DE GRUYTER

Demonstratio Mathematica 2024; 57: 20230134

Research Article

Cihan Unal*

On existence and multiplicity of solutions for a biharmonic problem with weights via Ricceri's theorem

https://doi.org/10.1515/dema-2023-0134
received June 12, 2023; accepted November 14, 2023

Abstract: In this work, we consider a special nondegenerate equation with two weights. We investigate multiplicity result of this biharmonic equation. Mainly, our purpose is to obtain this result using an alternative Ricceri's theorem. Moreover, we give some compact embeddings in variable exponent Sobolev spaces with second order to prove the main idea.

Keywords: $p(.)$-biharmonic operator, Ricceri's variational principle, variational methods

MSC 2020: 35J35, 46E35, 35J60, 47J30

1 Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded smooth domain. Assume that ω and ϑ are two different weight functions. The aim of this study is to discuss the three solutions for the following equation:

$$\begin{cases} \Delta^2_{p(\cdot),\omega} u + \lambda\vartheta(x)|u|^{q(x)-2}u = \mu f(x,u) + \eta g(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where p and q are the continuous functions on $\overline{\Omega}$, i.e., $p, q \in C(\overline{\Omega})$ with $\inf_{x\in\overline{\Omega}} p(x) > 1$, $\lambda, \mu, \eta \in [0,\infty)$, and $\Delta^2_{p(\cdot),\omega} u = \Delta(\omega(x)|\Delta u|^{p(x)-2}\Delta u)$ is the weighted $p(\cdot)$-biharmonic operator of fourth order. Let $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function and

$$F(x,u) = \int_0^u f(x,s)\mathrm{d}s$$

and

$$G(x,u) = \int_0^u g(x,s)\mathrm{d}s.$$

In recent years, the study of differential equations and variational problems with variable exponent growth conditions has been an interesting topic resulting from nonlinear electrorheological fluids and elastic mechanics. Moreover, since the problem involving $p(x)$-biharmonic operator, which is not homogeneous, possesses more complicated nonlinearities than the constant case, several variational methods such as the Lagrange multiplier theorem, may not be practicable (see [1–11]). Note that these physical problems were facilitated by the development of the variable exponent Lebesgue and Sobolev spaces.

*** Corresponding author: Cihan Unal,** Department of Mathematics, Faculty of Arts and Sciences, Sinop University, Sinop, Türkiye, e-mail: cihanunal88@gmail.com

Ricceri's three critical point theorem is a powerful tool to study the boundary problem of a differential equation (see, e.g., [12–15]). Particularly, Mihăilescu [16] used three critical points theorem of Ricceri [17] to study a particular $p(x)$-Laplacian equation. He proved the existence of three solutions for the problem. Liu [18] studied the solutions of the general $p(x)$-Laplacian equations with Neumann or Dirichlet boundary condition on a bounded domain, and obtained three solutions under appropriate hypotheses. Shi and Ding [19] generalized the corresponding result of [16]. To our best knowledge, there are no results of multiple solutions to $p(x)$-biharmonic equation under sublinear condition.

In 2001, problem (1) was studied by Drábek and Ôtani [20] for $p(x) = q(x) = p = \text{constant}$, $\vartheta(x) = \omega(x) = 1$, and $\mu = \eta = 0$. They proved that the problem has a principal positive eigenvalue, which is simple and isolated. Later, a nondecreasing sequence of positive eigenvalues of the weighted p-biharmonic operator was obtained, and the simplicity and isolation of the first positive eigenvalue were proved by Talbi and Tsouli [21] for $p(x) = q(x) = p = \text{constant}$ and two weight functions.

In 2009, Ayoujil and El Amrouss [1] obtained the eigenvalues of the following fourth-order elliptic equation, which is the particular case of (1):

$$\begin{cases} \Delta^2_{p(\cdot)}u = \lambda|u|^{p(x)-2}u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

in the space $W^{2,p(\cdot)}(\Omega) \cap W^{1,p(\cdot)}_0(\Omega)$ for $p(x) = q(x)$, $\vartheta(x) = \omega(x) = 1$, and $\mu = \eta = 0$. Problem (1) is also examined for $\vartheta(x) = 1$ by Ge et al. [5]. They demonstrate the existence of a continuous family of eigenvalues for the problem.

Li et al. [22] gave an approach to show the existence of at least three solutions for (1) with a non-weighted case. They used a three critical points theorem presented by Ricceri. Moreover, Aydin [23] considered a new compact embedding between a defined intersection space and a weighted Lebesgue space. The author investigated the existence and multiplicity of a weak solution for (1) with $\mu = \eta = 0$. In this regard, our study is a generalized version of [22,23] and some references therein.

Aydin and Unal [24] presented the double-weighted variable exponent Sobolev space and investigated some basic and advanced properties including compact embeddings of these spaces. They considered several elliptic problems such as the Steklov and Robin problems. Moreover, the authors used different approaches (e.g., Ricceri's variational principle, Ekeland's variational principle, and Fountain theorem) on the mentioned problems. For more details, we refer to [24–28].

