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Research Article

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On the generalized Mellin integral operators

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Abstract: In this study, we give a modification of Mellin convolution-type operators. In this way, we obtain the rate of convergence with the modulus of the continuity of the mth-order Mellin derivative of function f, but without the derivative of the operator. Then, we express the Taylor formula including Mellin derivatives with integral remainder. Later, a Voronovskaya-type theorem is proved. In the last part, we state order of approximation of the modified operators, and the obtained results are restated for the Mellin-Gauss-Weierstrass operator.

Keywords: convolution operators, Mellin derivatives, logarithmic modulus of continuity, Mellin-Gauss-Weierstrass operator

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1 Introduction

The aim of this article is to define a generalization of Mellin convolution operators to obtain a higher order of approximation with respect to the usual one. Mellin convolution-type integral operators, acting on functions defined on the multiplicative group $\mathbb{R}^+$, have different structure compared with the classical convolution operator defined on a line group. The operators, more suitable for the use of a dilation operator and the Haar measure instead of usual translation operator and Lebesgue measure, ensure invariant under dilation. The singular integrals of Mellin convolution-type were first introduced by Kolbe and Nessel in [1]. In the original article [2], the authors proved convergence of operators in L_p space with the connection of Mellin analysis. In [3], by taking into account the notions of the logarithmic Taylor formula, Mellin derivatives, logarithmic uniform continuity, and logarithmic moment of kernel function, approximation results by Mellin convolution operators have been simplified. From the beginning of the 2000s to the present, Mellin convolution operators have been extensively studied, especially by Bardaro and Mantellini, and very significant contributions have been made to the literature. A special attention deserves their article [4], where the authors used a new and powerful approach that uses a Taylor formula in terms of Mellin derivatives. In [5], the behaviour of iterates of Mellin-Fejer-type operators with respect to pointwise and uniform convergence was examined by Bardaro and Mantellini, and also, they introduced a new method in the construction of linear combinations of Mellin-type operators using the iterated kernels to acquire better approximation order. In [6], a Voronovskaya formula of high order for linear combinations of the Mellin-Gauss-Weierstrass operators was given. It is very respectable in terms of improving the order of approximation.

The authors in the series articles [5,7–9] regarded Mellin convolution operators of type:

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$$T_w f(s) = \int_0^\infty K_w(t) f(ts) \frac{\mathrm{d}t}{t}, \quad s \in \mathbb{R}^+,$$

where f belongs to the domain of the operator T_w and $K_w : (0, \infty) \to \mathbb{R}$ is a set of the kernels.

In this study, we develop a different method in the construction of linear combinations, using Mellin derivatives of approximated function of the form:

$$T_{w,m} f(s) = \int_0^\infty \sum_{j=0}^{m} \frac{\Theta^j f(t)}{j!} \ln^j \frac{s}{t} K_w(ts^{-1}) \frac{\mathrm{d}t}{t}, \quad s > 0,$$

where $\Theta^j f$ is the jth-order Mellin derivative of the function f.

We point out that in some cases, this method gives a better order of approximation with respect to the classical operator T_w, especially for what concerns the Voronovskaya formula and quantitative point of view. As we show an application, this happens, for example, for the moment operators and the Mellin-Gauss-Weierstrass operators.

Additionally, in [10–14], some investigations have been performed recently on sampling operators in some different spaces, such as Mellin-Lebesgue, Orlicz, etc.

The rest of the study is organized as follows. In the next section, primary notations and preliminaries related to the subject are reminded. Also, we give the definition of modified singular integral of Mellin convolution type. In this way, we express the approximation rate with the modulus of the continuity of the mth-order Mellin derivative of function f, but without the derivative of the operator. Then, we express the Taylor formula including Mellin derivatives with integral remainder. Later, a Voronovskaya-type theorem is proved. In the last part, we state the order of approximation of the modified operators, and the obtained results are restated for the Mellin-Gauss-Weierstrass operator.

2 Basic notations and preliminary results

Let us remark some notations related to Mellin operators given in [2], or Mellin derivative Θf of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by:

$$\Theta f(s) = s f'(s), \quad s > 0,$$

provided the usual derivative $f'(s)$ exists and Mellin differential operator of order $i \in \mathbb{N}$ is defined inductively by putting

$$\Theta^1 = \Theta, \quad \Theta^i = \Theta \circ \Theta^{i-1}, \quad \text{for } i \geq 2.$$

If $x > 0$, the $\ln x = \log_e x$, where e is the Euler number.

