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Research Article

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Existence and regularity of solutions for non-autonomous integrodifferential evolution equations involving nonlocal conditions

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Abstract: In this article, we investigate the existence and regularity of solutions for non-autonomous integrodifferential evolution equations involving nonlocal conditions. Using the theory of resolvent operators, some fixed point theorems, and an estimation technique of Kuratowski measure of noncompactness, we first establish some existence results of mild solutions for the proposed equation. Subsequently, we show by applying a newly established lemma that these solutions have regularity property under some conditions. Finally, as a sample of application, the obtained results are applied to a class of non-autonomous nonlocal partial integrodifferential equations.

Keywords: non-autonomous integrodifferential evolution equation, resolvent operator, fixed point theorem, nonlocal condition

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1 Introduction

In this article, we study the existence and regularity of solutions for non-autonomous integrodifferential evolution equations involving nonlocal conditions of the form:

$$\begin{cases} x'(t) = A(t)x(t) + \int_0^t B(t,s)x(s)\mathrm{d}s + G(t, x(t), x(r(t, x(t)))), \quad t \in [0, T], \\ x(0) + H(x, x) = x_0, \end{cases} \tag{1}$$

where $x(\cdot)$ is the state variable taking values in a Banach space X. For any $t \in [0, T]$, $\{A(t)\}_{t\in[0,T]}$ is the infinitesimal generator of an analytic semigroup $S_t(s)$, $s \geq 0$ and has fixed domain $D(A)$, and $\{B(t, s)\}_{(t,s)\in\triangle}$ ($\triangle$ defined in Section 2) is a family of closable linear operators with domains $D(B(t, s)) \supset D(A)$ satisfying some additional regularity conditions. The delay function $r(\cdot,\cdot) : [0, T] \times X \to [0, T]$ is continuous. $G(\cdot,\cdot,\cdot)$, and $H(\cdot,\cdot)$ are given functions to be defined later.

Partial integrodifferential equations, which can be transformed into abstract integrodifferential evolution equations in Banach space X, arise in electronics, biological models, heat flow in materials with fading memory, viscoelastic materials, and a lot of other physical phenomena [1–4]. In fact, Grimmer and Kappel [5], Grimmer and Pritchard [6] gained the resolvent operator associated with the following linear homogeneous equation in Banach space X

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$$\begin{cases} v'(t) = Av(t) + \int_0^t \gamma(t-s)v(s)\mathrm{d}s, & \text{for } t \ge 0, \\ v(0) = v_0 \in X, \end{cases}$$

and using resolvent operator theory, the existence of solutions of the following integrodifferential evolution equation:

$$\begin{cases} v'(t) = Av(t) + \int_0^t \gamma(t-s)v(s)\mathrm{d}s + f(t), & \text{for } t \ge 0, \\ v(0) = v_0 \in X, \end{cases} \tag{2}$$

was proved, where f is a continuous function. The resolvent operator, which replaces the role of C_0-semigroup for evolution equations, has an important effect on solving equation (2) in weak and strict senses. For more classical resolvent operator theory, we refer to Prüss's book [7]. In these years, several articles have been devoted to discussing the existence, regularity, stability, almost periodicity, and controllability of solutions for semilinear integrodifferential evolution equations by using the theory of resolvent operators (see [8–15] and references therein).

We noted that all the problems under considerations in the aforementioned works are autonomous integrodifferential evolution equation (2), i.e., the operator A is independent of time t. However, when we treat some non-autonomous integrodifferential evolution equations, it is usually supposed that the operator A is related to time t (i.e., $A = A(t)$), since this kind of operator is widely used in the applications [16]. Thus, it is interesting and important to research non-autonomous integrodifferential evolution equations. Actually, some topics for this class of equations, such as existence, uniqueness, regularity, and controllability of solutions, have been studied by many mathematicians (see [17–24]).

As we know, fixed point theory is a very powerful and important tool for studying the qualitative theory of differential equations. Concerning the literature, we cite the recent works by Younis et al. [25–27] for differential and integral equations, and the articles [28,29] for partial and abstract differential equations.

On the other hand, the nonlocal initial condition, which is a generalization of the classical initial condition $x(0) = x_0$, was motivated by physical problems. For example, Deng [30] described the diffusion phenomenon of a small amount of gas in a transparent tube by making use of the formula:

$$g(x) = \sum_{i=1}^{p} c_i x(t_i), \tag{3}$$

where c_i, $i = 1, \ldots, p$ are given constants and $0 < t_1 < \cdots < t_p < a$. In this situation, condition (3) allows the additional measurement at t_i, $i = 1, \ldots, p$, which is more accurate than the measurement just at $t = 0$. The pioneering work on evolution equations with the nonlocal condition (3) is due to Byszewski [31]. Since then, (integrodifferential) evolution equations with nonlocal conditions have been considered by many authors and several interesting results on some themes of nonlocal problems have been achieved (see [32–42] among others).

Meanwhile, evolution equations with the nonlocal condition $H(x, x)$ are also used to represent mathematical models for evolution of various phenomena. For instance, McKibben [43] considered the following nonlocal combustion model:

$$\begin{cases} \frac{\partial}{\partial t} z(\xi, t) = \frac{\partial^2}{\partial \xi^2} z(\xi, t) - 2\left(\frac{\partial}{\partial \xi} z(\xi, t)\right)^{\frac{5}{3}}, & -a \le \xi \le a,\ 0 < t < T, \\ z(a, t) = z(-a, t), 0 < t < T, \\ z(\xi, 0) = \sum_{i=1}^{n} \beta_i z(\xi, t_i) + \int_0^T \vartheta(s)\chi(s, z(\xi, s))\mathrm{d}s, & -a \le \xi \le a, \end{cases} \tag{4}$$

where β_i $(i = 1, 2, \ldots, n)$ are the positive real numbers. $\vartheta(\cdot)$ and $\chi(\cdot,\cdot)$ are the suitable functions. If we take $z(\cdot, t) = x(t)(\cdot)$, then we can reformulate (4) abstractly as

$$\begin{cases} x'(t) = \mathcal{A}x(t), & t \in (0, T), \\ x(0) = H(x, x), \end{cases}$$

in the space $E = C_p([-a, a]; \mathbb{R}) = \{h : [-a, a] \to \mathbb{R} :\ h$ is continuous and $h(a) = h(-a)\}$ by identifying the operator $\mathcal{A} : D(\mathcal{A}) \subset E \to E$ as

$$\mathcal{A}z = \frac{\partial^2 z}{\partial \xi^2} - 2\left(\frac{\partial z}{\partial \xi}\right)^{\frac{5}{3}},$$

with the domain

$$D(\mathcal{A}) = \left\{z \in E : \frac{\partial z}{\partial \xi}, \frac{\partial^2 z}{\partial \xi^2} \in C_p([-a, a]; \mathbb{R})\right\}.$$

Based on the aforementioned analysis, it is obvious that the nonlocal condition $H(x, x)$ is more general and complicated than the condition (3) and standard initial condition $x(0) = x_0$. For more models about evolution equations with the nonlocal condition $H(x, x)$, one can refer to [44].