The energy functional corresponding to problem (1) is defined on X as

$$E(u) = J(u) + \mu\Psi(u) + \eta\Phi(u),$$

where

$$J(u) = \int_\Omega \left(\frac{1}{p(x)} |\Delta u|^{p(x)}\omega(x) + \frac{\lambda}{q(x)} |u|^{q(x)}\vartheta(x) \right) dx,$$

$$\Psi(u) = -\int_\Omega F(x, u)dx,$$

$$\Phi(u) = -\int_\Omega G(x, u)dx.$$

Let us recall that a weak solution of (1) is any $u \in X$ such that

$$\int_\Omega \omega(x)|\Delta u|^{p(x)-2}\Delta u \Delta v dx + \lambda\int_\Omega \vartheta(x)|u|^{q(x)-2}uv dx = \mu\int_\Omega f(x, u)v dx + \eta\int_\Omega g(x, u)v dx,$$

for all $v \in X$.

This article is divided into four sections, organized as follows: in Section 2, we list some well-known definitions and some basic properties of the weighted (or non-weighted) variable exponent Lebesgue and Sobolev spaces. Moreover, we study boundary trace embedding theorems for variable exponent Sobolev space

$W_{\omega}^{2,p(x)}(\Omega)$, and we present some important properties of the $p(x)$-biharmonic operator. In Section 3, we recall an alternative form of Ricceri's theorem and give the proof of the main theorem.

Throughout this article, we use the notation X^* as the dual space of X.

2 Notation and preliminaries

In this section, we give some definitions and basic information about weighted variable Lebesgue and Sobolev spaces to find out the solution of the problem (1). Moreover, we need some properties of $p(x)$-biharmonic operator, which we will use later.

A normed space $(X, \|\cdot\|_X)$ is called a Banach function space (BF space), if Banach space $(X, \|\cdot\|_X)$ is continuously embedded into $L^1_{\text{loc}}(\Omega)$, briefly $X \hookrightarrow L^1_{\text{loc}}(\Omega)$, i.e., for any compact subset $K \subset \Omega$, there is some constant $C_1 > 0$ such that $\|f\chi_K\|_{L^1(\Omega)} \leq C_1 \|f\|_X$ for every $f \in X$. Moreover, a normed space X is compactly embedded in a normed space Y, briefly $X \hookrightarrow\hookrightarrow Y$, if $X \hookrightarrow Y$ and the identity operator $I : X \longrightarrow Y$ is compact, equivalently, I maps every bounded sequence $(x_i)_{i\in\mathbb{N}}$ into a sequence $(I(x_i))_{i\in\mathbb{N}}$ that contains a subsequence converging in Y. Suppose that X and Y are two Banach spaces and X is reflexive. Then, $I : X \longrightarrow Y$ is a compact operator if and only if I maps weakly convergent sequences in X onto convergent sequences in Y. More details can be found in [29].

Set

$$C_+(\overline{\Omega}) = \left\{p \in C(\overline{\Omega}) : \inf_{x\in\overline{\Omega}} p(x) > 1\right\},$$

where $C(\overline{\Omega})$ composes all continuous functions on $\overline{\Omega}$. Denote

$$p^- = \inf_{x\in\overline{\Omega}} p(x) \quad p^+ = \sup_{x\in\overline{\Omega}} p(x)$$

and

$$p^* = \begin{cases} \dfrac{Np(x)}{N - p(x)}, & p(x) < N, \\ +\infty, & p(x) \geq N, \end{cases}$$

for any $x \in \overline{\Omega}$. Let $p \in C_+(\overline{\Omega})$ and $1 < p^- \leq p(x) \leq p^+ < \infty$. Now, we define the modular function by

$$\varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\mathrm{d}x.$$

Then, the space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\varrho_{p(\cdot)}(u) < \infty$. Moreover, the modular space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a Banach space with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

(see [30]).

The weighted Lebesgue space $L^{p(\cdot)}_\omega(\Omega)$ is defined by

$$L^{p(\cdot)}_\omega(\Omega) = \left\{u \;\middle|\; u : \Omega \longrightarrow \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}\omega(x)\mathrm{d}x < \infty\right\}$$

such that $\|u\|_{p(\cdot),\omega} = \left\|u\omega^{\frac{1}{p(\cdot)}}\right\|_{p(\cdot)} < \infty$ for $u \in L^{p(\cdot)}_\omega(\Omega)$, where ω is a weight function from Ω to $(0, \infty)$. If $\omega \in L^\infty(\Omega)$, then $L^{p(\cdot)}_\omega = L^{p(\cdot)}$ (see [31,32]).

Let $\omega^{-\frac{1}{p(\cdot)-1}} \in L^1_{\text{loc}}(\Omega)$. We set the weighted variable exponent Sobolev space $W^{k,p(\cdot)}_\omega(\Omega)$ by

$$W^{k,p(\cdot)}_\omega(\Omega) = \{u \in L^{p(\cdot)}_\omega(\Omega) : D^\alpha u \in L^{p(\cdot)}_\omega(\Omega), 0 \leq |\alpha| \leq k\}$$

equipped with the norm

$$\|u\|_{k,p(\cdot),\omega} = \sum_{0\le|\alpha|\le k} \|D^\alpha u\|_{p(\cdot),\omega},$$

where $\alpha \in \mathbb{N}_0^N$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}}$. It is known that $W_\omega^{k,p(\cdot)}(\Omega)$ is a reflexive Banach space. In particular, the space $W_\omega^{1,p(\cdot)}(\Omega)$ is defined by

$$W_\omega^{1,p(\cdot)}(\Omega) = \{u \in L_\omega^{p(\cdot)}(\Omega) : |\nabla u| \in L_\omega^{p(\cdot)}(\Omega)\}.$$