Let $\mathbb{R}^+$ be the set of all positive real numbers endowed with the measure $\mu(E) = \int_E (\mathrm{d}t/t)$, where E is any (Lebesgue) measurable set. We will denote by $L_p(\mu, \mathbb{R}^+) = L_p(\mu), 1 \leq p < +\infty$, the Lebesgue spaces with respect to the measure μ and by $L_\infty(\mu)$ the space of all the essentially bounded functions. We will denote by $\|f\|_p$ and $\|f\|_\infty$ the corresponding norms.

For every $m \in \mathbb{N}_0$, we denote by $C^m = C^m(\mathbb{R}^+)$ the space of all functions $f : \mathbb{R}^+ \to \mathbb{R}$ with continuous mth-order derivative in $\mathbb{R}^+$. Moreover, we will say that $f \in C^m$ locally at the point $s \in \mathbb{R}^+$ if there is a neighbourhood U_s of the point s such that f is $(m-1)$-times continuously differentiable in U_s and the derivative of order m exists at the point s.

In the following, we will say that a function f is log-uniformly continuous in $\mathbb{R}^+$ if for every $\varepsilon > 0$, there exists $\eta > 0$ such that $|f(s_1) - f(s_2)| < \varepsilon$ whenever $|\ln s_1 - \ln s_2| < \eta$ for every $s_1, s_2 \in \mathbb{R}^+$.

We denote by BC^0 the subspace of C^0 containing the log-uniformly continuous and bounded functions in $\mathbb{R}^+$ and by BC^m the space of all the functions with mth-order Mellin derivative in BC^0.

For the set of kernels $K : \mathbb{R}^+ \to \mathbb{R}_0^+$, which satisfies the conditions:

(1) $K \in L_1(\mu)$ and $\int_0^\infty K(t)\frac{dt}{t} = 1$.
(2) $K_w(t) = wK(t^w)$.

For $j \in \mathbb{N}$, we define the logarithmic moments and the absolute logarithmic moments of order j of family (K_w) by, respectively:

$$m_j(K_w) := \int_0^\infty K_w(t) \ln^j t \frac{dt}{t} \quad \text{and} \quad M_j(K_w) := \int_0^\infty K_w(t) |\ln t|^j \frac{dt}{t}.$$

For fixed $m \in \mathbb{N}_0$ and $1 \le p \le +\infty$, we denote $L_{p,m}(\mu, \mathbb{R}^+) = L_{p,m}(\mu)$ the set of all $f \in L_{p,m}(\mu)$ whose Mellin derivatives $\Theta f, \ldots, \Theta^m f$ also belong to $L_p(\mu)$. If $p = \infty$, we define the norm as usually $\|f\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^+} |f(x)|$. Moreover, for $f \in L_{p,m}(\mu)$, there exist norms $\|\Theta^k f\|_p$, $0 \le k \le m$.

Definition 1. For $f \in L_{p,m}(\mu)$, we define that the modified singular integral of Mellin convolution-type is defined by:

$$T_{w,m} f(s) = \int_0^\infty \sum_{j=0}^m \frac{\Theta^j f(t)}{j!} \ln^j \frac{s}{t} K_w(ts^{-1}) \frac{dt}{t}, \quad s > 0. \tag{2.1}$$

In particular, $T_{w,0} f(s) = T_w f(s)$ for $f \in L_p(\mu)$. It is noted that (2.1) is not always positive.

Lemma 1. *Let $m \in \mathbb{N}_0$, $1 \le p \le \infty$, and $M_j(K_w) < \infty$ for $0 \le j \le m$. For $f \in L_{p,m}(\mu)$, we have*

$$\|T_{w,m} f\|_\infty \le \sum_{j=0}^m \frac{\|\Theta^j f\|_\infty}{j!} M_j(K_w)$$

and

$$\|T_{w,m} f\|_p \le \sum_{j=0}^m \frac{\|\Theta^j f\|_p}{j!} M_j(K_w).$$

Proof. First, if $p = \infty$, then using (2.1), we can write

$$\begin{aligned} T_{w,m} f(s) &= \int_0^\infty \sum_{j=0}^m \frac{\Theta^j f(t)}{j!} (-1)^j \ln^j \frac{t}{s} K_w(ts^{-1}) \frac{dt}{t} \\ &= \int_0^\infty \sum_{j=0}^m \frac{\Theta^j f(ts)}{j!} (-1)^j \ln^j t K_w(t) \frac{dt}{t}. \end{aligned}$$

