

Research Article

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Some fixed point results on ultrametric spaces endowed with a graph

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Abstract: The present article deals with new fixed point theorems by means of G -strongly contractive maps. The research findings are demonstrated in a spherically complete ultrametric space with a graph for single-valued mappings. The special cases of the results that extend the current ones are offered, along with some examples that illustrate our results. Besides, an application utilized in dynamic programming that endorses the acquired observations is also provided.

Keywords: dynamic programming, fixed point, F -contraction, functional equation, ultrametric space

MSC 2020: 54H25, 47H10

1 Introduction

Fixed point theory is an active and important branch of mathematics, and this theory is a powerful tool for investigating solutions to mathematical problems with various types of applications. Because of its simple application to several disciplines of mathematics, Banach [1] proposed the Banach fixed point theorem a century ago, which is a crucial source for the development of metric fixed point theory. Accordingly, there has been and continues to be a great deal of interest and demand for this hypothesis. The Banach fixed point theorem not only guarantees the existence and uniqueness of a fixed point of contraction self-mapping but also gives an effective approach to find the fixed point and can be expressed as “the contraction self-mapping T on a complete metric space (X, d) , i.e., for all $x, y \in X$, the inequality

$$d(Tx, Ty) \leq \mu d(x, y), \quad \text{where } \mu \in (0, 1) \quad (1)$$

is satisfied, then T owns a unique fixed point, and for every $x_0 \in X$, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to this fixed point.”

When investigating metric fixed point theory, generalizing the contraction mapping is a particularly interesting area of research. Many generalizations in this sense are available in the literature, and the main ones are [2–4]. Aside from such mappings, fixed point theorems employing contraction mappings established with auxiliary functions have been among the theory’s building blocks, and new functions are continuously being defined nowadays. For more details, see [5–10].

The study of the fixed point occupies a prominent role in many aspects of a metric space equipped with a graph structure. Echenique [11] implemented graphs to demonstrate Tarski’s fixed point theorem, integrating fixed point theory with graph theory. Espinola and Kirk [12] then utilized the fixed point findings in graph theory. Two key findings for fixed point theory with a graph have recently been presented. The first result was

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given by Jachymski [13] for single-valued mappings, and subsequently, Beg et al. [14] extended Jachymski's result for set-valued mappings. Following that, Sultana and Vetrivel established a fixed point theorem for Mizoguchi-Takahashi contraction in [15], while Sistani and Kazemipour [16] proved several theorems for (α, φ) -contractions in these directions. One can also see [17,18,38,39].

Specifically, we concentrate on one of the generalized metric spaces, the ultrametric space theory, an exciting field of research that has emerged in mathematics in the last 35 years. The notion of ultrametric space arose in many different works [19–24]. On the other hand, Gajić [25], considering the spherically completeness of a given ultrametric space, obtained a fixed point theorem with uniqueness for the mappings where the distance between Tx and Ty satisfies the following inequality:

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty)\} \quad \text{for all } x, y \in X, x \neq y.$$

We must mention that Gajić [26] extended this result for multivalued maps.

Another important study in fixed point theory is determining coincidence point results for single-valued and set-valued mappings. The analog of such results in ultrametric spaces was obtained by Rao et al. [27] for coincidence point theorems from single-valued maps to set-valued contractive maps. In addition, Zhang and Song [28] put forward some results for weak contraction in ultrametric space.

This study seeks to provide fixed point theorems for single-valued mappings on a spherical complete ultrametric space equipped with a graph using the principles outlined above. An application in dynamic programming is provided to convey the validity of the results. The outcomes of this study generalize, extend, improve, and unify several known results in ultrametric space settings.

2 Preliminaries

We present some fundamental definitions and outcomes that will be utilized throughout the study.

2.1 Some aspects of ultrametric spaces

This subsection explains fundamental concepts and terminologies related to ultrametric spaces.

Definition 2.1. [25] Let (X, d) be a metric space. If the metric d fulfils the ensuing inequality such that

$$d(x, y) \leq \max\{d(x, \omega), d(\omega, y)\}$$

for all $x, y, \omega \in X$, then d is called an ultrametric in X , and the pair (X, d) is an ultrametric space.

Example 2.2. [25] Suppose that d is a discrete metric on $X(X \neq \emptyset)$. Then, d is ultrametric on X .

Example 2.3. [25] Consider $[x]$ is the entire part of x for $x \in \mathbb{R}$. For any $e \in \mathbb{R} \setminus \mathbb{Q}$ and for all $x, y \in \mathbb{Q}$,

$$d(x, y) = \inf\{2^{-n} : n \in \mathbb{Z}, [2^n(x - e)] = [2^n(y - e)]\}$$

is ultrametric on \mathbb{Q} .