Inspired by the aforementioned discussions, in this study, we are going to discuss the existence and regularity of solutions for non-autonomous integrodifferential evolution equations involving nonlocal conditions (1). Combining the theory of resolvent operators for linear integrodifferential systems founded in [45], some fixed point principles, and an estimation technique of Kuratowski measure of noncompactness established by Chen and Cheng [46] (see Lemma 2.4), we first establish the existence of mild solutions for equation (1) under the situation that the nonlinear function $G(\cdot,\cdot,\cdot)$ and nonlocal function $H(\cdot,\cdot)$ satisfy different conditions, respectively. We would like to mention here that, in Theorem 3.2, we do not require the compactness condition for the resolvent operator $R(\cdot,\cdot)$, which is quite different from [19,24]. Based on these preparations, we further discuss the regularity of mild solutions of System (1) by using a newly established Lemma 4.1. Namely, we give some sufficient conditions to prove that each mild solution may be a strict solution. It is also worth mentioning that, in [37,40], the authors studied the regularity of solutions for non-autonomous semilinear integrodifferential equations with the nonlocal condition (3) under the condition that the nonlinear function satisfies continuously differentiable. However, the technique used in [37,40] becomes invalid for equation (1) since we only require that the nonlinear function $G(\cdot,\cdot,\cdot)$ satisfies Lipschitz or Hölder continuous (see (H_5) or (H_8)). Obviously, the obtained results in this article extend and develop the existing results, such as the aforementioned ones.

This article is built up as follows: in Section 2, we state briefly the basic theory of resolvent operators and Kuratowski measure of noncompactness. In Section 3, we are devoted to discussing the existence of mild solutions of (1) using Schauder and Darbo-Sadovskii fixed point theorems, respectively. The regularity problems are investigated in Section 4. Finally, an example is provided to demonstrate the applications of the obtained results in Section 5.

2 Preliminaries

In this section, we introduce some preliminary facts on theory of resolvent operators and Kuratowski measure of noncompactness to be used in this article. Let $(X, \|\cdot\|)$ and $(W, \|\cdot\|_W)$ be two Banach spaces. We shall suppose for each $t \in [0, T]$ that $A(t) : D(A) \subseteq X \to X$ is a closed linear operator with fixed domain. Next, we denote by Y the Banach space $(D(A), \|\cdot\|_1)$ equipped with the graph norm $\|x\|_1 = \|A(0)x\| + \|x\|$, for $x \in D(A)$. The symbol $\mathscr{L}(W, X)$ represents the space of bounded linear operators $W \to X$ endowed with norm $\|\cdot\|_{W.0}$ $(X_0 = X)$, and we write $\mathscr{L}(X)$ and $\|\cdot\|$ if $W = X$. The sets $\triangle = \{(t, s) : 0 \le s \le t \le T\}$ and $\triangle^0 = \{(t, s) : 0 \le s < t \le T\}$ are frequently used. Hereafter, $C([0, T]; X)$ is the Banach space of continuous functions from $[0, T]$ into X with the norm of supremum $\|\cdot\|_C$. $C_s(\triangle; W, X)$ denotes the space of all strongly continuous functions $H : \triangle \to \mathscr{L}(W, X)$, i.e., $H(\cdot,\cdot)y$ is continuous in X on $\triangle$, for any fixed $y \in W$, and we let $C_s(\triangle; X) = C_s(\triangle; X, X)$ for short.

We now introduce the theory of resolvent operators for linear homogeneous integrodifferential equation related to equation (1).

Definition 2.1. ([45], Definition 5) A two-parameter family of linear operators $\{R(t,s)\}_{(t,s)\in\triangle}$ in $\mathscr{L}(X)$ is said to be a resolvent operator for the equation

$$\begin{cases} x'(t) = A(t)x(t) + \int_0^t B(t,s)x(s)\mathrm{d}s, \quad t \in [0,T], \\ x(0) = x_0 \in X, \end{cases} \tag{5}$$

if it fulfills the following properties:

(P_1) $R \in C_s(\triangle; X)$ and $R(t,t) = I$ on $[0,T]$;

(P_2) $R(t,s)X \subset Y$ for $(t,s) \in \triangle^0$, $AR \in C_s(\triangle^0; X)$, $ARA^{-1} \in C_s(\triangle; X)$, and

$$\|A(t)R(t,s)\| \le N(t-s)^{-1}, \tag{6}$$

for some constant $N > 0$ and $(t,s) \in \triangle^0$;

(P_3) $\frac{\partial}{\partial t}R(t,s) = A(t)R(t,s) + \int_s^t B(t,r)R(r,s)\mathrm{d}r$ strongly for $t > s$;

(P_4) $\frac{\partial^+}{\partial s}R(t,s) = -R(t,s)A(s) - \int_s^t R(t,r)B(r,s)\mathrm{d}r$ on $D(A(s))$, $t > s$.

In this article, we always suppose that equation (5) fulfills the following assumptions.

(V_1) For any $0 \le t \le T$, $A(t)$ is closed linear densely defined operator in X such that

$$\|(\lambda I - A(t))^{-1}\| \le C_1/(|\lambda| + 1), \quad \text{for all } t \in [0,T], \ \mathrm{Re}\lambda \ge 0, \tag{7}$$

holds for some constant $C_1 \ge 1$.

(V_2) The domains $D(A(t)) = D(A)$ are independent of $t \in [0,T]$ and

$$\|(A(t) - A(s))A^{-1}(\tau)\| \le C_2\,|t-s|^{\alpha}, \quad \text{for all } t, s, \tau \in [0,T],$$

where $C_2 > 0$ and $\alpha \in (0,1]$ are the constants.

(V_3) For any $(t,s) \in \triangle$, $B(t,s)$ is a closable linear operator in X such that $D(B(t,s)) \supset D(A)$. Furthermore, $K : \triangle \to \mathscr{L}(X)$ defined by $K(t,s) = B(t,s)A^{-1}(s)$ satisfies

$$\|K(t,s) - K(\bar{t} - \bar{s})\| \le C_3(|t-\bar{t}|^{\alpha} + |s-\bar{s}|^{\alpha}), \quad \text{for all } (t,s), (\bar{t},\bar{s}) \in \triangle,$$

where $C_3 > 0$ and $\alpha \in (0,1]$ are the constants.

Then, in view of [45, Theorem 2], the aforementioned $(V_1)-(V_3)$ insure that there exists a unique resolvent operator $R(t,s)$ for equation (5). Moreover, the resolvent operator $R(t,s)$ is also analytic and there exists $M \ge 1$ such that

$$\|R(t,s)\| \le M, \quad \text{for } (t,s) \in \triangle. \tag{8}$$

Remark 1. According to [47, Theorem 6.1], we observe that condition (V_1) implies that for every $t \in [0,T]$, $A(t)$ is the infinitesimal generator of an analytic semigroup $S_t(s)$, $s \ge 0$, and it also follows from (V_1) that there exists an angle $\vartheta \in (0, \frac{\pi}{2})$ such that

$$\rho(A(t)) \supset \Sigma = \{\lambda : |\arg\lambda| \ge \vartheta\} \cup \{0\} \tag{9}$$

and that (7) holds for all $\lambda \in \Sigma$, possibly with a different constant C_1. Here, $\rho(A(t))$ stands for the resolvent set of operator $A(t)$.

Remark 2. From (9), it follows that operator $A^{-1}(t)$ exists for $t \in [0, T]$. Furthermore, we note that if, for each $t \in [0, T]$, $A^{-1}(t)$ is a compact operator, then $R(t, s)$ is a compact operator for any $(t, s) \in \triangle^0$ (see [45, Corollary 4]). In addition, the compactness of $R(t, s)$, $(t, s) \in \triangle^0$ indicates the continuity in uniform operator topology.

Based on Remark 2, we can establish the following lemma.