From this, we obtain

$$|T_{w,m} f(s)| \le \int_0^\infty \sum_{j=0}^m \frac{|\Theta^j f(ts)|}{j!} |\ln^j t| K_w(t) \frac{dt}{t} \le \sum_{j=0}^m \frac{1}{j!} \|\Theta^j f\|_\infty M_j(K_w),$$

and, later for $1 \le p < \infty$, by the generalized Minkowsky inequality, we have

$$\begin{aligned} \|T_{w,m} f\|_p &= \left(\int_0^\infty \left| \int_0^\infty \sum_{j=0}^m \frac{\Theta^j f(ts)}{j!} (-1)^j \ln^j t K_w(t) \frac{dt}{t} \right|^p \frac{ds}{s} \right)^{1/p} \\ &\le \sum_{j=0}^m \frac{1}{j!} \int_0^\infty |\ln t|^j K_w(t) \left(\int_0^\infty |\Theta^j f(ts)|^p \frac{ds}{s} \right)^{1/p} \frac{dt}{t} \\ &= \sum_{j=0}^m \frac{1}{j!} \|\Theta^j f\|_p \int_0^\infty |\ln t|^j K_w(t) \frac{dt}{t} \\ &= \sum_{j=0}^m \frac{1}{j!} \|\Theta^j f\|_p M_j(K_w). \end{aligned}$$

The desired results are obtained. □

A special attention deserves this article from 2011, [15] where the authors used a new and powerful approach that uses a Taylor formula in terms of Mellin derivatives. Inspired from here, we can express the following lemma.

Lemma 2. *Suppose that f is locally of class $C^{(m+1)}$ at a point $s_0 \in \mathbb{R}^+$. We have the following equality:*

$$\begin{aligned} f(s) = f(s_0) + \Theta f(s_0)\ln\frac{s}{s_0} + \frac{1}{2!}\Theta^2 f(s_0)\ln^2\frac{s_0}{s} + \cdots + \\ + \frac{1}{m!}\Theta^m f(s_0)\ln^m\frac{s_0}{s} + \frac{1}{m!}\int_{s_0}^{s}\ln^m\frac{s}{t}\Theta^{m+1}f(t)\frac{dt}{t}. \end{aligned} \tag{2.2}$$

Proof. We use the induction method. For $m = 0$, using the fundamental theorem of calculus, we have

$$f(s) = f(s_0) + \int_{s_0}^{s}\Theta f(t)\frac{dt}{t} = f(s_0) + \int_{s_0}^{s} f'(t)dt.$$

Applying integration by parts with

$$\begin{aligned} u &= \Theta f(t) \quad \text{and} \quad dv = \frac{dt}{t}, \\ du &= (f'(t) + tf''(t))\frac{dt}{t} v = \ln t - \ln s, \end{aligned}$$

we obtain

$$\begin{aligned} f(s) &= f(s_0) + \int_{s_0}^{s}\Theta f(t)\frac{dt}{t} \\ &= f(s_0) + \Theta f(t)\ln\frac{t}{s}\Big|_{s_0}^{s} - \int_{s_0}^{s}\ln\frac{t}{s}\Theta^2 f(t)\frac{dt}{t} \\ &= f(s_0) + \Theta f(s_0)\ln\frac{s}{s_0} + \int_{s_0}^{s}\ln\frac{s}{t}\Theta^2 f(t)\frac{dt}{t}. \end{aligned}$$

Now, let us assume that equation (2.2) is true for $m = k$ and show it to be true for $m = k + 1$. So, we have to show that

$$f(s) = f(s_0) + \Theta f(s_0)\ln\frac{s}{s_0} + \frac{1}{2!}\Theta^2 f(s_0)\ln^2\frac{s_0}{s} + \cdots + \frac{1}{(k+1)!}\Theta^{k+1}f(s_0)\ln^{k+1}\frac{s_0}{s} + + \frac{1}{(k+1)!}\int_{s_0}^{s}\ln^{k+1}\frac{s}{t}\Theta^{k+2}f(t)\frac{dt}{t}.$$

In case $m = k$, for the integral

$$\frac{1}{k!}\int_{s_0}^{s}\ln^k\frac{s}{t}\Theta^{k+1}f(t)\frac{dt}{t},$$

applying again integration by parts with

$$\begin{aligned} u &= \Theta^{k+1}f(t) \quad \text{and} \quad dv = \frac{1}{k!}\ln^k\frac{s}{t}\frac{dt}{t} \\ du &= t\frac{d}{dt}\Theta^{k+1}f(t)\frac{dt}{t} \\ &= \Theta^{k+2}f(t)\frac{dt}{t} \end{aligned}$$

and

$$v=\frac{(-1)^{k+1}}{(k+1)!}\ln^{k+1}\frac{s}{t},$$

we obtain

$$\frac{1}{k!}\int_{s_0}^{s}\ln^k\frac{s}{t}\Theta^{k+1}f(t)\frac{dt}{t}=(-1)^{k+1}\frac{\Theta^{k+1}f(t)}{(k+1)!}\ln^{k+1}\frac{s}{t}\bigg|_{s_0}^{s}+\frac{1}{(k+1)!}\int_{s_0}^{s}\ln^{k+1}\frac{s}{t}\Theta^{k+2}f(t)\frac{dt}{t}.$$

Thus, we obtain the desired result. □

3 Order of approximation

In this section, we examine the order of approximation of the family of operators $(T_{w,m})$.

