Research Article

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Holomorphic curves into projective spaces with some special hypersurfaces

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Abstract: In this article, we establish some truncated second main theorems for holomorphic curves into projective spaces with some special hypersurfaces and give some applications. In addition, the defect relation, the algebraically degenerate conditions and uniqueness theorem for holomorphic curves with some special divisors may be improved.

Keywords: holomorphic curves, Nevanlinna theory, hypersurfaces

MSC 2020: 32H30, 32A22

1 Introduction and main results

We first recall the standard notation in Nevanlinna theory. For the details, please refer to [1,2]. Let $E = \sum_i \alpha_i a_i$ be a divisor on $\mathbb{C}$, where $\alpha_i \geq 0$, $a_i \in \mathbb{C}$, and let $M \in \mathbb{N} \cup \{\infty\}$. Denoted by Δ_t, the disk $\{z \in \mathbb{C}, |z| < t\}$. Summing the M-truncated degrees of the divisor on disks by:

$$n^{[M]}(t, E) \coloneqq \sum_{a_i \in \Delta_t} \min\{\alpha_i, M\}, \quad (t > 0),$$

the *truncated counting function* at level M of E is defined by taking the logarithmic average:

$$N^{[M]}(r, E) \coloneqq \int_1^r \frac{n^{[M]}(t, E)}{t} \mathrm{d}t, \quad (r > 1).$$

When $M = \infty$, we write $n(t, E)$ and $N(r, E)$ instead of $n^{[\infty]}(t, E)$, $N^{[\infty]}(r, E)$.

Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be a holomorphic curve having a reduced representation $f = [f_0 : f_1 : \cdots : f_N]$ in the homogeneous coordinates $[z_0 : \cdots : z_N]$ of $\mathbb{P}^N(\mathbb{C})$. Let $D = \{Q = 0\}$ be a divisor in $\mathbb{P}^N(\mathbb{C})$ defined by a homogeneous polynomial $Q \in \mathbb{C}[z_0, \ldots, z_N]$ of degree $d \geq 1$. If $f(\mathbb{C}) \not\subset D$, we define the *truncated counting function* of f with respect to D as:

$$N_f^{[M]}(r, D) \coloneqq N^{[M]}(r, (Q \circ f)_0),$$

where $(Q \circ f)_0$ denotes the zero divisor of $Q \circ f$.

The *proximity function* of f for the divisor D is defined as:

$$m_f(r, D) \coloneqq \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^d \|Q\|}{|Q \circ f(re^{i\theta})|} \frac{\mathrm{d}\theta}{2\pi},$$

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where $||Q||$ is the maximum absolute value of the coefficients of Q and

$$||f(z)|| = \max\{|f_0(z)|, \ldots, |f_N(z)|\}.$$

Since $|Q(f)| \le ||Q|| ||f||^d$, one has $m_f(r, D) \ge 0$.

The *Nevanlinna-Cartan characteristic function* $T_f(r)$ is defined by:

$$\begin{aligned} T_f(r) &:= \int_0^{2\pi} \log||f(re^{i\theta})|| \frac{\mathrm{d}\theta}{2\pi} - \log||f(0)|| \\ &= \int_1^r \frac{\mathrm{d}t}{t} \int_{\Delta_t} f^* \omega_N + O(1), \end{aligned}$$

where ω_N is the Fubini-study form on $\mathbb{P}^N(\mathbb{C})$.

The Nevanlinna theory consists of two fundamental theorems [2].The Poisson-Jensen formula implies the first main theorem.

Theorem 1.1. (First main theorem [2]) *Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be a holomorphic curve and D be a hypersurface of degree d in $\mathbb{P}^N(\mathbb{C})$ such that $f(\mathbb{C}) \not\subset D$. Then, for every $r > 1$, the following holds*:

$$dT_f(r) = N_f(r, D) + m_f(r, D) + O(1),$$

whence

$$N_f(r, D) \le dT_f(r) + O(1). \tag{1}$$

Hence, the first main theorem gives an upper bound on the counting function in terms of the characteristic function. The lower bound for the sum of certain counting functions is called the second main theorem. Such types of estimates were given in several situations.