Lemma 2.1. *If the operator $R(t, s)$ is compact for all $(t, s) \in \triangle^0$, then $\|R(\nu, 0)R(t, s) - R(t, s)\| \to 0$ as $\nu \to 0^+$ for each $(t, s) \in \triangle^0$.*

Proof. Suppose that there exists a constant $N_1 > 0$ such that $\|R(z, 0)\| \le N_1$ for every $z \in (0, \frac{1}{2})$, and let $\varepsilon > 0$ be given. Since the set $U_t = \{R(t, s)x : \|x\| \le 1\}$ is compact, and therefore, there are $x_1, x_2, \ldots x_K$ such that the open balls $B(\varepsilon_1, R(t, s)x_m)(1 \le m \le K)$ with radius $\varepsilon_1 = \frac{\varepsilon}{2(N_1+1)}$ centered at $R(t, s)x_m$ cover U_t. According to the strong continuity of $R(\nu, 0)$, there is $0 < \delta \le \frac{1}{2}$ such that

$$\|R(\nu, 0)R(t, s)x_m - R(t, s)x_m\| < \frac{\varepsilon}{2}, \quad \text{for } 0 \le \nu \le \delta \quad \text{and} \quad m = 1, 2, \ldots K. \tag{10}$$

For each $x \in X$ with $\|x\| \le 1$, there exists a x_m such that

$$\|R(t, s)x - R(t, s)x_m\| < \varepsilon_1. \tag{11}$$

Then, from (10) and (11), for $0 \le \nu \le \delta$, we find

$$\begin{aligned}\|R(\nu, 0)R(t, s)x - R(t, s)x\| &\le \|R(\nu, 0)\|\|R(t, s)x - R(t, s)x_m\| + \|R(\nu, 0)R(t, s)x_m - R(t, s)x_m\| \\ &\quad + \|R(t, s)x - R(t, s)x_m\| \\ &< N_1\varepsilon_1 + \frac{\varepsilon}{2} + \varepsilon_1 \\ &= \varepsilon.\end{aligned}$$

Thus, we conclude the assertion. The proof is finished. □

Next, we turn to state the definition and basic properties of Kuratowski measure of noncompactness, which will be used in Section 3.

Definition 2.2. [48] The Kuratowski measure of noncompactness $\eta(\cdot)$ defined on bounded set S of Banach space X is

$$\eta(S) \coloneqq \inf\{r > 0 : S = \cup_{i=1}^{n} S_i \text{ and } \operatorname{diam}(S_i) \le r \quad \text{for } i = 1, 2, \ldots, n\}.$$

Lemma 2.2. [48,49] *Let X be a Banach space, and $U, V \subset X$ be bounded, then the following properties hold*:
(1) $\eta(U) \le \eta(V)$ *if $U \subset V$*;
(2) $\eta(U) = 0$ *if and only if $\overline{U}$ is compact, where $\overline{U}$ means the closure hull of U*;
(3) $\eta(U) = \eta(\overline{U}) = \eta(conv\ U)$, *where conv U means the convex hull of U*;
(4) $\eta(\lambda U) = |\lambda|\eta(U)$, *where $\lambda \in \mathbb{R}$*;
(5) $\eta(U \cup V) = \max\{\eta(U), \eta(V)\}$;
(6) $\eta(U + V) \le \eta(U) + \eta(V)$, *where $U + V = \{z | z = u + v, u \in U, v \in V\}$*;
(7) $\eta(U + z) = \eta(U)$, *for any $z \in X$*.

In what follows, we denote by $\alpha(\cdot)$ the Kuratowski measure of noncompactness on the bounded set of Banach space X and denote by $\alpha_C(\cdot)$ the Kuratowski measure of noncompactness on the bounded set of $C([0, T]; X)$. To discuss our main results, we need the following lemmas.

Lemma 2.3. [50, Lemma 2.1] *Let $D \subset C([0, T]; X)$ be bounded and equicontinuous, then $\overline{co}D \subset C([0, T]; X)$ is also bounded and equicontinuous (where $\overline{co}\ D$ denotes the closed convex hull of D).*

Lemma 2.4. [46, Theorem 5.3] *For each bounded subset $D \subset X$, there exists a countable subset D_0 of D such that $\alpha(D) = \alpha(D_0)$.*

Lemma 2.5. [51, Theorem 2.1] *If $\{x_m\}_{m=1}^{\infty} \subset L^1([0,T];X)$ is uniformly integrable, i.e., there exists a function $\psi \in L^1([0,T];\mathbb{R}^+)$ such that*

$$\|x_m(t)\| \le \psi(t), \quad a.e.\ t \in [0,T], m = 1, 2, \ldots.$$

Then, the function $t \to \alpha(\{x_m(t)\}_{m=1}^{\infty})$ is measurable and

$$\alpha\left(\left\{\int_0^t x_m(s)\mathrm{d}s\right\}_{m=1}^{\infty}\right) \le 2\int_0^t \alpha(\{x_m(s)\}_{m=1}^{\infty})\mathrm{d}s.$$

Lemma 2.6. [48] *Let $D \subset C([0,T];X)$ be a bounded set. Then, $\alpha(D(t)) \le \alpha_C(D)$ for all $t \in [0,T]$, where $D(t) = \{x(t) : x \in D\}$. Moreover, if D is equicontinuous on $t \in [0,T]$, then $\alpha(D(t))$ is continuous on $[0,T]$, and $\alpha_C(D) = \max_{t\in[0,T]}\alpha(D(t))$.*

Lemma 2.7. [52, Lemma 12] *Let $\{R_m\}_{m\ge1}$ be a sequence of bounded linear maps on X converging pointwise to $R \in \mathscr{L}(X)$. Then, for any compact set K in X, R_m converges to R uniformly in K, namely,*

$$\sup_{x\in K}\|R_m x - Rx\| \to 0, \quad as\ m \to \infty.$$

Lemma 2.8. ([49] Darbo-Sadovskii's fixed point theorem) *Suppose that Γ is a α-condensing operator on a Banach space X, i.e., Γ is continuous and there exists a positive constant $k \in [0,1)$ such that $\alpha(\Gamma(D)) \le k\alpha(D)$ for every bounded subset D of X. If $\Gamma(B) \subseteq B$ for a convex, closed, and bounded subset B of X, then Γ has at least one fixed point in B.*

Lemma 2.9. [53, Theorem 1.3.2] *If function $F : [a,b] \to X$ is continuous and differentiable on (a,b), then there exists $\xi \in (a,b)$ such that $\|F(b) - F(a)\| \le \|F'(\xi)\|(b-a)$.*

3 Mild solutions

Our main objective in this section is to study the existence of mild solutions for equation (1) using Schauder and Darbo-Sadovskii fixed point theorems, respectively. The mild solutions of equation (1) expressed by the resolvent operator are defined as follows.

Definition 3.1. A function $x(\cdot) \in C([0,T];X)$ is called a mild solution of equation (1), if it verifies

$$x(t) = R(t,0)[x_0 - H(x,x)] + \int_0^t R(t,s)G(s,x(s),x(r(s,x(s))))\mathrm{d}s, \quad \text{for } t \in [0,T].$$

To guarantee the existence of mild solutions, we make the following hypotheses:

(H_1) $R(t,s)$ is compact for each $(t,s) \in \triangle^0$.

(H_2) The function $G : [0,T] \times X \times X \to X$ fulfills the following conditions:

- (i) For each $t \in [0,T]$, the function $G(t,\cdot,\cdot) : X \times X \to X$ is continuous and for each $x, y \in X$, the function $G(\cdot,x,y) : [0,T] \to X$ is strongly measurable;
- (ii) For each positive number ρ, there exists a function $f_\rho(\cdot) \in L^1([0,T];\mathbb{R}^+)$ such that

$$\sup_{\|x\|,\|y\|\le\rho}\|G(t,x,y)\| \le f_\rho(t), \quad \text{for } t \in [0,T],$$

and

$$\liminf_{\rho\to+\infty} \frac{\|f_\rho\|_{L^1([0,T];\,\mathbb{R}^+)}}{\rho} := \varphi < +\infty.$$

(H_3) $H : C([0,T];X) \times C([0,T];X) \to X$ is a continuous function and satisfies the following conditions:
(i) $H(\cdot,\cdot)$ is a compact map;
(ii) There are constants $L_i > 0$, $i = 1, 2$ such that for $u, w \in C([0,T];X)$,

$$\|H(u,w)\| \le L_1\|u\|_C + L_2\|w\|_C.$$

For any $l > 0$, let

$$B_l = \{x \in C([0,T];X) : \|x\|_C \le l\},$$

then B_l is a bounded, closed, and convex set in $C([0,T];X)$.