The function $\varrho_{1,p(\cdot),\omega} : W_\omega^{1,p(\cdot)}(\Omega) \longrightarrow [0,\infty)$ is shown as $\varrho_{1,p(\cdot),\omega}(u) = \varrho_{p(\cdot),\omega}(u) + \varrho_{p(\cdot),\omega}(\nabla u)$. Also, the norm $\|u\|_{1,p(\cdot),\omega} = \|u\|_{p(\cdot),\omega} + \|\nabla u\|_{p(\cdot),\omega}$ makes the space $W_\omega^{1,p(\cdot)}(\Omega)$ a Banach space.

Moreover, the space $C_0^\infty(\Omega)$ is a subspace of $W_\omega^{1,p(\cdot)}(\Omega)$. Thus, we can obtain the closure of $C_0^\infty(\Omega)$ in $W_\omega^{1,p(\cdot)}(\Omega)$, which is denoted by $W_{0,\omega}^{1,p(\cdot)}(\Omega)$. The space $W_{0,\omega}^{1,p(\cdot)}(\Omega)$ can also be defined as

$$W_{0,\omega}^{1,p(\cdot)}(\Omega) = \{u \in W_\omega^{1,p(\cdot)}(\Omega) : u\,|_{\partial\Omega} = 0\}.$$

Finally, the space $W_{0,\omega}^{1,p(\cdot)}(\Omega)$ is equipped with the norm

$$\|u\|_{W_{0,\omega}^{1,p(\cdot)}} = \|\nabla u\|_{p(\cdot),\omega}$$

by the Poincaré inequality (see [33, Proposition 3.4]). Therefore, the norms $\|\cdot\|_\omega^{1,p(\cdot)}$ and $\|\cdot\|_{W_{0,\omega}^{1,p(\cdot)}}$ are equivalent on $W_{0,\omega}^{1,p(\cdot)}(\Omega)$. The space $(W_{0,\omega}^{1,p(\cdot)}(\Omega), \|\cdot\|_{W_{0,\omega}^{1,p(\cdot)}})$ is also a separable and reflexive Banach space (see [30,33–38]).

Let $f \in L_{\mathrm{loc}}^1(\Omega)$. Then, the maximal function of f is defined by

$$Mf(x) = \sup_{x\in B} \frac{1}{|B|} \int_{B\cap\Omega} |f(y)|\mathrm{d}y,$$

where the supremum is taken over all ball centers at x. It is proved that the operator Mf is bounded on $L_\omega^{p(\cdot)}(\Omega)$. In addition, $C_0^\infty(\mathbb{R}^N)$ is dense in $W_\omega^{1,p(\cdot)}(\mathbb{R}^N)$ and $L_\omega^{p(\cdot)}(\mathbb{R}^N)$ (see [34,39]).

Proposition 2.1. (See [40]) *If $1 < q(x) \le p(x) < \infty$, $0 < \vartheta(x) \le \omega(x)$ a.e. $x \in \Omega$, and $|\Omega| < \infty$, then the inequality*

$$\|u\|_{q(\cdot),\vartheta} \le C_2 \,\|u\|_{p(\cdot),\omega}$$

holds for $u \in L_\omega^{p(\cdot)}(\Omega)$, i.e., the embeddings $L_\omega^{p(\cdot)}(\Omega) \hookrightarrow L_\vartheta^{q(\cdot)}(\Omega)$ are continuous, where C_2 is independent of u.

Let E and F be Banach spaces. Define the sum norm on the space $Y = E \cap F$ as $\|u\|_Y = \|u\|_E + \|u\|_F$ for $u \in Y$. To solve the problem (1.1), we call for an equivalent norm, and a new compact embedding theorem for $X = W_\omega^{2,p(\cdot)}(\Omega) \cap W_{0,\omega}^{1,p(\cdot)}(\Omega)$. For $u \in X$, we have

$$\|u\|_X = \|\nabla u\|_{p(\cdot),\omega} + \|u\|_\omega^{2,p(\cdot)} = \|u\|_{p(\cdot),\omega} + \|\nabla u\|_{p(\cdot),\omega} + \sum_{|\alpha|=2} \|D^\alpha u\|_{p(\cdot),\omega}.$$

In 2008, Zang and Fu [41] showed that the norms $\|u\|_X$ and $\|\Delta u\|_{p(\cdot)}$ are equivalent on X for $\omega(x) = 1$. To improve and prove the weighted version of this theorem, we have required conditions for X. Thus, if we use the techniques and methods in [41, Theorem 4.4], then we obtain the following more general norm for X.