A holomorphic curve $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ is said to be algebraically (linearly) nondegenerate if its image is not contained in any hypersurface (hyperplane). A family of $q > N + 1$ hypersurfaces $\{D_i\}_{i=1}^q$ in $\mathbb{P}^N(\mathbb{C})$ is in general position if any $N + 1$ hypersurfaces in this family have empty intersection:

$$\bigcap_{i \in I} \mathrm{supp}(D_i) = \varnothing \quad (\text{for all } I \subset \{1, \ldots, q\}, |I| = N + 1).$$

In 1933, Cartan [3] proved the following second main theorem with truncated counting functions.

Theorem 1.2. (Cartan's second main theorem, [3]) *Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be a linearly nondegenerate holomorphic curve. Let $H_1, \ldots, H_q$ be the hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position. Assume that $f(\mathbb{C}) \not\subset H_j \quad (j = 1, \ldots, q)$. Then,*

$$||(q - N - 1)T_f(r) \le \sum_{i=1}^{q} N_f^{[N]}(r, H_j) + o(T_f(r)),$$

where the notation "$||P$" means that the assertion P holds for all $r \in [1, +\infty)$ excluding a Borel subset E of $(1, +\infty)$ with $\int_E \mathrm{d}r < +\infty$.

In the one-dimensional case, Cartan recovered the classical Nevanlinna second main theorem. Since then, many authors tried to extend the result of Cartan to the case of (possible) nonlinear hypersurface, e.g., Erëmenko and Sodin [4] and Ru [5]. Note that it is still an open question of truncating the counting functions in the generalizations of Cartan's second main theorem. Some results in this direction are obtained recently but one requires the presence of more targets or big truncated level (see, for instance, [6–8]).

In [9], Yang et al. obtained the following second main theorem for a holomorphic curve intersecting a fixed hypersurface without the level of truncation.

Theorem 1.3. (Theorem 3.1, [9]) *Let $f:\mathbb{C}\to\mathbb{P}^N(\mathbb{C})$ be an algebraically nondegenerate holomorphic curve, and let m and n be the integers with $n\le m<(1+\frac{1}{N})n-(N+1)$. Let D be a hypersurface defined by a homogeneous polynomial in I_m with coefficients of nonzero polynomials. Then,*

$$||T_f(r)\le\frac{1}{n-(m-n+N+1)}N_f(r,D)+o(T_f(r)).$$

Thin [10] obtained the truncated second main theorem for holomorphic curves intersecting some special hypersurfaces as follows.

Theorem 1.4. (Theorem 3, [10]) *Let $f:\mathbb{C}\to\mathbb{P}^N(\mathbb{C})$ be an algebraically nondegenerate holomorphic curve. Let d and n be the integers satisfying $n>N(d+N+1)$. Let $D_j=\{z\in\mathbb{P}^N(\mathbb{C}):Q_j(z)=0\}, 0\le j\le N$ be the hypersurfaces of degree d such that the hypersurfaces $\{z_j^nQ_j(z)=0\}_{j=0}^N$ are in general position in $\mathbb{P}^N(\mathbb{C})$. Let $D=\{z\in\mathbb{P}^N(\mathbb{C}):\sum_{j=0}^N z_j^nQ_j(z)=0\}$. Then,*

$$||(n-(d+N+1))T_f(r)+\sum_{j=0}^N(N_f(r,D_j)-N_f^{[N]}(r,D_j))\le N_f^{[N]}(r,D)+o(T_f(r)).$$

In this article, we want to improve Thin's work. Our main result is stated as follows.

Theorem 1.5. *Let $f:\mathbb{C}\to\mathbb{P}^N(\mathbb{C})$ be an algebraically nondegenerate holomorphic curve. Let d_j and k_j $(0\le j\le N)$ be the integers satisfying $\sum_{j=0}^N\frac{k_j-N}{d_j}>N$. Suppose $D_j=\{z\in\mathbb{P}^N(\mathbb{C}):z_j^{k_j}Q_j(z)=0\}, 0\le j\le N$ be the hypersurfaces of degree d_j in general position in $\mathbb{P}^N(\mathbb{C})$. We assume that $d=\mathrm{lcm}(d_0,\ldots,d_N)$ and $\sum_{j=0}^N(z_j^{k_j}Q_j)^{\frac{d}{d_j}}\not\equiv 0$, which is denoted by $D=\{z\in\mathbb{P}^N(\mathbb{C}):\sum_{j=0}^N(z_j^{k_j}Q_j(z))^{\frac{d}{d_j}}=0\}$. Then,*

$$||\left(\sum_{j=0}^N\frac{k_j-N}{d_j}-N\right)T_f(r)\le\frac{1}{d}N_f^{[N]}(r,D)+o(T_f(r)).$$