In the light of the aforementioned conditions, we establish the first existence result.

Theorem 3.1. *Suppose that the conditions $(H_1)-(H_3)$ hold, then equation (1)has a mild solution provided that*

$$M(L_1 + L_2 + \varphi) < 1. \tag{12}$$

Proof. Let the operator $\Gamma : C([0,T];X) \to C([0,T];X)$ defined by

$$(\Gamma x)(t) = R(t,0)[x_0 - H(x,x)] + \int_0^t R(t,s)G(s,x(s),x(r(s,x(s))))\mathrm{d}s. \tag{13}$$

From the strong continuity of $R(t,s)$, $(t,s) \in \triangle$, and $(H_2)-(H_3)$, we deduce that $\Gamma x \in C([0,T];X)$. Subsequently, we will prove that the operator Γ satisfies the conditions of Schauder fixed point theorem, and hence, Γ has a fixed point, which is a solution of equation (1). For the sake of convenience, we give the proof in several steps.

Step 1. We declare that there exists a positive number l_0 such that $\Gamma(B_{l_0}) \subseteq B_{l_0}$.

As a matter of fact, if it is not true, then for any $l > 0$, there are $x_l(\cdot) \in B_l$ and $t_l \in [0,T]$, such that $\|(\Gamma x_l)(t_l)\| > l$. On the other hand, however, by (8), (H_2) (ii) and (H_3) (ii), we have that

$$\begin{aligned} l &< \|(\Gamma x_l)(t_l)\| \\ &\le M[\|x_0\| + \|H(x_l,x_l)\|] + M\int_0^{t_l} \|G(s,x_l(s),x_l(r(s,x_l(s))))\|\mathrm{d}s \\ &\le M[\|x_0\| + L_1\|x_l\|_C + L_2\|x_l\|_C] + M\int_0^{t_l} f_l(s)\mathrm{d}s \\ &\le M[\|x_0\| + L_1 l + L_2 l] + M\|f_l\|_{L^1([0,T];\,\mathbb{R}^+)}. \end{aligned}$$

Dividing on both sides by the l and taking the lower limit as $l \to +\infty$, we obtain that

$$1 < M(L_1 + L_2 + \varphi).$$

This contradicts (12). Thus, there is a positive number l_0 such that $\Gamma(B_{l_0}) \subseteq B_{l_0}$.

Step 2. We show that Γ is continuous on B_{l_0}.

Let $\{x_n\} \subseteq B_{l_0}$ with $x_n \to x$ in B_{l_0}. Then, by the continuity of $r(\cdot,\cdot)$ and the fact that $0 \le r(s,x(s)) \le T$, we know that for each $s \in [0,T]$,

$$\begin{aligned} \|x_n(r(s,x_n(s))) - x(r(s,x(s)))\| &\le \|x_n(r(s,x_n(s))) - x(r(s,x_n(s)))\| + \|x(r(s,x_n(s))) - x(r(s,x(s)))\| \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{14}$$

By the continuity of $H(\cdot,\cdot)$ and $G(\cdot,\cdot)$ combined with (14), we see that

$$H(x_n,x_n) \to H(x,x), \quad n \to \infty,$$

and

$$G(s, x_n(s), x_n(r(s, x_n(s)))) \to G(s, x(s), x(r(s, x(s)))), \quad n \to \infty.$$

Applying condition (H_2) (ii), we have

$$\|G(s, x_n(s), x_n(r(s, x_n(s)))) - G(s, x(s), x(r(s, x(s))))\| \le 2f_{l_0}(s).$$

Using the Lebesgue dominated convergence theorem, we obtain that

$$\begin{aligned}\|\Gamma x_n - \Gamma x\|_C &\le \sup_{0\le t\le T}\left(M\|H(x_n, x_n) - H(x, x)\| + M\int_0^t \|G(s, x_n(s), x_n(r(s, x_n(s)))) - G(s, x(s), x(r(s, x(s))))\|\mathrm{d}s\right)\\ &\to 0, \quad \text{as } n \to \infty,\end{aligned}$$

i.e., Γ is continuous.

Step 3. We certify that $\{\Gamma x : x \in B_{l_0}\}$ is a family of equicontinuous functions.

We let $0 < t_1 < t_2 \le T$ and $\varepsilon > 0$ be small enough, then, for any $x \in B_{l_0}$,

$$\begin{aligned}\|(\Gamma x)(t_2) - (\Gamma x)(t_1)\| &\le \|R(t_2, 0) - R(t_1, 0)\|[\|x_0\| + \|H(x, x)\|]\\ &\quad + \int_0^{t_1-\varepsilon} \|R(t_2, s) - R(t_1, s)\|\|G(s, x(s), x(r(s, x(s))))\|\mathrm{d}s\\ &\quad + \int_{t_1-\varepsilon}^{t_1} \|R(t_2, s) - R(t_1, s)\|\|G(s, x(s), x(r(s, x(s))))\|\mathrm{d}s\\ &\quad + \int_{t_1}^{t_2}\|R(t_2, s)\|\|G(s, x(s), x(r(s, x(s))))\|\mathrm{d}s\\ &\le \|R(t_2, 0) - R(t_1, 0)\|[\|x_0\| + (L_1 + L_2)l_0]\\ &\quad + \sup_{s\in[0,t_1-\varepsilon]} \|R(t_2, s) - R(t_1, s)\| \int_0^{t_1-\varepsilon} f_{l_0}(s)\mathrm{d}s\\ &\quad + 2M\int_{t_1-\varepsilon}^{t_1} f_{l_0}(s)\mathrm{d}s + M\int_{t_1}^{t_2} f_{l_0}(s)\mathrm{d}s.\end{aligned}$$

The right-hand side is independently of $x \in B_{l_0}$ and tends to zero as $t_2 - t_1 \to 0$ and $\varepsilon \to 0$ since, by Remark 2 and (H_1), $R(t, s)$ is continuous in the uniform operators topology for all $(t, s) \in \triangle^0$. Furthermore, according to the compactness of $\overline{H(B_{l_0}, B_{l_0})}$ (see (H_3) (i)), the strong continuity of $R(t, s), (t, s) \in \triangle$ and Lemma 2.7, we obtain that

$$\begin{aligned}\|(\Gamma x)(t) - (\Gamma x)(0)\| &\le \|R(t, 0)[x_0 - H(x, x)] - [x_0 - H(x, x)]\| + \int_0^t\|R(t, s)\|\|G(s, x(s), x(r(s, x(s))))\|\mathrm{d}s\\ &\le \|R(t, 0)x_0 - x_0\| + \|R(t, 0)H(x, x) - H(x, x)\| + M\int_0^t f_{l_0}(s)\mathrm{d}s\\ &\le \|R(t, 0)x_0 - x_0\| + \sup_{y\in\overline{H(B_{l_0},B_{l_0})}} \|R(t, 0)y - y\| + M\int_0^t f_{l_0}(s)\mathrm{d}s\\ &\to 0, \quad \text{as } t \to 0,\end{aligned}$$

which implies that $\{\Gamma x : x \in B_{l_0}\}$ are equi-continuous at $t = 0$. Therefore, the operator Γ maps B_{l_0} into a family of equicontinuous functions on $[0, T]$.