Definition 2. We give the modulus of continuity of $f\in L_p(\mu)$, $0\le p<\infty$, with:

$$\omega_p(f;\,\delta)=\sup_{|\ln t|\le\delta}\left(\int_0^\infty|f(ts)-f(s)|^p\frac{ds}{s}\right)^{1/p}$$

and $f\in BC^0$ with

$$\omega_\infty(f;\,\delta)=\sup_{|\ln t|\le\delta}|f(ts)-f(s)|.$$

They provide the properties of the classical modulus of continuity. These properties can be shown analogously to [16, Theorem 2.4] and [3], respectively.

Theorem 1. *Let $f\in L_{p,m}(\mu)$. Let also*

$$M_{m+1}(K_w)=\int_0^\infty K_w(t)|\ln t|^{m+1}\,\frac{dt}{t}<\infty$$

and

$$M_m(K_w)=\int_0^\infty K_w(t)|\ln t|^{m}\,\frac{dt}{t}>0.$$

Then, we have

$$\|f-T_{w,m}f\|_p\le\frac{2}{m!}M_m(K_w)\omega_p\left(\Theta^m f;\,\frac{M_{m+1}(K_w)}{M_m(K_w)}\right).$$

Also, if $p=\infty$, for $f\in BC^m\cap L_\infty(\mu)$, we have

$$\|f-T_{w,m}f\|_\infty\le\frac{2}{m!}M_m(K_w)\omega_\infty\left(\Theta^m f;\,\frac{M_{m+1}(K_w)}{M_m(K_w)}\right).$$

Proof. We use the following modified Taylor formula:

$$f(s)=\sum_{j=0}^{m}\frac{\Theta^j f(t)}{j!}\ln^j\frac{s}{t}+\frac{1}{(m-1)!}\int_t^s\ln^{m-1}\frac{s}{u}[\Theta^m f(u)-\Theta^m f(t)]\frac{du}{u}. \tag{3.1}$$

If we multiply both sides of equation (3.1) by $K_w(ts^{-1})$ and take the integral, we obtain

$$f(s) = T_{w,m}f(s) + \frac{1}{(m-1)!}\int_0^\infty\left[\int_t^s \ln^{m-1}\frac{s}{u}[\Theta^m f(u) - \Theta^m f(t)]\frac{du}{u}\right]K_w(ts^{-1})\frac{dt}{t}. \tag{3.2}$$

First, let us show the second inequality. If $p = \infty$, then we have

$$\begin{aligned}|\Theta^m f(u) - \Theta^m f(t)| &\le \omega_\infty(\Theta^m f;\ |\ln u - \ln t|)\\ &\le \left(1 + \frac{1}{\delta}|\ln u - \ln t|\right)\omega_\infty(\Theta^m f;\ \delta).\end{aligned}$$

Using (3.2), we obtain

$$\begin{aligned}|T_{w,m}f(s) - f(s)| &\le \frac{\omega_\infty(\Theta^m f;\ \delta)}{(m-1)!}\int_0^\infty\left[\int_t^s \ln^{m-1}\frac{s}{u}\left(1 + \frac{1}{\delta}|\ln u - \ln t|\right)\frac{du}{u}\right]K_w(ts^{-1})\frac{dt}{t}\\
&= \frac{\omega_\infty(\Theta^m f;\ \delta)}{(m-1)!}\int_0^\infty\left[\int_t^1 \ln^{m-1}\frac{1}{u}\left(1 + \frac{1}{\delta}|\ln us - \ln ts|\right)\frac{du}{u}\right]K_w(t)\frac{dt}{t}\\
&\le \frac{\omega_\infty(\Theta^m f;\ \delta)}{m!}\int_0^\infty K_w(t)|\ln t|^m\left(1 + |\ln t|\frac{1}{\delta(m+1)}\right)\frac{dt}{t}\\
&= \frac{\omega_\infty(\Theta^m f;\ \delta)}{m!}M_m(K_w)\left(1 + \frac{1}{\delta(m+1)}\frac{M_{m+1}(K_w)}{M_m(K_w)}\right)\\
&\le \frac{\omega_\infty(\Theta^m f;\ \delta)}{m!}M_m(K_w)\left(1 + \frac{1}{\delta}\frac{M_{m+1}(K_w)}{M_m(K_w)}\right).\end{aligned}$$

Choosing $\delta = \frac{M_{m+1}(K_w)}{M_m(K_w)}$, we obtain the desired result.