Example 1.6. If $k_j=d_j$, $D_j=\{z\in\mathbb{P}^N(\mathbb{C}):z_j^{d_j}=0\}, 0\le j\le N$ be the hypersurfaces of degree d_j. We see that the hypersurfaces $D_j(0\le j\le N)$ are in general position in $\mathbb{P}^N(\mathbb{C})$. Set $d=\mathrm{lcm}(d_0,\ldots,d_N)$. Then, $D=\{z\in\mathbb{P}^N(\mathbb{C}):\sum_{j=0}^N z_j^d=0\}$ satisfies Theorem 1.5 with $\sum_{j=0}^{N+1}\frac{1}{d_j}<\frac{1}{N}$. In particular,$d_j=d>N(N+1)$ $(0\le j\le N)$; Theorem 1.5 holds for $D=\{z\in\mathbb{P}^N(\mathbb{C}):\sum_{j=0}^N z_j^d=0\}$. This example shows that the hypersurfaces that we discussed in Theorem 1.5 exist.

When $d_0=\cdots=d_N=d$ in Theorem 1.5, we have

Corollary 1.7. *Let $f:\mathbb{C}\to\mathbb{P}^N(\mathbb{C})$ be an algebraically nondegenerate holomorphic curve. Let d and $k_j(0\le j\le N)$ be the integers satisfying $\sum_{j=0}^N k_j>(d+N+1)N$. Suppose $D_j=\{z\in\mathbb{P}^N(\mathbb{C}):z_j^{k_j}Q_j(z)=0\}, 0\le j\le N$ be the hypersurfaces of degree d in general position in $\mathbb{P}^N(\mathbb{C})$, which is denoted by $D=\{z\in\mathbb{P}^N(\mathbb{C}):\sum_{j=0}^N(z_j^{k_j}Q_j(z))=0\}$. Then,*

$$||\left(\sum_{j=0}^N k_j-(d+N+1)N\right)T_f(r)\le N_f^{[N]}(r,D)+o(T_f(r)).$$

Under the assumption of Theorem 1.4, since $N_f(r,D_j)\le dT_f(r)$,

$$(n-(d+N+1))T_f(r)+\sum_{j=0}^N(N_f(r,D_j)-N_f^{[N]}(r,D_j))\le(n+(N+1)d-(d+N+1))T_f(r).$$

We note $n + (N+1)d - (d+N+1) \le (N+1)n - (d+N+1)N$ if $n > N(d+N+1)$ $(N \ge 2)$. Therefore, Corollary 1.5 improves Theorem 1.4 when $k_0 = \cdots = k_N = n$.

We define the *defect* and *defect with level of truncation* M of f intersecting hypersurface D of degree d by:

$$\delta_f(D) := 1 - \limsup_{r\to+\infty} \frac{N_f(r,D)}{dT_f(r)}, \quad \text{and} \quad \delta_f^{[M]}(D) := 1 - \limsup_{r\to+\infty} \frac{N_f^{[M]}(r,D)}{dT_f(r)}.$$

From Cartan's second main theorem, we have a defect relation $\sum_{j=1}^{q} \delta_f^{[N]}(H_j) \le N+1$ for a linearly nondegenerate holomorphic curve $f: \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ and hyperplanes $H_j, 1 \le j \le q$, in $\mathbb{P}^N(\mathbb{C})$ in general position. For the hypersurfaces case, we will prove

Theorem 1.8. *Let $D_j = \{z \in \mathbb{P}^N(\mathbb{C}) : Q_j(z) = 0\}$, $0 \le j \le N$ be the hypersurfaces of degree d_j in general position in $\mathbb{P}^N(\mathbb{C})$. Suppose that $d = \operatorname{lcm}(d_0, \ldots, d_N)$ and $\sum_{j=0}^{N} Q_j^{\frac{d}{d_j}} \not\equiv 0$, which is denoted by $D_{N+1} := \{z \in \mathbb{P}^N(\mathbb{C}) : \sum_{j=0}^{N} Q_j^{\frac{d}{d_j}}(z) = 0\}$. Then,*