Step 4. We verify that for $t \in [0, T]$, the set $(\Gamma B_{l_0})(t) = \{(\Gamma x)(t) : x \in B_{l_0}\}$ is relatively compact in X.

Clearly, due to (H_3) (i), $(\Gamma B_{l_0})(0) = \{x_0 - H(x, x) : x \in B_{l_0}\}$ is relatively compact in X. Let $t \in (0, T]$ be fixed, $0 < \varepsilon < t$, for $x \in B_{l_0}$, define

$$(\Gamma_\varepsilon x)(t) = R(t,0)[x_0 - H(x,x)] + R(\varepsilon,0)\int_0^{t-\varepsilon} R(t,s)G(s,x(s),x(r(s,x(s))))\mathrm{d}s.$$

Note that

$$\left\| \int_0^{t-\varepsilon} R(t,s)G(s,x(s),x(r(s,x(s))))\mathrm{d}s \right\| \le M\int_0^{t-\varepsilon} f_{l_0}(s)\mathrm{d}s \le M\|f_{l_0}\|_{L^1([0,T];\,\mathbb{R}^+)} < +\infty,$$

from which and the fact $R(\varepsilon,0)$ is compact, we deduce that, for any $t \in (0,T]$, the set

$$\left\{ R(\varepsilon,0)\int_0^{t-\varepsilon} R(t,s)G(s,x(s),x(r(s,x(s))))\mathrm{d}s : x \in B_{l_0} \right\}$$

is relatively compact in X, and thus, $(\Gamma_\varepsilon B_{l_0})(t)$ does so. By Lemma 2.1, we obtain that

$$\begin{aligned}
\|(\Gamma x)(t) - (\Gamma_\varepsilon x)(t)\| &\le \left\| \int_0^{t} R(t,s)G(s,x(s),x(r(s,x(s))))\mathrm{d}s - R(\varepsilon,0)\int_0^{t-\varepsilon} R(t,s)G(s,x(s),x(r(s,x(s))))\mathrm{d}s \right\| \\
&\le \left\| \int_0^{t-\varepsilon} (R(t,s) - R(\varepsilon,0)R(t,s))G(s,x(s),x(r(s,x(s))))\mathrm{d}s \right\| \\
&\quad + \left\| \int_{t-\varepsilon}^{t} R(t,s)G(s,x(s),x(r(s,x(s))))\mathrm{d}s \right\| \\
&\le \int_0^{t-\varepsilon} \|R(t,s) - R(\varepsilon,0)R(t,s)\| f_{l_0}(s)\mathrm{d}s + M\int_{t-\varepsilon}^{t} f_{l_0}(s)\mathrm{d}s \to 0, \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Then, there are relatively compact sets arbitrarily close to the set $(\Gamma B_{l_0})(t)$. Hence, the set $(\Gamma B_{l_0})(t)$ is also relatively compact in X for every $t \in (0,T]$.

In view of Steps 2–4 together with the infinite-dimensional version of Ascoli-Arzela theorem, we can infer that Γ is a completely continuous map on B_{l_0}, and thus, there exists a $x \in B_{l_0}$ such that $\Gamma x = x$ via the Schauder fixed point theorem. Consequently, equation (1) has a mild solution $x(\cdot)$ on $[0,T]$, and the proof is complete. □

Our next purpose is to show that we can avoid condition (H_1), i.e., we will use the Darbo-Sadovskii fixed point theorem and Kuratowski measure of noncompactness to get rid of the assumption for compactness of the resolvent operator $R(\cdot,\cdot)$. To this end, the function $G(\cdot,\cdot,\cdot)$ is assumed to satisfy that

(H_4) (i) For each $t \in [0,T]$, the function $G(t,\cdot,\cdot) : X \times X \to X$ is continuous, and for each $x, y \in X$, the function $G(\cdot,x,y) : [0,T] \to X$ is strongly measurable;

(ii) There is a positive function $f(\sigma)$ such that

$$\sup_{\|x\|,\|y\|\le\sigma} \|G(t,x,y)\| \le f(\sigma), \quad \text{for } t \in [0,T],$$

and

$$\liminf_{\sigma\to+\infty} \frac{f(\sigma)}{\sigma} := \tau < +\infty.$$

(iii) There exist functions $\mu_1, \mu_2 \in L^1([0,T];\mathbb{R}^+)$ such that, for any bounded and countable sets $D_1, D_2 \in X$ and $t \in [0,T]$,

$$\alpha(G(t,D_1,D_2)) \le \mu_1(t)\alpha(D_1) + \mu_2(t)\alpha(D_2).$$

Then, we have the following.

Theorem 3.2. *Assume that the conditions* (H_3) *and* (H_4) *are fulfilled. Assume further that* $R(t,s)$ *is uniform operator topology continuous for all* $(t,s) \in \triangle^0$*, then equation* (1) *admits at least one mild solution provided that*

$$M(L_1 + L_2 + \tau T) < 1, \tag{15}$$

and

$$2M[||\mu_1||_{L^1([0,T];\,\mathbb{R}^+)} + ||\mu_2||_{L^1([0,T];\,\mathbb{R}^+)}] < 1. \tag{16}$$

Proof. Define the operator Γ on $C([0,T];X)$ by (13). Proceeding as in Steps $1-3$ of Theorem 3.1, applying Inequality (15), (H_3), (H_4) (i)–(ii), and the uniform operator topology continuity of $R(\cdot,\cdot)$, we can verify that there exists a $l_1 > 0$ such that $\Gamma(B_{l_1}) \subseteq B_{l_1}$, Γ is continuous on B_{l_1}, and $\Gamma(B_{l_1}) = \{\Gamma x : x \in B_{l_1}\}$ is a family of equicontinuous functions.

We next prove that $\Gamma : \Theta \to \Theta$ is a α-condensing operator, where $\Theta = \overline{co}\,\Gamma(B_{l_1})$ and $\overline{co}\,\Gamma(B_{l_1})$ mean the closed convex hull of $\Gamma(B_{l_1})$. By Lemma 2.3, we can know that $\Theta \subset B_{l_1}$ is bounded and equicontinuous, and operator $\Gamma : \Theta \to \Theta$ is continuous. For any $D \subset \Theta$, $\Gamma(D)$ is bounded. Consequently, by Lemma 2.4, there exists a countable set $D_0 = \{x_m\}_{m=1}^{\infty} \subset D$, such that

$$\alpha_C(\Gamma(D)) = \alpha_C(\Gamma(D_0)). \tag{17}$$

By the equicontinuity of D, we know that $D_0 \subset D$ is also equicontinuous. Then, using Lemma 2.5, Lemma 2.6, and the conditions (H_3) (i), (H_4) (iii), we have that

$$\begin{aligned}
\alpha(\Gamma(D_0)(t)) &\le M\alpha(\{x_0\}) + M\alpha(\{H(x_m,x_m)\}_{m=1}^{\infty}) + 2M\int_0^t \alpha(\{G(s,x_m(s),x_m(r(s,x_m(s))))\}_{m=1}^{\infty})\mathrm{d}s \\
&= 2M\int_0^t \alpha(\{G(s,x_m(s),x_m(r(s,x_m(s))))\}_{m=1}^{\infty})\mathrm{d}s \\
&\le 2M\int_0^t [\mu_1(s)\alpha(D_0(s)) + \mu_2(s)\alpha(D_0(r(s,D_0(s))))]\mathrm{d}s \\
&\le 2M\int_0^t [\mu_1(s) + \mu_2(s)]\mathrm{d}s\,\alpha_C(D_0) \\
&\le 2M[||\mu_1||_{L^1([0,T];\,\mathbb{R}^+)} + ||\mu_2||_{L^1([0,T];\,\mathbb{R}^+)}]\alpha_C(D).
\end{aligned}$$

