}(\Omega)$.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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