$$\sum_{j=0}^{N+1} \delta_f^{[N]}(D_j) \le N+1.$$

Example 1.9. Let $D_j := \{z \in \mathbb{P}^N(\mathbb{C}) : z_1^{d_j} + \cdots + z_{N+1}^{d_j} = 0\}, 1 \le j \le q$, be the hypersurfaces of degree $d_j > N(N+1)$ in $\mathbb{P}^N(\mathbb{C})$. From Theorem 1.5, we obtain $\delta_f^{[N]}(D_j) \le \frac{N(N+1)}{d_j}$. Thus,

$$\sum_{j=1}^{q} \delta_f^{[N]}(D_j) \le N(N+1) \sum_{j=1}^{q} \frac{1}{d_j}.$$

If $\sum_{j=1}^{q} \frac{1}{d_j} < \frac{1}{N}$, we obtain

$$\sum_{j=1}^{q} \delta_f^{[N]}(D_j) < N+1.$$

This is an example where Shiffman's conjecture for defect with level of truncation holds.

In the other context, Green and Griffths [11] conjectured that every holomorphic curve in a complex projective hypersurface of general type is degenerate. It is a topic that is paid more attention (see [12–14] etc.). In this article, we will obtain some results of algebraically degeneracy with special hypersurfaces as follows.

Theorem 1.10. *Let $D_j = \{z \in \mathbb{P}^N(\mathbb{C}) : Q_j(z) = 0\}, 0 \le j \le N$, be the hypersurfaces of degree d_j in general position in $\mathbb{P}^N(\mathbb{C})$. Suppose that $d = \operatorname{lcm}(d_0, \ldots, d_N)$ and $\sum_{j=0}^{N} Q_j^{\frac{d}{d_j}} \not\equiv 0$, which is denoted by $D_{N+1} := \{z \in \mathbb{P}^N(\mathbb{C}) : \sum_{j=0}^{N} Q_j^{\frac{d}{d_j}}(z) = 0\}$. Then, every holomorphic curve $f: \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ whose image intersecting D_j $(0 \le j \le N+1)$ with multiplicity at least l_j. If*

$$\sum_{j=0}^{N+1} \frac{1}{l_j} < \frac{1}{N},$$

then f is algebraically degenerate.

Theorem 1.11. *Let d_j and $k_j (0 \le j \le N)$ be the integers satisfying $\sum_{j=0}^{N} \frac{k_j - N}{d_j} > N$. Let D be the hypersurface as in Theorem 1.5. Then, every holomorphic curve $f: \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ whose image f intersecting D with multiplicity at least $l > \frac{N}{\sum_{j=0}^{N} \frac{k_j - N}{d_j} - N}$ must be algebraically degenerate.*

When $d_0 = \cdots = d_N = d$, we obtain

Corollary 1.12. *Let d and $k_j (0 \le j \le N)$ be the integers satisfying $\sum_{j=0}^{N} k_j > (d + N + 1)N$. Let D be the hypersurface as in Corollary 1.7. Then, every holomorphic curve $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ whose image f intersecting D with multiplicity at least $l > \frac{Nd}{\sum_{j=0}^{N} k_j - (d+N+1)N}$ is algebraically degenerate.*

Finally, as an application of Theorem 1.5, we prove a uniqueness result as follows.

Theorem 1.13. *Let $f, g : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be algebraically nondegenerate holomorphic curves. Let D be the hypersurface as in Theorem 1.5. Assume that $f(z) = g(z)$ on $f^{-1}(D) \cup g^{-1}(D)$. If $N + \frac{2N}{d} < \sum_{j=0}^{N} \frac{k_j - N}{d_j}$, then $f \equiv g$.*

We note that if $k_0 = \cdots = k_N$ and $d = d_0 = \cdots = d_N$, Theorem 1.13 gives back Theorem 10 in [10].

2 Proof of theorems

The following lemma is inspired by [10]. We will give the proof for completeness.