Since $\Gamma(D_0) \subset \Gamma(B_{l_1})$ is bounded and equicontinuous, from Lemma 2.6, we obtain that

$$\alpha_C(\Gamma(D_0)) = \max_{t\in[0,T]} \alpha(\Gamma(D_0)(t)). \tag{18}$$

Thus, using (17) and (18), we see that

$$\alpha_C(\Gamma(D)) \le 2M[||\mu_1||_{L^1([0,T];\,\mathbb{R}^+)} + ||\mu_2||_{L^1([0,T];\,\mathbb{R}^+)}]\alpha_C(D).$$

Due to (16), Γ is a α-condensing map on Θ, and by the Darbo-Sadovskii fixed point theorem, Γ has at least one fixed point $x(\cdot)$ on Θ, which is a mild solution of equation (1). The proof is complete. □

4 Regularity of solutions

The main purpose of this section is to discuss the regularity of mild solutions for equation (1), i.e., we will provide conditions that allow the existence of strict solutions for equation (1). We introduce the following definition:

Definition 4.1. [45, Definition 3] A function $x \in C([0,T];X)$ is said to be a strict solution of equation (1) if $x \in C^1([0,T];X)$, $x(t) \in D(A)$ on $[0,T]$, $A(t)x(t)$ is continuous on $[0,T]$, and (1) holds in $[0,T]$.

When the function $G(\cdot,\cdot,\cdot)$ satisfies Lipschitz condition, i.e.,

(H_5) The function $G:[0,T]\times X\times X\to X$ is a Lipschitz continuous, namely, there exists $L_3>0$ such that

$$\|G(t_1,x_1,y_1)-G(t_2,x_2,y_2)\| \le L_3(|t_1-t_2|+\|x_1-x_2\|+\|y_1-y_2\|),\ t_1,t_2\in[0,T],\ x_1,x_2,y_1,y_2\in X,$$

we have the following result.

Theorem 4.1. *Assume the conditions of Theorem* 3.1 *and* (H_5) *hold true. Also, the following conditions are fulfilled:*

(H_6) *There is a Banach space* $(Z,\|\cdot\|_Z)$ *continuously included in* X *such that* $B(t,s)\in\mathscr{L}(Z,X)$ *and* $R(t,s)\in\mathscr{L}(X,Z)$ *for* $(t,s)\in\triangle$, *i.e., there exist positive numbers* M_1 *and* M_2 *such that*

$$\|B(t,s)\|_{Z.0}\le M_1,\ (t,s)\in\triangle, \tag{19}$$

and

$$\|R(t,s)\|_{0.Z}\le M_2,\ (t,s)\in\triangle. \tag{20}$$

(H_7) $r:[0,T]\times X\to[0,T]$ *is a Hölder continuous function, i.e., there exist* $L_4>0$, $0<\theta<1$ *such that for* $t_1,t_2\in[0,T]$, $x,y\in X$,

$$|r(t_1,x)-r(t_2,y)|\le L_4(|t_1-t_2|^\theta+\|x-y\|^\theta).$$

Then, for every $x_0-H(x,x)\in D(A)$, *equation* (1) *has a strict solution.*

We first give a crucial lemma before proving Theorem 4.1.

Lemma 4.1. *Let condition* (H_6) *holds, and* $f(s)$ *is a continuous function on* $[0,T]$, *and then, the function* $\int_0^t R(t,s)f(s)\mathrm{d}s$ *is uniformly Hölder continuous in* $t\in[0,T]$ *with any exponent* $0<\gamma<1$.

Proof. We show initially that if $0<h<1$, $|t-s|\ge h$, then

$$\|R(t+h,s)-R(t,s)\|\le\frac{Ch^\gamma}{|t-s|^\gamma}, \tag{21}$$

where $C>0$ is a constant.

In fact, for each $x\in X$ and for each bounded linear functional $p(X\to\mathbb{R})$, the mean value theorem gives, for some $\tilde h\in(0,h)$,

$$|p(R(t+h,s)x)-p(R(t,s)x)|\le\left|p\left(\frac{\partial}{\partial t}R(t+\tilde h,s)x\right)\right|h\le\|p\|\left\|\frac{\partial}{\partial t}R(t+\tilde h,s)\right\|\|x\|h. \tag{22}$$

Combining Definition 2.1-(P_3), (6), (19), (20), and (22), we have

$$\begin{aligned}|p(R(t+h,s)x)-p(R(t,s)x)| &\le \|p\|\left(\|A(t)R(t,s)\|+\left\|\int_s^t B(t,r)R(r,s)\mathrm{d}r\right\|\right)\|x\|h\\ &\le\|p\|\left(N|t-s|^{-1}+\int_s^t\|B(t,r)\|_{Z.0}\|R(r,s)\|_{0.Z}\mathrm{d}r\right)\|x\|h\\ &\le\|p\|(N|t-s|^{-1}+M_1M_2T)\|x\|h\\ &\le\|p\|(N|t-s|^{-1}+M_1M_2T^2|t-s|^{-1})\|x\|h\\ &=\|p\|(N+M_1M_2T^2)\|x\|h|t-s|^{-1}\\ &\coloneqq\|p\|C\|x\|h|t-s|^{-1}\\ &\le\|p\|C\|x\|h^\gamma|t-s|^{-\gamma},\end{aligned}$$

which indicates that Inequality (21) holds.

We next show that the function $\int_0^t R(t,s)f(s)\mathrm{d}s$ is uniformly Hölder continuous in $t \in [0,T]$ with any exponent $0 < \gamma < 1$. Using (8) and (21), we obtain that

$$\begin{aligned}
&\left\| \int_0^{t+h} R(t+h,s)f(s)\mathrm{d}s - \int_0^t R(t,s)f(s)\mathrm{d}s \right\| \\
&\quad \leq \int_0^{t-h} \|R(t+h,s) - R(t,s)\| \|f(s)\|\mathrm{d}s + \int_{t-h}^{t+h} \|R(t+h,s)\|\|f(s)\|\mathrm{d}s + \int_{t-h}^{t} \|R(t,s)\|\|f(s)\|\mathrm{d}s \\
&\quad \leq \int_0^{t-h} \frac{Ch^\gamma}{|t-s|^\gamma}\mathrm{d}s \sup_{t\in[0,T]} \|f(t)\| + M \sup_{t\in[0,T]} \|f(t)\| 2h + M \sup_{t\in[0,T]} \|f(t)\| h \\
&\quad \leq \frac{1}{1-\gamma} T^{1-\gamma} \sup_{t\in[0,T]} \|f(t)\| Ch^\gamma + M \sup_{t\in[0,T]} \|f(t)\| 2h + M \sup_{t\in[0,T]} \|f(t)\| h.
\end{aligned}$$

This implies that we establish our assertion. The proof is completed. □

We now turn back to prove Theorem 4.1.