Now, we can prove the first one in a similar way. For $f \in L_{p,m}(\mu)$, we have

$$\begin{aligned}\|f - T_{w,m}f\|_p &= \left(\int_0^\infty\left|\int_0^\infty\left[\frac{1}{(m-1)!}\int_t^s \ln^{m-1}\frac{s}{u}[\Theta^m f(u) - \Theta^m f(t)]\frac{du}{u}\right]K_w(ts^{-1})\frac{dt}{t}\right|^p\frac{ds}{s}\right)^{1/p}\\
&\le \left(\int_0^\infty K_w(t)\int_0^\infty\left|\left[\frac{1}{(m-1)!}\int_{ts}^s \ln^{m-1}\frac{s}{u}[\Theta^m f(u) - \Theta^m f(ts)]\frac{du}{u}\right]\right|^p\frac{ds}{s}\right)^{1/p}\frac{dt}{t}\\
&\le \int_0^\infty K_w(t)\left(\frac{1}{(m-1)!}\left|\int_t^1\left|\ln\frac{1}{u}\right|^{m-1}\left(\int_0^\infty|\Theta^m f(us) - \Theta^m f(ts)|^p\frac{ds}{s}\right)^{1/p}\frac{du}{u}\right|\right)\frac{dt}{t}\\
&\le \int_0^\infty K_w(t)\left(\frac{1}{(m-1)!}\left|\int_t^1\left|\ln\frac{1}{u}\right|^{m-1}\omega_p\left(\Theta^m f;\left|\ln\frac{u}{t}\right|\right)\frac{du}{u}\right|\right)\frac{dt}{t}\\
&\le \omega_p(\Theta^m f;\ \delta)\int_0^\infty K_w(t)\left(\frac{1}{(m-1)!}\left|\int_t^1\left|\ln\frac{1}{u},\right|^{m-1}\left(1 + \frac{1}{\delta}\left|\ln\frac{u}{t}\right|\right)\frac{du}{u}\right|\right)\frac{dt}{t}.\end{aligned}$$

By performing the necessary calculations, we obtain

$$\|f - T_{w,m}f\|_p \le \frac{\omega_p(\Theta^m f;\,\delta)}{m!}\int_0^\infty |K_w(t)|\,|\ln t|^m\left(1 + |\ln t|\frac{1}{\delta(m+1)}\right)\frac{dt}{t}$$
$$= \frac{\omega_p(\Theta^m f;\,\delta)}{m!}\left(M_m(K_w) + \frac{1}{\delta(m+1)}M_{m+1}(K_w)\right)$$
$$= \frac{\omega_p(\Theta^m f;\,\delta)}{m!}M_m(K_w)\left(1 + \frac{1}{\delta(m+1)}\frac{M_{m+1}(K_w)}{M_m(K_w)}\right)$$
$$\le \frac{\omega_p(\Theta^m f;\,\delta)}{m!}M_m(K_w)\left(1 + \frac{1}{\delta}\frac{M_{m+1}(K_w)}{M_m(K_w)}\right).$$

Choosing $\delta = \frac{M_{m+1}(K_w)}{M_m(K_w)}$, we obtain the desired result. □

4 Voronovskaya-type theorem

This section devoted to Voronovskaya-type theorem by means of Mellin derivatives. The theorem determines the precise order of pointwise approximation.

Theorem 2. *Let $s \in \mathbb{R}^+$, $M_{m+2}(K_w) < \infty$, $f \in L_{\infty,m}(\mu)$, and $\Theta^j f \in L_{\infty,m+2-j}(\mu)$, $0 \le j \le m$ for fixed $m \in \mathbb{N}_0$.*

(1) (a) If $m_{m+1}(K_w) = 0$, we have

$$T_{w,m}f(s) - f(s) = (-1)^m\frac{(m+1)\Theta^{m+2}f(s)}{(m+2)!}m_{m+2}(K_w) + o\left(w^{-(m+2)}\right).$$

(b) If $m_{m+1}(K_w) \ne 0$, we have

$$T_{w,m}f(s) - f(s) = (-1)^m\frac{\Theta^{m+1}f(s)}{(m+1)!}m_{m+1}(K_w) + (-1)^m\frac{(m+1)\Theta^{m+2}f(s)}{(m+2)!}m_{m+2}(K_w) + o\left(w^{-(m+1)}\right).$$