Lemma 2.1. *Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be an algebraically nondegenerate holomorphic curve. Let $D_j = \{z \in \mathbb{P}^N(\mathbb{C}) : Q_j(z) = 0\}$, $0 \le j \le N$, be the hypersurfaces of degree d_j in general position in $\mathbb{P}^N(\mathbb{C})$. Suppose that $d = \mathrm{lcm}(d_0, \ldots, d_N)$ and $\sum_{j=0}^{N} Q_j^{\frac{d}{d_j}} \not\equiv 0$, which is denoted by $D := \{z \in \mathbb{P}^N(\mathbb{C}) : \sum_{j=0}^{N} Q_j^{\frac{d}{d_j}}(z) = 0\}$. Then,*

$$||T_f(r) \le \sum_{j=0}^{N} \frac{1}{d_j} N_f^{[N]}(r, D_j) + \frac{1}{d} N_f^{[N]}(r, D) + o(T_f(r)).$$

Proof. First, we suppose that hypersurfaces $D_0, \ldots, D_N$ have the same degree d. Since they are in general position in $\mathbb{P}^N(\mathbb{C})$, we have

$$\mathrm{supp} D_0 \cap \cdots \cap \mathrm{supp} D_N = \emptyset.$$

Thus, by Hilbert's Nullstellensatz [15], for any integer i, $0 \le i \le N$, there is an integer $m > d$ such that

$$x_i^m = \sum_{j=0}^{N} b_{ij}(x_0, \ldots, x_N) Q_j(x_0, \ldots, x_N),$$

where $b_0, \ldots, b_N$ are the homogeneous polynomials with coefficients in $\mathbb{C}$ of degree $m - d$.

Let $f = [f_0 : \cdots : f_N]$ be a reduced representation of f, where $f_0, \ldots, f_N$ are the entire functions on $\mathbb{C}$ and have no common zeros. Then,

$$|f_i(z)|^m \le c||f(z)||^{m-d} \max\{|Q_0(f(z))|, \ldots, |Q_N(f(z))|\},$$

where c is a positive constant depending only on the coefficients of b_{ij}, $0 \le i, j \le N$, thus depending only on the coefficients of Q_j, $0 \le j \le N$. Therefore,

$$|f_i(z)|^d \le c \max\{|Q_0(f(z))|, \ldots, |Q_N(f(z))|\}, \quad 0 \le i \le N. \tag{2}$$

We consider the holomorphic curve $F : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ induced by the map:

$$(Q_0(f(z)), \ldots, Q_N(f(z))).$$

Since f is algebraically nondegenerate and hypersurfaces D_j, $j = 0, \ldots, N$ are in general position in $\mathbb{P}^N(\mathbb{C})$, $F = [Q_0(f(z)) : \cdots : Q_N(f(z))]$ is a reduced representation of F and F is linearly nondegenerate. Hence, Inequality (2) implies that

$$T_F(r) \geq dT_f(r) + O(1). \tag{3}$$

On the other hand, applying Theorem 1.2 to F with the hyperplanes:

$$H_j := \{z \in \mathbb{P}^N(\mathbb{C}) : z_j = 0\} \ (0 \leq j \leq N), \ \ H_{N+1} = \left\{z \in \mathbb{P}^N(\mathbb{C}) : \sum_{j=0}^{N} z_j = 0\right\},$$

we have

$$||T_F(r) \leq \sum_{j=0}^{N+1} N_F^{[N]}(r, H_j) + o(T_f(r)).$$

Note that

$$N_F^{[N]}(r, H_j) = N_f^{[N]}(r, D_j) \ (0 \leq j \leq N), \ \ N_F^{[N]}(r, H_{N+1}) = N_f^{[N]}(r, D).$$

Hence,

$$||T_F(r) \leq \sum_{j=0}^{N} N_f^{[N]}(r, D_j) + N_f^{[N]}(r, D) + o(T_f(r)).$$

Combining Inequality (3), we have the conclusion of the lemma for the case of hypersurfaces with the same degrees.

If $D_0, \ldots, D_N$ have not the same degree, then the hypersurfaces $D_0^{d/d_0}, \ldots, D_N^{d/d_N}$ have the same degree d. Hence,

$$||dT_f(r) \leq \sum_{j=0}^{N} N_f^{[N]}(r, D_j^{d/d_j}) + N_f^{[N]}(r, D) + o(T_f(r)).$$

Since $N_f^{[N]}(r, D_j^{d/d_j}) \leq \frac{d}{d_j} N_f^{[N]}(r, D_j)$, $0 \leq j \leq N$, it yields that

$$||T_f(r) \leq \sum_{j=0}^{N} \frac{1}{d_j} N_f^{[N]}(r, D_j) + \frac{1}{d} N_f^{[N]}(r, D) + o(T_f(r)). \qquad \square$$