Proof of Theorem 4.1. By Theorem 3.1, we know that equation (1) has a mild solution $x(\cdot)$ on $[0,T]$. Set

$$\begin{aligned}
o(t) &= R(t,0)[x_0 - H(x,x)], \\
q(t) &= \int_0^t R(t,s)G(s,x(s),x(r(s,x(s))))\mathrm{d}s.
\end{aligned}$$

It follows from Lemmas 2.9 and 4.1 that $o(t)$ and $q(t)$ are Hölder continuous on $[0,T]$, respectively. Thus, we see that $x(\cdot)$ is Hölder continuous on $[0,T]$, i.e., for each $t_1, t_2 \in [0,T]$, there are constants $L_5 > 0$, $0 < \theta_1 < 1$, such that

$$\|x(t_1) - x(t_2)\| \leq L_5 \, |t_1 - t_2|^{\theta_1}, \tag{23}$$

from which, together with (H_5) and (H_7), we obtain

$$\begin{aligned}
&\|G(t+h,x(t+h),x(r(t+h,x(t+h)))) - G(t,x(t),x(r(t,x(t))))\| \\
&\quad \leq L_3(h + \|x(t+h) - x(t)\| + \|x(r(t+h,x(t+h))) - x(r(t,x(t)))\|) \\
&\quad \leq L_3(h + L_5 h^{\theta_1} + L_5 \, |r(t+h,x(t+h)) - r(t,x(t))|^{\theta_1}) \\
&\quad \leq L_3(h + L_5 h^{\theta_1} + L_5 [L_4(h^\theta + \|x(t+h) - x(t)\|^\theta)]^{\theta_1}) \\
&\quad \leq L_3(h + L_5 h^{\theta_1} + L_5 [L_4(h^\theta + L_5^\theta h^{\theta\theta_1})]^{\theta_1}),
\end{aligned}$$

which implies that the map $t \to G(t,x(t),x(r(t,x(t))))$ is Hölder continuous on $[0,T]$. Hence, from Corollary 5 of [45], the mild solution $x(\cdot)$ of equation (1) is a strict solution, and this completes the proof. □

In the following, we give another regularity result in the case where (H_5) is replaced by weaker condition:
(H_8) $G : [0,T] \times X \times X \to X$ is a Hölder continuous function, i.e., there exist $L_6 > 0$, $0 < \theta_2 < 1$ such that

$$\|G(t_1,x_1,y_1) - G(t_2,x_2,y_2)\| \leq L_6(|t_1 - t_2|^{\theta_2} + \|x_1 - x_2\|^{\theta_2} + \|y_1 - y_2\|^{\theta_2}),\ t_1,t_2 \in [0,T],\ x_1,x_2,y_1,y_2 \in X.$$

Theorem 4.2. *Suppose that the conditions of Theorem* 3.1 *and* (H_6)–(H_8) *are satisfied. Then, equation* (1) *has a strict solution for every* $x_0 - H(x,x) \in D(A)$.

Proof. The proof is similar to that of Theorem 4.1. Therefore, we only demonstrate the differences in the proof. From Theorem 4.1, we know that $x(\cdot)$ is Hölder continuous on $[0,T]$. Applying (H_7), (H_8), and (23), we have that

$$\begin{aligned}\|G(t+h, x(t+h), x(r(t+h))) - G(t, x(t), x(r(t)))\| &\le L_6(h^{\theta_2} + L_5^{\theta_2} h^{\theta_1\theta_2} + L_5^{\theta_2}\,|r(t+h, x(t+h)) - r(t, x(t))|^{\theta_1\theta_2})\\ &\le L_6(h^{\theta_2} + L_5^{\theta_2} h^{\theta_1\theta_2} + L_5^{\theta_2}[L_4(h^{\theta} + \|x(t+h) - x(t)\|^{\theta})]^{\theta_1\theta_2})\\ &\le L_6(h^{\theta_2} + L_5^{\theta_2} h^{\theta_1\theta_2} + L_5^{\theta_2}[L_4(h^{\theta} + L_5^{\theta} h^{\theta\theta_1})]^{\theta_1\theta_2}).\end{aligned}$$

Then, the map $t \to G(t, x(t), x(r(t, x(t))))$ is Hölder continuous on $[0, T]$, and hence, $x(\cdot)$ is a strict solution of equation (1) by [45, Corollary 5]. The proof is completed. □

5 Application

In this section, we apply the obtained abstract results to investigate the existence and regularity problems for the following non-autonomous nonlocal partial integrodifferential equation:

$$\begin{cases}\dfrac{\partial}{\partial t} z(t,x) = \dfrac{\partial^2}{\partial x^2} z(t,x) - a(t)z(t,x) + \displaystyle\int_0^t b(t,s)\frac{\partial^2}{\partial x^2} z(s,x)\mathrm{d}s + c\left(t, z(t,x), z\left(\frac{t\cos(z(t,x))}{\sqrt{\pi}}, x\right)\right),\\ \quad t \in [0,1], \quad x \in [0,\pi], \quad z(t,0) = z(t,\pi) = 0, \quad t \in [0,1],\\ z(0,x) + \displaystyle\int_0^\pi K_1(x,y)\sin(z(t,y))\mathrm{d}y + \int_0^\pi K_2(x,y)\ln(1 + |z(t,y)|)\mathrm{d}y = z_0(x), \quad x \in [0,\pi],\end{cases} \tag{24}$$

where $a : [0,1] \to \mathbb{R}$ is a Hölder continuous function and satisfies $\underline{a} := \min_{t\in[0,1]}|a(t)| > -1$. $b(\cdot,\cdot) : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function. The functions $c(\cdot,\cdot,\cdot)$ and $K_i(\cdot,\cdot)$, $i = 1, 2$, will be described in the following.

Let $X = L^2([0,\pi]; \mathbb{R})$ with the norm $\|\cdot\|$, and we consider the operator $(A, D(A))$, which is given by

$$Az = z''$$

with the domain

$$D(A) = \{z(\cdot) \in X : z', z'' \in X, \quad \text{and} \quad z(0) = z(\pi) = 0\}.$$

It is well known that A generates a compact analytic semigroup $(T(t))_{t\ge0}$. Furthermore, A has eigenvalues $-n^2$, $n \in \mathbb{N}$, with the corresponding normalized eigenvectors $z_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, $n = 1, 2, \dots$, and it can be expressed as

$$Az = \sum_{n=1}^{\infty} - n^2\langle z, z_n\rangle z_n,\ z \in D(A).$$

Next, let operator family $\{A(t) : 0 \le t \le 1\}$ on X be defined by

$$\begin{aligned}&D(A(t)) = D(A), \quad t \in [0,1],\\ &A(t)z = Az - a(t)z, \quad z \in D(A).\end{aligned}$$

Then, $A(t)$ generates an evolution operator $\{U(t,s) : 0 \le s \le t \le 1\}$ fulfilling (V_1)–(V_2) (see [54]), and

$$U(t,s)z = \sum_{n=1}^{\infty} e^{-n^2(t-s) - \int_s^t a(\tau)\mathrm{d}\tau}\langle z, e_n\rangle e_n, \quad z \in X,$$

for $0 \le s \le t \le 1$.

We also consider the operator $B(t,s) : D(A) \subset X \to X$, $0 \le s \le t \le 1$, which is defined by

$$D(B(t,s)) = D(A), B(t,s)z = b(t,s)Az, z \in D(A).$$

It is easy to see that $B(t,s) \in \mathscr{L}(Y, X)$ for $0 \le s \le t \le 1$, where Y is defined as in Section 2. Furthermore, if we suppose that $b(t,s)$ is Hölder continuous, then the condition (V_3) is well verified, and thus, the corresponding linear system of (24) has an analytic resolvent operator $(R(t,s))_{t\ge s}$. Moreover, using the similar method as in

[55], one can easily certify that the resolvent $(\lambda I - A(t))^{-1}$ is compact for all $t \in [0,1]$ and some $\lambda \in \rho(A(t))$ ($\rho(A(t))$ is defined as in Remark 1. Hence, the resolvent operator $R(t,s)$ is compact for $t > s$ by virtue of Remark 2.

We impose the following conditions on System (24):

(A1) The function $c : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exists a constant L_c such that

$$|c(t_1, x_1, y_1) - c(t_2, x_2, y_2)| \le L_c(|t_1 - t_2| + |x_1 - x_2| + |y_1 - y_2|),$$

and there is a function $\xi \in L^1([0,1]; \mathbb{R}^+)$ such that

$$|c(t,x,y)| \le \xi(t)(|x| + |y|),$$

for any $t_1, t_2 \in [0,1]$, $x, y, x_i, y_i \in \mathbb{R}$, $i = 1, 2$.