Theorem 2.2. *Let Ω be a bounded domain with Lipschitz boundary. If the operator Mf is bounded on $L_\omega^{p(\cdot)}(\Omega)$, then the norm $\|\Delta u\|_{p(\cdot),\omega}$ is equivalent to the norm $\|u\|_X$ in $W_\omega^{2,p(\cdot)}(\Omega) \cap W_{0,\omega}^{1,p(\cdot)}(\Omega)$.*

Then, we take the norm for the space X as $\|u\|_X = \|\Delta u\|_{p(\cdot),\omega}$, where

$$\|u\|_X = \inf\left\{\gamma > 0 : \int_\Omega \omega(x) \left|\frac{\Delta u}{\gamma}\right|^{p(x)} \mathrm{d}x \le 1\right\},$$

for $u \in X$. It is noted that the space $(X, \|\cdot\|_X)$ is a separable and reflexive Banach space. Throughout this article, we will take the norm $\|\cdot\|_X$ and the modular $\varrho^{\omega}_{p(\cdot)}$, which is defined as $\varrho^{\omega}_{p(\cdot)} : X \to \mathbb{R}$ by

$$\varrho^{\omega}_{p(\cdot)}(u) = \int_\Omega \omega(x)|\Delta u(x)|^{p(x)}\mathrm{d}x.$$

Proposition 2.3. (See [5,6,42]) *For any $u, u_k \in X$ $(k = 1, 2, \ldots)$, we have*

(i) $\|u\|_X < 1(=1, >1)$ *if and only if* $\varrho^{\omega}_{p(\cdot)}(u) < 1(=1, >1)$,

(ii) $\|u\|_X^{p^-} \le \varrho^{\omega}_{p(\cdot)}(u) \le \|u\|_X^{p^+}$ *with* $\|u\|_X \ge 1$,

(iii) $\|u\|_X^{p^+} \le \varrho^{\omega}_{p(\cdot)}(u) \le \|u\|_X^{p^-}$ *with* $\|u\|_X \le 1$,

(iv) $\min\{\|u\|_X^{p^-}, \|u\|_X^{p^+}\} \le \varrho^{\omega}_{p(\cdot)}(u) \le \max\{\|u\|_X^{p^-}, \|u\|_X^{p^+}\}$,

(v) $\|u - u_k\|_X \to 0$ *if and only if* $\varrho^{\omega}_{p(\cdot)}(u - u_k) \to 0$ *as* $k \to \infty$,

(vi) $\|u_k\|_X \to \infty$ *if and only if* $\varrho^{\omega}_{p(\cdot)}(u_k) \to \infty$ *as* $k \to \infty$.

Let

$$p_2^*(x) = \begin{cases} \dfrac{Np(x)}{N - 2p(x)}, & \text{if } p(x) < \dfrac{N}{2}, \\ +\infty, & \text{if } p(x) \ge \dfrac{N}{2}, \end{cases}$$

for every $x \in \overline{\Omega}$. It is obvious that $p(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$. In 2009, Ayoujil and Amrouss [1] proved that the space $W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$ is compactly embedded in $L^{q(\cdot)}(\Omega)$ with $p, q \in C_+(\overline{\Omega})$ such that $q(x) < p_2^*(x)$ for any $x \in \overline{\Omega}$. Moreover, Aydin [23] proved the compact embedding from X into $L_{\vartheta}^{q(\cdot)}(\Omega)$, where $p(x) \le q(x) < p_2^*(x)$ for $x \in \overline{\Omega}$.

Let us consider the functional

$$J(u) = \int_\Omega \left(\frac{1}{p(x)} |\Delta u|^{p(x)}\omega(x) + \frac{\lambda}{q(x)} |u|^{q(x)}\vartheta(x) \right)\mathrm{d}x.$$

It is easy to see that $J(u)$ is well defined and continuous differentiable in X.

Proposition 2.4. *For any $u, u_k \in X(k = 1, 2, \ldots)$, we have*

(i) $\|u\|_X < 1(=1, >1) \Leftrightarrow J(u) < 1(=1, >1)$,

(ii) $\|u\|_X^{p^-} \le J(u) \le \|u\|_X^{p^+}$ *with* $\|u\|_X \ge 1$,

(iii) $\|u\|_X^{p^+} \le J(u) \le \|u\|_X^{p^-}$ *with* $\|u\|_X \le 1$,

(iv) $\|u_k\|_X \to 0$ *if and only if* $J(u_k) \to 0$, *as* $k \to \infty$,

(v) $\|u_k\|_X \to \infty$ *if and only if* $J(u_k) \to \infty$, *as* $k \to \infty$.

The proof of Proposition 2.4 can be seen by the same method in [37, Theorem 1.3]. Also, the operator $T := J' : X \longrightarrow X^*$ defined as

$$\langle T(u), v\rangle = \int_\Omega (\omega(x)|\Delta u|^{p(x)-2}\Delta u\Delta v + \lambda\vartheta(x)|u|^{q(x)-2}uv)\mathrm{d}x,$$

for any $u, v \in X$. Moreover, the function $u \in X$ is a weak solution of the problem (1) if and only if u is a critical point of J.

It is noted that Zhikov and Surnachev [43] proved a sufficient condition for the density of smooth functions in weighted variable exponent Sobolev space. Therefore, the weak solution of problem (1) is well defined.