Proof. By the Taylor formula, we can write

$$\Theta^j f(t) = \sum_{i=0}^{m+2-j}\frac{\Theta^{i+j}f(s)}{i!}\ln^i\frac{t}{s} + \varphi_j(t,s)\ln^{m+2-j}\frac{t}{s},$$

where $\varphi_j(t) \equiv \varphi_j(t,s)$, $\varphi_j(t)\ln^{m+2-j}t \in L_{\infty,m+2-j}(\mu)$ for $0 \le j \le m$ and $\lim_{t\to s}\varphi_j(t) = \varphi_j(s)$. Using (2.1), we obtain

$$\begin{aligned} T_{w,m}f(s) &= \int_0^\infty K_w(ts^{-1})\sum_{j=0}^m\frac{(-1)^j}{j!}\sum_{i=0}^{m+2-j}\frac{\Theta^{i+j}f(s)}{i!}\ln^{i+j}\frac{t}{s}\frac{dt}{t} \\ &\quad + \int_0^\infty K_w(ts^{-1})\ln^{m+2}\frac{t}{s}\sum_{j=0}^m\frac{(-1)^j}{j!}\varphi_j(t)\frac{dt}{t} \\ &= A_{w,m}f(s) + B_{w,m}f(s). \end{aligned} \tag{4.1}$$

First, we consider $A_{w,m}f$. We obtain

$$A_{w,m}f(s) = \int_0^\infty K_w(ts^{-1})\left(\sum_{j=0}^m\frac{(-1)^j}{j!}\sum_{l=j}^m\frac{\Theta^l f(s)}{(l-j)!}\ln^l\frac{t}{s} + \frac{\Theta^{m+1}f(s)}{(m+1)!}\ln^{m+1}\frac{t}{s}\sum_{j=0}^m\binom{m+1}{j}(-1)^j\right.$$
$$\left. + \frac{\Theta^{m+2}f(s)}{(m+2)!}\ln^{m+2}\frac{t}{s}\sum_{j=0}^m\binom{m+2}{j}(-1)^j\right)\frac{dt}{t}.$$

Using the equalities

$$\sum_{j=0}^{m}\binom{m+1}{j}(-1)^j=(-1)^m,\quad \sum_{j=0}^{m}\binom{m+2}{j}(-1)^j=(m+1)(-1)^m,$$

$$\sum_{j=0}^{m}\frac{(-1)^j}{j!}\quad\sum_{l=j}^{m}\frac{\Theta^l f(s)}{(l-j)!}\ln^l\frac{t}{s}=\sum_{l=0}^{m}\frac{\Theta^l f(s)}{l!}\ln^l\frac{t}{s}\sum_{j=0}^{l}\binom{l}{j}(-1)^j,$$

and

$$\sum_{j=0}^{l}\binom{l}{j}(-1)^j=\begin{cases}1, & l=0\\ 0, & l\neq 0\end{cases},$$

we can write

$$A_{w,m}f(s)=f(s)\int_0^\infty K_w(ts^{-1})\frac{\mathrm{d}t}{t}+(-1)^m\frac{\Theta^{m+1}f(s)}{(m+1)!}\int_0^\infty K_w\left(ts^{-1}\right)\ln^{m+1}\frac{t}{s}\frac{\mathrm{d}t}{t}$$
$$+(-1)^m\frac{(m+1)\Theta^{m+2}f(s)}{(m+2)!}\int_0^\infty K_w\left(ts^{-1}\right)\ln^{m+2}\frac{t}{s}\frac{\mathrm{d}t}{t}$$

and

$$A_{w,m}f(s)=f(s)+(-1)^m\frac{\Theta^{m+1}f(s)}{(m+1)!}m_{m+1}(K_w)+(-1)^m\frac{(m+1)\Theta^{m+2}f(s)}{(m+2)!}m_{m+2}(K_w).$$

Now, we estimate the last term $B_{w,m}f$. Let $\sum_{j=0}^{m}\frac{(-1)^j}{j!}\varphi_j(t)=\phi_j(t)$. Then,

$$|B_{\omega,m}f(s)|\leq\left(\int_0^{1/\delta}+\int_{1/\delta}^{\delta}+\int_{\delta}^{\infty}\right)K_w(t)|\phi_j(ts)||\ln^{m+2}t|\frac{\mathrm{d}t}{t}=I_1+I_2+I_3.$$