Proof of Theorem 1.5. Put $\tilde{D}_j := \{z \in \mathbb{P}^N(\mathbb{C}) : Q_j(z) = 0\}$, $0 \leq j \leq N$, which are the hypersurfaces of degree $d_j - k_j \geq 0$. According Inequality (1),

$$N_f^{[N]}(r, \tilde{D}_j) \leq N_f(r, \tilde{D}_j) \leq (d_j - k_j)T_f(r), \quad 0 \leq j \leq N.$$

Since $D_j = \{z \in \mathbb{P}^N(\mathbb{C}) : z_j^{k_j} Q_j(z) = 0\}$, $0 \leq j \leq N$, for every $0 \leq j \leq N$,

$$\begin{aligned} N_f^{[N]}(r, D_j) &\leq N_f^{[N]}(r, \tilde{D}_j) + N^{[N]}\left(r, \frac{1}{f_j^{k_j}}\right) \\ &\leq (d_j - k_j)T_f(r) + NN^{[1]}\left(r, \frac{1}{f_j^{k_j}}\right) \\ &= (d_j - k_j)T_f(r) + NN^{[1]}\left(r, \frac{1}{f_j}\right) \leq (d_j - k_j + N)T_f(r). \end{aligned}$$

From Lemma 2.1, we have

$$\begin{aligned}\|T_f(r) &\le \sum_{j=0}^{N}\frac{1}{d_j}N_f^{[N]}(r,D_j)+\frac{1}{d}N_f^{[N]}(r,D)+o(T_f(r))\\ &\le\left(\sum_{j=0}^{N}\frac{d_j-k_j+N}{d_j}\right)T_f(r)+\frac{1}{d}N_f^{[N]}(r,D)+o(T_f(r)).\end{aligned}$$

Thus,

$$\|\left(\sum_{j=0}^{N}\frac{k_j-N}{d_j}-N\right)T_f(r)\le\frac{1}{d}N_f^{[N]}(r,D)+o(T_f(r)).$$

Theorem 1.5 is proved. □

Proof of Theorem 1.8. By Lemma 2.1 and the definition of truncated defect of holomorphic curves, we obtain Theorem 1.8. □

Proof of Theorems 1.10 and 1.11. If the holomorphic curve $f:\mathbb{C}\to\mathbb{P}^N(\mathbb{C})$ whose image intersecting hypersurface D in $\mathbb{P}^N(\mathbb{C})$ with multiplicity at least l, then

$$N_f^{[N]}(r,D)\le NN_f^{[1]}(r,D)\le\frac{N}{l}N_f(r,D).$$

Suppose that f is algebraically nondegenerate. By Lemma 2.1 Inequality (1), we have

$$\begin{aligned}\|T_f(r) &\le \sum_{j=0}^{N}\frac{1}{d_j}N_f^{[N]}(r,D_j)+\frac{1}{d}N_f^{[N]}(r,D_{N+1})+o(T_f(r))\\ &\le\sum_{j=0}^{N}\frac{N}{l_jd_j}N_f(r,D_j)+\frac{N}{dl_{N+1}}N_f(r,D_{N+1})+o(T_f(r))\\ &\le\left(\sum_{j=0}^{N+1}\frac{N}{l_j}\right)T_f(r)+o(T_f(r)).\end{aligned}$$

Hence,

$$\sum_{j=0}^{N+1}\frac{1}{l_j}\ge\frac{1}{N}.$$

This is a contradiction. The proof of Theorem 1.10 is completed.

Suppose that f is algebraically nondegenerate. By Theorem 1.5 and Inequality (1), we obtain

$$\begin{aligned}\|\left(\sum_{j=0}^{N}\frac{k_j-N}{d_j}-N\right)T_f(r) &\le\frac{1}{d}N_f^{[N]}(r,D)+o(T_f(r))\\ &\le\frac{N}{ld}N_f(r,D)+o(T_f(r))\le\frac{N}{l}T_f(r)+o(T_f(r)).\end{aligned}$$