Theorem 2.5. *$T \coloneqq J' : X \longrightarrow X^*$ is a*
(*i*) *continuous, bounded, mapping of type (S_+), and strictly monotone operator,*
(*ii*) *homeomorphism.*

Proof. Using the same method in [22], it is easy to see that (*i*) holds. Now, we prove that T is a homeomorphism. Since T is the strictly monotone operator, the operator T is an injective operator. Moreover, T is a coercive operator. To prove this, let $u \in X$ and $\|u\|_X > 1$. If we consider Proposition 2.3, [40, Proposition 2.2] and the fact that X compactly embedded into $L_{\vartheta}^{q(\cdot)}(\Omega)$, then we obtain

$$\begin{aligned} J(u) &= \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} \omega(x) \mathrm{d}x + \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} \vartheta(x) \mathrm{d}x \\ &\geq \frac{1}{p^+} \int_{\Omega} |\Delta u|^{p(x)} \omega(x) \mathrm{d}x + \frac{\lambda}{q^+} \int_{\Omega} |u|^{q(x)} \vartheta(x) \mathrm{d}x \\ &\geq \frac{1}{p^+} \varrho_{p(\cdot)}^{\omega}(u) + \frac{\lambda}{q^+} \min\{\|u\|_{q(\cdot),\vartheta}^{q^-}, \|u\|_{q(\cdot),\vartheta}^{q^+}\} \\ &\geq \frac{1}{p^+} \|u\|_X^{p^-} + \frac{\lambda}{q^+} \min\{\|u\|_X^{q^-}, \|u\|_X^{q^+}\} \\ &= \frac{1}{p^+} \|u\|_X^{p^-} + \frac{\lambda}{q^+} \|u\|_X^{q^-}. \end{aligned}$$

This follows that $J(u) \longrightarrow \infty$ as $\|u\|_X \longrightarrow \infty$, i.e., J is coercive on X. As a result, by Minty-Browder theorem (see [44]), the operator J is an surjection and admits an inverse mapping. It is enough to see that J^{-1} is continuous. Assume that $(f_n)_{n\in\mathbb{N}}$ is a sequence of X^* satisfying $f_n \longrightarrow f$ in X^*. Assume that u_n and u are in X such that

$$J^{-1}(f_n) = u_n \quad \text{and} \quad J^{-1}(f) = u.$$

Since the functional J is coercive, the sequence (u_n) is bounded in the reflexive space X. For a subsequence denoted by (u_n), we obtain $u_n \rightharpoonup \hat{u}$ in X. This yields

$$\lim_{n\longrightarrow+\infty} \langle J(f_n) - J(f), u_n - \hat{u} \rangle = \lim_{n\longrightarrow+\infty} \langle f_n - f, u_n - \hat{u} \rangle = 0.$$

This follows by the continuity of J that

$$u_n \longrightarrow \hat{u} \text{ in } X \quad \text{and} \quad J(u_n) \longrightarrow J(\hat{u}) = J(u) \text{ in } X^*.$$

By the injectivity of J, we obtain that $u = \hat{u}$. □

3 Main results

Before giving our main result, we need to the alternative form of Ricceri's theorem (see [45,46]).

Theorem 3.1. *X is a reflexive real Banach space. $J : X \longrightarrow \mathbb{R}$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* and J is bounded on every bounded subset of X; $\Psi : X \longrightarrow \mathbb{R}$ is a continuously Gâteaux differentiable whose Gâteaux derivative is compact; $I \subseteq \mathbb{R}$ an interval. Let*

$$\lim_{\|x\|\longrightarrow+\infty} (J(x) + \mu\Psi(x)) = +\infty \tag{2}$$

for every $\mu \in I$, and that there is a $h \in \mathbb{R}$ satisfying

$$\sup_{\mu\in I} \inf_{x\in X} (J(x) + \mu(\Psi(x) + h)) < \inf_{x\in X} \sup_{\mu\in I} (J(x) + \mu(\Psi(x) + h)). \tag{3}$$

Thus, there is an open interval $\Lambda \subseteq I$ and a positive real number ρ with the following property, i.e., for all $\mu \in \Lambda$ and every C^1 functional $\Phi : X \longrightarrow \mathbb{R}$ with compact derivative, there is a positive real number δ satisfying for every $\beta \in [0, \delta]$, the expression

$$J'(x) + \mu\Psi'(x) + \beta\Phi'(x) = 0$$

has at least three solutions in X whose norms are less than ρ.

Proposition 3.2. *X is a non-empty set and J and Ψ are two real functions on X. Suppose that there exist $r > 0$ and $x_0, x_1 \in X$ satisfying*

$$J(x_0) = -\Psi(x_0) = 0, J(x_1) > r, \sup_{x \in J^{-1}((-\infty, r])} (-\Psi(x)) < r\frac{-\Psi(x_1)}{J(x_1)}.$$

Thus, for every h such that

$$\sup_{x \in J^{-1}((-\infty, r])} (-\Psi(x)) < h < r\frac{-\Psi(x_1)}{J(x_1)},$$

the expression

$$\sup_{\mu \geq 0} \inf_{x \in X} (J(x) + \mu(\Psi(x) + h)) < \inf_{x \in X} \sup_{\mu \geq 0} (J(x) + \mu(\Psi(x) + h))$$

is satisfied.

Now, we are ready to give our motivation.