We take into account I_2. Let $\varepsilon>0$ be fixed and choose $\delta>1$ such that $|\phi_j(t)|<\varepsilon$ for $t\in U_\delta$. Hence,

$$|I_2|\leq\varepsilon\int_{1/\delta}^{\delta}K_w(t)|\ln^{m+2}t|\frac{\mathrm{d}t}{t}\leq\varepsilon M_{m+2}(K_w).$$

As regards I_1 (and analogously for I_3), we have

$$|I_1|\leq\frac{\|\phi_j\|_\infty}{w^{m+2}}\int_0^{1/\delta^w}K(t)|\ln^{m+2}t|\frac{\mathrm{d}t}{t}=o\left(w^{-(m+1)}\right),$$

so $\lim_{w\to+\infty}w^{m+1}|I_1|=0$. The proof follows by limsup and liminf arguments. □

5 Application to the Mellin-Gauss-Weierstrass operator

Let us consider the function

$$K(t)=\frac{1}{\sqrt{4\pi}}e^{-\left(\frac{1}{2}\log t\right)^2},\quad t\in\mathbb{R}^+. \tag{5.1}$$

This function generates the modified Mellin-Gauss-Weierstrass operator:

$$\mathcal{T}_{w,m}f(s)=\frac{w}{\sqrt{4\pi}}\int_0^\infty\sum_{j=0}^{m}\frac{\Theta^j f(t)}{j!}\ln^j\frac{s}{t}e^{-\left(\frac{w}{2}\log\frac{t}{s}\right)^2}\frac{\mathrm{d}t}{t}.$$

It is easy to see that Property (1) holds and $m_1(K) = 0$. We know that if m is odd, $m_m(K) = 0$, and if m is even, $m_m(K) = 2^{m/2}(m-1)!!$, where in this case $n!! = 3.5 \ldots n$ with n odd.

From Theorem 1, we obtain

Corollary 1. *Let $f \in L_{p,m}(\mu)$ with $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$. We have,*

$$\|f - \mathcal{T}_{w,m}f\|_p \leq \frac{2}{m!}\frac{2^m}{\sqrt{\pi}\,w^m}\left(\Gamma\left(\frac{m+1}{2}\right) + 2\Gamma\left(\frac{m+2}{2}\right)\right)\omega_p\left(\Theta^m f;\ \frac{1}{w}\right).$$

Also, if $p = \infty$, for $f \in BC^m \cap L_\infty(\mu)$, we have

$$\|f - T_{w,m}f\|_\infty \leq \frac{2}{m!}\frac{2^m}{\sqrt{\pi}\,w^m}\left(\Gamma\left(\frac{m+1}{2}\right) + 2\Gamma\left(\frac{m+2}{2}\right)\right)\omega_\infty\left(\Theta^m f;\ \frac{1}{w}\right).$$

Proof. We have

$$\int_0^\infty K_w(t)|\ln t|^m\ \frac{dt}{t} = \frac{2^m}{\sqrt{\pi}\,w^m}\Gamma\left(\frac{m+1}{2}\right)$$

and

$$\lim_{w\to\infty}\frac{\int_0^\infty K_w(t)|\ln t|^{m+1}\ \frac{dt}{t}}{\int_0^\infty K_w(t)|\ln t|^m\ \frac{dt}{t}} = 0.$$

Therefore, via Theorem 1, we obtain the desired result. □

Corollary 2. *Let $f \in L_{p,m}(\mu)$ with $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$.* If *$\Theta^m f \in Lip_\alpha(L_p)$ for fixed $0 < \alpha \leq 1$, i.e. $\omega_p(\Theta^m f;\ \delta) = O(\delta^\alpha)$. Then, we have for $w > 0$*

$$\|f - \mathcal{T}_{w,m}f\|_p = O\left(w^{-(m+\alpha)}\right).$$

Remark 1. Corollaries 1 and 2 show that the order of approximation by the modified Mellin-Gauss-Weierstrass operator $\mathcal{T}_{w,m}$ is better than the approximation by classical operator T_w.

From Theorem 2, we obtain

Corollary 3. *Let $s \in \mathbb{R}^+$, $f \in L_{\infty,m+2}(\mu)$, and $\Theta^j f \in L_{\infty,m+2-j}(\mu)$, $0 \leq j \leq m$ for fixed $m \in \mathbb{N}_0$. If m is even, then we have*

$$\lim_{w\to\infty} w^{m+2}[\mathcal{T}_{w,m}f(s) - f(s)] = \frac{(m+1)\Theta^{m+2}f(s)}{(m+2)!}m_{m+2}(K).$$

Corollary 4. *Let $s \in \mathbb{R}^+$, $f, \Theta^2 f \in L_{\infty,2}(\mu)$. Then, we have*

$$\lim_{w\to\infty} w^2[\mathcal{T}_w f(s) - f(s)] = \Theta^2 f(s).$$

Remark 2. Corollaries 3 and 4 show that the modified Mellin-Gauss-Weierstrass operators $\mathcal{T}_{w,m}$ provide better order of pointwise approximation.