Thus,

$$l\le\frac{N}{\sum_{j=0}^{N}\frac{k_j-N}{d_j}-N}.$$

This contradicts our assumption. Theorem 1.11 is proved. □

Proof of Theorem 1.13. We suppose that $f\not\equiv g$. Then, there are two numbers $\alpha,\beta\in\{0,1,\ldots,N\}$, $\alpha\ne\beta$ such that

$$f_\alpha g_\beta\not\equiv f_\beta g_\alpha.$$

Assume that $z_0 \in f^{-1}(D) \cup g^{-1}(D)$. The condition $f(z) = g(z)$ on $z \in f^{-1}(D) \cup g^{-1}(D)$ implies that z_0 is a zero of $\frac{f_\alpha}{f_\beta} - \frac{g_\alpha}{g_\beta}$. Therefore, we have

$$N_f^{[N]}(r, D) \le NN_f^{[1]}(r, D) \le NN_{\frac{f_\alpha}{f_\beta} - \frac{g_\alpha}{g_\beta}}(r, 0) \le N(T_f(r) + T_g(r)) + O(1).$$

Applying Theorem 1.5, we have

$$\|\left(\sum_{j=0}^{N} \frac{k_j - N}{d_j} - N\right) T_f(r) \le \frac{1}{d} N_f^{[N]}(r, D) + o(T_f(r)) \le \frac{N}{d}(T_f(r) + T_g(r)) + o(T_f(r)).$$

Similarly,

$$\|\left(\sum_{j=0}^{N} \frac{k_j - N}{d_j} - N\right) T_g(r) \le \frac{N}{d}(T_f(r) + T_g(r)) + o(T_f(r)).$$

Therefore,

$$\|\sum_{j=0}^{N} \frac{k_j - N}{d_j} \le N + \frac{2N}{d}.$$

This is a contradiction. Hence, $f \equiv g$. □

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References

[1] M. Ru, *Nevanlinna Theory and Its Relation to Diophantine Approximation*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2021.

[2] J. Noguchi and J. Winkelmann, *Nevanlinna Theory in Several Complex Variables and Diophantine Approximation*, Springer Science and Business Media, Tokyo, 2013.

[3] H. Cartan, *Sur les zeros des combinaisions linearires de p fonctions holomorpes donnees*, Mathematica **7** (1933), 80–103.

[4] A. È. Erëmenko and M. L. Sodin, *Distribution of values of meromorphic functions and meromorphic curves from the standpoint of potential theory*, Algebra Anal. **3** (1991), no. 1, 131–164.

[5] M. Ru, *A defect relation for holomorphic curves intersecting hypersurfaces*, Am. J. Math. **126** (2004), no. 1, 215–226.

[6] S. D. Quang, *Degeneracy second main theorems for meromorphic mappings into projective varieties with hypersurfaces*. Trans. Amer. Math. Soc. **371** (2019), no. 4, 2431–2453.

[7] S. D. Quang, *Generalizations of degeneracy second main theorem and Schmidt's subspace theorem*. Pacific J. Math. **318** (2022), no. 1, 153–188.

[8] M. Ru, *Holomorphic curves into algebraic varieties*. Ann. Math. (2), **169** (2009), no. 1, 255–267.

[9] L. Yang, L. Shi, and X. Pang, *Remarks to Cartan's second main theorem for holomorphic curves into* $\mathbb{P}^N(\mathbb{C})$, Proc. Am. Math. Soc. **145** (2017), no. 8, 3437–3445.

[10] N. V. Thin, *A note on Cartan's second main theorem for holomorphic curve intersecting hypersurface*. J. Math. Anal. Appl. **452** (2017), no. 1, 488–494.
[11] M. Green and Ph. Griffiths, *Two applications of algebraic geometry to entire holomorphic mappings*, In: The Chern Symposium 1979 (Proc. Inter. Sympos. Beckeley, California (1979), Springer-Verlag, New York, 1980, pp. 41–74.
[12] L. Darondeau, *On the logarithmic Green-Griffiths conjecture*, Int. Math. Res. Not. **2016** (2016), no. 6, 1871–1923.
[13] D. T. Huynh, D. V. Vu, and S. Y. Xie, *Entire holomorphic curves into projective spaces intersecting a generic hypersurface of high degree*, Annales de la Institut Fourier **69** (2019), no. 2, 653–671.
[14] A. Nadel, *Hyperbolic surfaces in* P^3, Duke Math. **58** (1989), 749–771.
[15] V. D. Waerden, *Algebra*, vol. 2, 7th ed., Springer-Verlag, New York, 1991.