Theorem 3.3. *Let $\sup_{(x,s) \in \Omega \times \mathbb{R}} \frac{|f(x,s)|}{1+|s|^{z(x)-1}} < +\infty$, where $z \in C(\overline{\Omega})$ and $z(x) < q^*(x)$ for every $x \in \overline{\Omega}$ and there are $\rho, \varphi > 0$ and a function $\gamma(x) \in C(\overline{\Omega})$ with $1 < \gamma^- \leq \gamma^+ \leq \max\{q^-, p^-\}$ such that*

(*i*) *$F(x, s) > 0$ for a.e. $x \in \Omega$ and all $s \in (0, \varrho]$,*

(*ii*) *there are $r(x) \in C(\overline{\Omega})$ and $q^+ < r^- \leq r(x) < q^*(x)$ satisfying*

$$\limsup_{s \to 0} \sup_{x \in \Omega} \frac{F(x, s)}{|s|^{r(x)}} < +\infty,$$

(*iii*) *The inequality*

$$|F(x, s)| \leq \varphi(1 + |s|^{\gamma(x)}),$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$.

Then, there is an open interval $\Lambda \subseteq (0, +\infty)$ and a positive number ρ with the following property: for each $\mu \in \Lambda$ and each function $g(x, s) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\sup_{(x,s) \in \Omega \times \mathbb{R}} \frac{|g(x, s)|}{1 + |s|^{t(x)-1}} < +\infty,$$

where $t(x) \in C(\overline{\Omega})$ and $t(x) < q^*(x)$ for all $x \in \overline{\Omega}$, there is a positive constant δ such that for every $\beta \in [0, \delta]$, problem (1) *has at least three weak solutions whose norms in X are less than ρ.*

Proof. The equalities

$$\langle J'(u), v\rangle = \int_{\Omega} (\omega(x)|\Delta u|^{p(x)-2}\Delta u \Delta v + \lambda\vartheta(x)|u|^{q(x)-2}uv)\mathrm{d}x$$

$$\langle \Psi'(u), v\rangle = -\int_\Omega f(x,u)v\,dx$$

$$\langle \Phi'(u), v\rangle = -\int_\Omega g(x,u)v\,dx$$

hold for all $u, v \in X$. By Theorem 2.5, J is a continuous Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*. Therefore, Ψ and Φ are continuously Gâteaux differentiable functionals whose Gâteaux derivative is compact. Also, it is clear that J is bounded on every bounded subset of X.

By Theorem 2.5, it is clear that

$$J(u) \geq \frac{1}{p^+}\|u\|_X^{p^-} + \frac{\lambda}{q^+}\|u\|_X^{q^-}, \tag{4}$$

for $\|u\|_X \geq 1$. Moreover, if we consider (iii) and [23, Theorem 6], then we have

$$\begin{aligned}\mu\Psi(u) &\geq -\mu\int_\Omega \varphi(1+|u|^{\gamma(x)})dx \geq -\mu\varphi(|\Omega| + \|u\|_{\gamma(\cdot)}^{\gamma^+})\\ &\geq -C_3(1+\|u\|_{\gamma(\cdot)}^{\gamma^+}) \geq -C_4(1+\|u\|_X^{\gamma^+}),\end{aligned} \tag{5}$$

for every $\mu \in \Lambda$ and $u \in X$. Here, C_3 and C_4 are positive constants. This follows by (4), (5), and use the fact $\gamma^+ \leq \max\{q^-, p^-\}$ that

$$J(u) + \mu\Psi(u) \geq \frac{1}{p^+}\|u\|_X^{p^-} + \frac{\lambda}{q^+}\|u\|_X^{q^-} - C_4(1+\|u\|_X^{\gamma^+}).$$

This yields

$$\lim_{\|u\|_X\to\infty}(J(u)+\mu\Psi(u)) = +\infty,$$

for every $u \in X$, $\mu \in [0,+\infty)$. This means that the assumption (2) is verified.

Now, the remaining part of the proof is verifying the assumption (3). To obtain this, we verify the assumptions of Proposition 3.2. Let $u_0 = 0$. Then, it is clear that

$$J(u_0) = -\Psi(u_0) = 0.$$

Now, we prove that (3) is satisfied. By (ii), there exist $\zeta \in [0,1]$ and $C_5 > 0$ such that

$$F(x,s) < C_5|s|^{r(x)} < C_5|s|^{r^-},$$

for a.e. $x \in \Omega$ and for every $s \in [-\zeta, \zeta]$. This follows by (iii) that there is a constant C_6 such that

$$F(x,s) < C_6|s|^{r^-},$$

for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$. If we consider the fact $X \hookrightarrow L^{r^-}(\Omega)$, then we obtain

$$-\Psi(u) = \int_\Omega F(x,u)dx < C_6\int_\Omega |u|^{r^-}dx \leq C_7\|u\|_X^{r^-} \leq C_7\theta^{\frac{r^-}{q^+}},$$

when $\frac{\|u\|_X^{q^+}}{q^+} \leq \theta$. Since $r^- > q^+$, we have

$$\lim_{\theta\to 0^+}\frac{\sup_{\frac{\|u\|_X^{q^+}}{q^+}\leq\theta}(-\Psi(u))}{\theta} = 0. \tag{6}$$