6 Conclusion

In this article, we introduced a modification of Mellin convolution-type operators. In this way, we obtained the rate of convergence with the modulus of the continuity of the mth-order Mellin derivative of function f, but without the derivative of the operator. Then, we expressed the Taylor formula including Mellin derivatives with integral remainder. Later, a Voronovskaya-type theorem was proved. Finally, we stated order of approximation of the modified operators, and the obtained results were restated for the Mellin-Gauss-Weierstrass operator.

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References

[1] W. Kolbe and R. J. Nessel, *Saturation theory in connection with Mellin transform methods*, SIAM J. Math. Anal. **3** (1972), 246–262, DOI: https://doi.org/10.1137/0503024.

[2] P. L. Butzer and S. Jansche, *A direct approach to the Mellin transform*, J. Fourier Anal. Appl. **3** (1997), 325–375, DOI: http://eudml.org/doc/59511.

[3] R. Mamedov, *The Mellin Transform and Approximation Theory*, Elm, Baku, 1991.

[4] C. Bardaro and I. Mantellini, *Voronovskaya-type estimates for Mellin convolution operators*, Results Math. **1** (2007), 1–16, DOI: https://doi.org/10.1007/s00025-006-0231-3.

[5] C. Bardaro and I. Mantellini, *On Voronovskaja formula for linear combinations of Mellin-Gauss-Weierstrass operators*, Appl. Math. Comput. **218** (2012), 10171–10179, DOI: https://doi.org/10.1016/j.amc.2012.03.097.

[6] C. Bardaro and I. Mantellini, *On the iterates of Mellin-Fejer convolution operators*, Acta Appl. Math. **121** (2012), 213–229, DOI: https://doi.org/10.1007/s10440-012-9704-4.

[7] C. Bardaro and I. Mantellini, *A quantitative Voronovskaya formula for Mellin convolution operators*, Mediterr. J. Math. **7** (2010), 483–501, DOI: https://doi.org/10.1007/s00009-010-0062-z.

[8] C. Bardaro and I. Mantellini, *Approximation properties for linear combinations of moment type operators*, Comput. Math. Appl. **62** (2011), 2304–2313, DOI: https://doi.org/10.1016/j.camwa.2011.07.017.

[9] C. Bardaro and I. Mantellini, *On Mellin convolution operators: a direct approach to the asymptotic formulae*, Integral Transforms Spec. Funct. **25** (2014), 182–195, DOI: https://doi.org/10.1080/10652469.2013.838755.

[10] T. Acar and S. Kursun, *Pointwise convergence of generalized Kantorovich exponential sampling series*, Dolomites Res. Notes Approx. **16** (2023), 1–10, DOI: https://doi.org/10.14658/pupj-drna-2023-2-1.

[11] L. Angeloni, N. Cetin, D. Costarelli, A. R. Sambucini, and G. Vinti, *Multivariate sampling Kantorovich operators: Quantitative estimates in Orlicz spaces*, Constr. Math. Anal. **4** (2021), 229–241, DOI: https://doi.org/10.33205/cma.876890.

[12] A. Aral, T. Acar, and S. Kursun, *Generalized Kantorovich forms of exponential sampling series*, Anal. Math. Phys. **12** (2022), 1–19, DOI: https://doi.org/10.1007/s13324-022-00667-9.

[13] C. Bardaro, G. Bevignani, I. Mantellini, and M. Seracini, *Bivariate generalized exponential sampling series and applications to seismic waves*, Constr. Math. Anal. **2** (2019), 153–167, DOI: https://doi.org/10.33205/cma.594066.

[14] D. Costarelli and G. Vinti, *A quantitative estimate for the sampling Kantorovich series in terms of the modulus of continuity in Orlicz spaces*, Constr. Math. Anal. **2** (2019), 8–14, DOI: https://doi.org/10.33205/cma.484500.

[15] C. Bardaro and I. Mantellini, *A note on the Voronovskaja theorem for Mellin-Fejer convolution operators*, Appl. Math. Lett. **24** (2011), 2064–2067, DOI: https://doi.org/10.1016/j.aml.2011.05.043.

[16] C. Bardaro, J. Musielak, and G. Vinti, *Nonlinear integral operators and applications*, in: de Gruyter Series in Nonlinear Analysis and Applications, vol. 9, Walter de Gruyter & Co., Berlin, 2003.