(A2) The functions $K_i : [0,\pi] \times [0,\pi] \to \mathbb{R}$, $i = 1, 2$, are measurable with $K_i(0,y) = K_i(\pi,y) = 0$, $y \in [0,\pi]$ and satisfy, respectively,

$$\sigma_1 := \left(\int_0^\pi\int_0^\pi K_1^2(x,y)\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}} < +\infty$$

and

$$\sigma_2 := \left(\int_0^\pi\int_0^\pi K_2^2(x,y)\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}} < +\infty.$$

Now, we take $u(t)(x) = z(t,x)$ and define the functions $G : [0,T] \times X \times X \to X$, $r : [0,1] \times X \to [0,1]$ and $H : C([0,1]; X) \times C([0,1]; X) \to X$, respectively, as

$$G(t,z,z)(x) = c(t, z(x), z(x)), \quad z \in X,$$

$$r(t,z)(x) = \frac{t\cos(z(x))}{\sqrt{\pi}}, \quad z \in X,$$

and

$$H(u,u)(x) = \int_0^\pi K_1(x,y)\sin(u(t)(y))\mathrm{d}y + \int_0^\pi K_2(x,y)\ln(1 + |u(t)(y)|)\mathrm{d}y, \quad u \in C([0,1]; X).$$

Then, with these notations, equation (24) can be rewritten as the abstract form (1).

In the sequel, we will verify that the conditions in Theorems 3.1 and 4.1 are all satisfied for System (24). Condition (A_1) implies that the function $G(\cdot,\cdot,\cdot)$ satisfies (H_2) and (H_5). In fact, for $t_1, t_2 \in [0,1]$ and $z_1, z_2, v_1, v_2 \in X$, we find

$$\begin{aligned}
\|G(t_1, z_1, v_1) - G(t_2, z_2, v_2)\| &= \left(\int_0^\pi |c(t_1, z_1(x), v_1(x)) - c(t_2, z_2(x), v_2(x))|^2\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \left(\int_0^\pi L_c^2(|t_1 - t_2| + |z_1(x) - z_2(x)| + |v_1(x) - v_2(x)|)^2\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \left(\int_0^\pi L_c^2\,|t_1 - t_2|^2\mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_0^\pi L_c^2\,|z_1(x) - z_2(x)|^2\mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_0^\pi L_c^2\,|v_1(x) - v_2(x)|^2\mathrm{d}x\right)^{\frac{1}{2}} \\
&= L_c\sqrt{\pi}|t_1 - t_2| + L_c\|z_1 - z_2\| + L_c\|v_1 - v_2\| \\
&\le L_c\sqrt{\pi}(|t_1 - t_2| + \|z_1 - z_2\| + \|v_1 - v_2\|).
\end{aligned}$$

Then, we obtain that $L_3 \coloneqq L_c\sqrt{\pi}$. Similarly, we also find

$$\begin{aligned}
\|G(t,z,v)\| &= \left(\int_0^\pi |c(t,z(x),v(x))|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \left(\int_0^\pi \xi^2(t)|z(x)+v(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \left(\int_0^\pi \xi^2(t)|z(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_0^\pi \xi^2(t)|v(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \xi(t)(\|z\| + \|v\|).
\end{aligned}$$

It can easily be seen that $f_\rho(t) \coloneqq 2\xi(t)\rho$ ($\|z\|, \|v\| \le \rho$) and $\|f_\rho\|_{L^1([0,1];\,\mathbb{R}^+)} \coloneqq 2\|\xi\|_{L^1([0,1];\,\mathbb{R}^+)}\rho$ (note that $\varphi \coloneqq \liminf_{\rho\to+\infty}\frac{\|f_\rho\|_{L^1([0,1];\,\mathbb{R}^+)}}{\rho} = 2\|\xi\|_{L^1([0,1];\,\mathbb{R}^+)}$). On the other hand, for $u, w \in C([0,1];X)$, we have that

$$\begin{aligned}
\|H(u,w)\| &\le \left(\int_0^\pi \left|\int_0^\pi K_1(x,y)\sin(u(t)(y))\mathrm{d}y\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_0^\pi \left|\int_0^\pi K_2(x,y)\ln(1+|w(t)(y)|)\mathrm{d}y\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \left(\int_0^\pi\left(\int_0^\pi |K_1(x,y)||u(t)(y)|\mathrm{d}y\right)^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_0^\pi\left(\int_0^\pi |K_2(x,y)||w(t)(y)|\mathrm{d}y\right)^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \left(\int_0^\pi\int_0^\pi K_1^2(x,y)\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}} \|u(t)\| + \left(\int_0^\pi\int_0^\pi K_2^2(x,y)\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}} \|w(t)\| \\
&= \sigma_1\|u(t)\| + \sigma_2\|w(t)\| \le \sigma_1\|u\|_C + \sigma_2\|w\|_C.
\end{aligned}$$

From the aforementioned inequalities, the function $H(\cdot,\cdot)$ clearly satisfies the condition (H_3) with $L_1 \coloneqq \sigma_1$ and $L_2 \coloneqq \sigma_2$, respectively (also see [56] for the compactness property). Furthermore, for $t_1, t_2 \in [0,1]$ and $z_1, z_2 \in X$, we find

$$\begin{aligned}
\|r(t_1,z_1) - r(t_2,z_2)\| &= \frac{1}{\sqrt{\pi}}\left(\int_0^\pi |t_1\cos(z_1(x)) - t_2\cos(z_2(x))|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \frac{1}{\sqrt{\pi}}\left[\left(\int_0^\pi |t_1\cos(z_1(x)) - t_2\cos(z_1(x))|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_0^\pi |t_2\cos(z_1(x)) - t_2\cos(z_2(x))|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}\right] \\
&\le \frac{1}{\sqrt{\pi}}\left[\left(\int_0^\pi |t_1 - t_2|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_0^\pi |\cos(z_1(x)) - \cos(z_2(x))|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}\right] \\
&\le \frac{1}{\sqrt{\pi}}\left[\sqrt{\pi}|t_2 - t_1| + \left(\int_0^\pi |z_1(x) - z_2(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}}\right] \\
&\le |t_1 - t_2| + \|z_1 - z_2\|,
\end{aligned}$$

which implies that the condition (H_7) is satisfied with $L_4 = 1$. Thus, on the basis of Theorems 3.1 and 4.1, we have the following conclusions for equation (24).

(1) If $M(\sigma_1 + \sigma_2 + 2\|\xi\|_{L^1([0,1];\,\mathbb{R}^+)}) < 1$ holds, then the non-autonomous nonlocal partial integro-differential equation (24) has a mild solution on $[0,1]$.
(2) If $z_0(x) \in D(A)$ and $K_i(\cdot,\cdot)$ are C^2 functions, $i = 1, 2$, then the non-autonomous nonlocal partial integro-differential equation (24) admits a strict solution on $[0,1]$ as long as (20) holds.

6 Conclusion

The purpose of this article is the study of non-autonomous integrodifferential evolution equations involving nonlocal conditions. The existence and regularity of mild solutions for the proposed equation are obtained by applying Schauder and Darbo-Sadovskii fixed point theorems, Kuratowski measure of noncompactness, resolvent operators theory, and Lemma 4.1. Finally, an example is provided to show the effectiveness of the obtained results. In the future, we will consider the regularity of neutral non-autonomous integrodifferential evolution equation with finite delay and will utilize resolvent operators technique on this system.

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