Assume that $u_1 \in C^2(\Omega)$ is a function satisfying $u_1|_{\partial\Omega} = 0$, $\max_{\overline{\Omega}} u_1 \leq d$, and positive in Ω. This follows that $u_1 \in X$ and $J(u_1) > 0$. By (i), we obtain

$$-\Psi(u_1) = \int_\Omega F(x, u_1(x))\mathrm{d}x > 0.$$

Thus, if we consider (6), then there exists $\theta \in \left(0, \min\left\{J(u_1), \frac{1}{p^+}\right\}\right)$ such that

$$\sup_{\frac{\|u\|_X^{q^+}}{q^+}\le\theta} (-\Psi(u)) < \theta\frac{-\Psi(u_1)}{J(u_1)}.$$

Now, let $u \in J^{-1}((-\infty, \theta])$. This yields

$$\int_\Omega\left(\frac{1}{p(x)}|\Delta u|^{p(x)}\omega(x) + \frac{\lambda}{q(x)}|u|^{q(x)}\vartheta(x)\right)\mathrm{d}x \le \theta < 1,$$

which, by Proposition 2.4, implies $\|u\|_X < 1$. Thus, we obtain

$$\frac{1}{p^+}\|u\|_X^{p^+} + \frac{\lambda}{q^+}\|u\|_X^{q^-} \le \int_\Omega\left(\frac{1}{p(x)}|\Delta u|^{p(x)}\omega(x) + \frac{\lambda}{q(x)}|u|^{q(x)}\vartheta(x)\right)\mathrm{d}x < \theta.$$

Therefore, we obtain

$$J^{-1}((-\infty, \theta]) \subset \left\{u \in X : \frac{1}{p^+}\|u\|_X^{p^+} + \frac{\lambda}{q^+}\|u\|_X^{q^-} < \theta\right\},$$

and then,

$$\sup_{u\in J^{-1}((-\infty,\theta])} (-\Psi(u)) < \theta\frac{-\Psi(u_1)}{J(u_1)}.$$

If we consider Theorem 3.1 and Proposition 3.2, then the desired result is completed. □

Acknowledgments: The author would like to thank the handling editor and referees for their helpful and useful comments and suggestions.

Funding information: The author states that there is no funding involved.

Author contributions: The author has accepted responsibility for the entire content of this manuscipt and approved its submission.

Conflict of interest: The author states that there is no conflict of interest.

References

[1] A. Ayoujil and A. R. El Amrouss, *On the spectrum of a fourth order elliptic equation with variable exponent*, Nonlinear Anal. **71** (2009), 4916–4926.
[2] A. C. Cavalheiro, *Existence of entropy solutions for degenerate quasilinear elliptic equations, Complex Var*. Elliptic Equ. **53** (2008), no. 10, 945–956.
[3] A. C. Cavalheiro, *Existence and uniqueness of solutions for some degenerate nonlinear elliptic equations*, Arc. Math. (Brno) Tomus **50** (2014), 51–63.
[4] A. C. Cavalheiro, *Existence and uniqueness of solutions for problems with weighted p-Laplacian and p-biharmonic operators*, Elec. J. Math. Anal. App. **3** (2015), no. 2, 215–226.
[5] B. Ge, Q. M. Ge, and Y. H. Wu, *Eigenvalues of the $p(x)$-biharmonic operator with indefinite weight*, Z. Angew. Math. Phys. **66** (2015), 1007–1021.
[6] K. Kefi, *For a class of $p(x)$-biharmonic operators with weights*, RACSAM **113** (2019), 1557–1570.
[7] K. Kefi and V. Rădulescu, *On a $p(x)$-biharmonic problem with singular weights*, Z. Angew. Math. Phys. **68** (2017), no. 4, 80.

[8] K. Kefi and K. Saoudi, *On the existence of a weak solution for some singular $p(x)$-biharmonic equation with Navier boundary conditions*, Adv. Nonlinear Anal. **8** (2019), 1171–1183.

[9] L. Kong, *On a fourth order elliptic problem with a $p(x)$-biharmonic operator*, Appl. Math. Lett. **27** (2014), 21–25.

[10] L. Kong, *Eigenvalues for a fourth order elliptic problem*, Proc. Amer. Math. Soc. **143** (2015), 249–258.

[11] L. Li and C. Tang, *Existence and multiplicity of solutions for a class of $p(x)$-biharmonic equations*, Acta Math. Sci. **33B** (2013), no. 1, 155–170.

[12] G. A. Afrouzi and S. Heidarkhani, *Three solutions for a Dirichlet boundary value problem involving the p-Laplacian*, Nonlinear Anal. **66** (2007), 2281–2288.

[13] G. Bonanno and P. Candito, *Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian*, Arch. Math. **80** (2003), 424–429.

[14] G. Bonanno and R. Livrean, *Multiplicity theorems for the Dirichlet problem involving the p-Laplacian*, Nonlinear Anal. **54** (2003), 1–7.

[15] G. Bonanno, G. Molica Bisci, and V. Rădulescu, *Multiple solutions of generalized Yamabe equations on Riemannian manifolds and applications to Emden-Fowler problems*, Nonlinear Anal. Real World Appl. **12** (2011), no. 5, 2656–2665.

[16] M. Mihăilescu, *Existence and multiplicity of solutions for a Neumann problem involving the p(x)-Laplace operator*, Nonlinear Anal. **67** (2007), 1419–1425.

[17] B. Ricceri, *On a three critical points theorem*, Arch. Math. (Basel) **75** (2000), 220–226.

[18] Q. Liu, *Existence of three solutions for $p(x)$-Laplacian equations*, Nonlinear Anal. **68** (2008), 2119–2127.

[19] X. Shi and X. Ding, *Existence and multiplicity of solutions for a general p(x)-Laplacian Neumann problem*, Nonlinear Anal. **70** (2009), 3715–3720.

[20] P. Drábek and M. Ôtani, *Global bifurcation result for the p-biharmonic operator*, Electron. J. Differential Equations **48** (2001), 1–19.

[21] M. Talbi and N. Tsouli, *On the spectrum of the weighted p-biharmonic operator with weight*, Mediterr. J. Math. **4** (2007), 73–86.

[22] L. Li, L. Ding, and W. Pan, *Existence of multiple solutions for a $p(x)$-biharmonic equation*, Electron. J. Differential Equations **2013** (2013), no. 139, 1–10.

[23] I. Aydin, *Almost all weak solutions of the weighted p(.)-biharmonic problem*, J. Anal. 2023, DOI: https://doi.org/10.1007/s41478-023-00628-w.

[24] I. Aydin and C. Unal, *Three solutions to a Steklov problem involving the weighted p(.)-Laplacian*, Rocky Mountain J. Math. **51** (2021), no. 1, 67–76.

[25] I. Aydin and C. Unal, *Weighted stochastic field exponent Sobolev spaces and nonlinear degenerated elliptic problem with nonstandard growth*, Hacettepe J. Math. Stat. **49** (2020), no. 4, 1383–1396.

[26] I. Aydin and C. Unal, *Existence and multiplicity of weak solutions for eigenvalue Robin problem with weighted p(.)-Laplacian*, Ric. Mat. **72** (2023), 511–528, DOI: https://doi.org/10.1007/s11587-021-00621-0.

[27] O. Kulak, I. Aydin, and C. Unal, *Existence of weak solutions for weighted Robin problem involving p(.)-biharmonic operator*, Differ. Equ. Dyn. Syst. 2022, DOI: https://doi.org/10.1007/s12591-022-00619-6.

[28] C. Unal and I. Aydin, *On some properties of relative capacity and thinness in weighted variable exponent Sobolev spaces*, Anal. Math. **46** (2020), no. 1, 147–167.

[29] R. A. Adams and J. J. Fournier, *Sobolev Spaces*, 2nd ed., vol. 305, Academic Press, Amsterdam, 2003.

[30] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), no. 116, 4, 592–618.

[31] S. G. Deng, *Eigenvalues of the $p(x)$-Laplacian Steklov problem*, J. Math. Anal. Appl. **339** (2008), 925–937.

[32] M. Hsini, N. Irzi, and K. Kefi, *Nonhomogeneous $p(x)$-Laplacian Steklov problem with weights*, Complex Var. Elliptic Equ. **65** (2020), no. 3, 440–454.

[33] C. Unal and I. Aydin, *Compact embeddings of weighted variable exponent Sobolev spaces and existence of solutions for weighted p (.)-Laplacian*, Complex Var. Elliptic Equ. **66** (2020), no. 10, 1755–1773.

[34] I. Aydin, *Weighted variable Sobolev spaces and capacity*, J. Funct. Spaces Appl. **2012** (2012), 132690, DOI: https://doi.org/10.1155/2012/132690.

[35] I. Aydin and C. Unal, *The Kolmogorov-Riesz theorem and some compactness criterions of bounded subsets in weighted variable exponent amalgam and Sobolev spaces*, Collect. Math. **71** (2020), 349–367.

[36] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer-Verlag, Berlin, 2011.

[37] X. L. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), 424–446.

[38] C. Unal and I. Aydin, *Compact embeddings on a subspace of weighted variable exponent Sobolev spaces*, Adv. Oper. Theory **4** (2019), no. 2, 388–405.

[39] M. Izuki, T. Nogayama, T. Noi, and Y. Sawano, *Wavelet characterization of local Muckenhoupt weighted Lebesgue spaces with variable exponent*, Nonlinear Anal. **198** (2020), 111930.

[40] Q. Liu, *Compact trace in weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega;\nu_0,\nu_1)$*, J. Math. Anal. Appl. **348** (2008), 760–774.

[41] A. Zang and Y. Fu, *Interpolation inequalities for derivatives in variable exponent Lebesgue-Sobolev spaces*, Nonlinear Anal. **69** (2008), 3629–3636.

[42] M. Hsini, N. Irzi, and K. Kefi, *Existence of solutions for a $p(x)$-biharmonic problem under Neumann boundary conditions*, Appl. Anal. **100** (2021), no. 10, 2188–2199.

[43] V. V. Zhikov and M. D. Surnachev, *On density of smooth functions in weighted Sobolev spaces with variable exponents*, St. Petersburg Math. J. **27** (2016), 415–436.
[44] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/B*, Springer-Verlag, New York, 1990.
[45] B. Ricceri, *Existence of three solutions for a class of elliptic eigenvalue problems*, Math. Comput. Modelling **32** (2000), 1485–1494.
[46] B. Ricceri, *A three critical points theorem revisited*, Nonlinear Anal. **70** (2009), 3084–3089.