Example 2.4. [29] Consider that the family of all sequences of non-negative integers is represented by X . For $x = \{x_n\}_{n \in \mathbb{N}}, y = \{y_n\}_{n \in \mathbb{N}} \in X$, set

$$u(x, y) = \inf\{n \in \mathbb{N} : x_n \neq y_n\}$$

and

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ \frac{1}{u(x, y)}, & \text{if } x \neq y. \end{cases}$$

Then, the pair (X, d) is a complete ultrametric space.

Remark 1. [25] Every ultrametric space is a metric space; however, the converse need not be true. For instance, (\mathbb{R}, d) is a usual space and is not an ultrametric space.

Definition 2.5. [30] Let (X, d) be an ultrametric space, $x \in X$ and $r > 0$. Then,

$$\mathfrak{B}(x, r) = \{y \in X : d(x, y) \leq r\},$$

with $\mathfrak{B}(x, 0) = \{x\}$, is called ball. It is denoted by $\mathfrak{B}(x, r)$.

Remark 2. [30] A well-known characteristic property of an ultrametric space is the following:

$$\text{if } x, y \in X, 0 \leq r \leq \varepsilon \quad \text{and} \quad \mathfrak{B}(x, r) \cap \mathfrak{B}(x, \varepsilon) \neq \emptyset, \quad \text{then} \quad \mathfrak{B}(x, r) \subset \mathfrak{B}(x, \varepsilon).$$

Definition 2.6. [30] Let (X, d) be an ultrametric space. If every shrinking collection of balls in X has a non-empty intersection, then an ultrametric space (X, d) is said to be spherically complete.

Definition 2.7. [31] Let X be a nonempty set. A pseudo metric on X is a function:

$$d : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$$

satisfying the following conditions for any $x, y, \omega \in X$:

- (i) $d(x, y) = 0 \Rightarrow x = y$,
- (ii) $d(x, y) = d(y, x)$,
- (iii) $d(x, y) \leq d(x, \omega) + d(\omega, y)$.

Definition 2.8. [32] A pseudo metric on X is an ultra pseudo metric if, in addition to conditions given in Definition 2.7, for any $x, y, \omega \in X$,

- (iii') $d(x, y) \leq \max\{d(x, \omega), d(\omega, y)\}$.

Proposition 2.9. [32] A pseudo metric (X, d) is an ultra pseudo metric if and only if, for any three points $x, y, \omega \in X$, one of the following conditions is satisfied:

- $d(\omega, y) \leq d(x, y) = d(x, \omega)$ or
- $d(x, \omega) \leq d(y, x) = d(y, \omega)$ or
- $d(x, y) \leq d(\omega, x) = d(\omega, y)$.

Remark 3. Every spherically complete ultrametric space is a complete metric space. The converse is not true in general. As an example, we mention the completion of \mathbb{C}_p of the algebraic closure of the field of rational p -adic numbers. According to Krasner, this field has nice algebraic properties because it is algebraically closed and even isomorphic to complex numbers (cf. [33], pp. 134–145), but it has also been shown that \mathbb{C}_p is not spherically complete. This is mainly due to the fact that the complex p -adic numbers are a separable, complete ultrametric space with dense valuation (cf. [33], pp. 143–144).

2.2 Fundamentals of graph theory

Let X be a nonempty set and Δ denotes the diagonal of cartesian product $X \times X$. A graph on X is an object $G = (V(G), E(G))$, where $V(G)$ is a vertex set whose elements are called vertices, and $E(G)$ is an edge set. We assume that G has no parallel edges and $\Delta \subset E(G)$.

If x and y are vertices of G , then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}_{n \in \{0,1,2,\dots,k\}}$ of vertices such that

$$x_0 = x, x_k = y \quad \text{and} \quad (x_{i-1}, x_i) \in E(G) \quad \text{for } i \in \{1, 2, \dots, k\}.$$

Note that a graph G is connected if there is a path between any two vertices, and it is weakly connected if \tilde{G} is connected, where \tilde{G} denotes the undirected graph obtained from G by ignoring the direction of edges.

Denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

As it is more convenient to treat \tilde{G} as a directed graph for which the set of its edges is symmetric, under this convention, we have

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

The pair (V', E') is a subgraph of G if $V' \subseteq V(G)$ and $E' \subseteq E(G)$, and for any edge $(x, y) \in E'$, $x, y \in V'$.

If G is symmetric and x is a vertex in G , then the subgraph G_x consisting of all edges and vertices that are contained in some path beginning at x is called the component of G containing x . In this case, $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation \mathcal{R} defined on $V(G)$ by the rule:

$$y \mathcal{R} z \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

For detailed information about graph theory, see [34,40].

3 Main results

In this section, initially, we define a generalized G -strongly contractive map and then establish a fixed point theorem for this type of mapping. Besides, an illustrative example supports the validity of the main result.

Definition 3.1. Let (X, d) be an ultrametric space endowed with a graph G . A self-mapping T on X is a generalized G -strongly contractive mapping if

- (i) T preserves the edges of G , that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$,
- (ii) for all $x, y \in X$, $x \neq y$ with $(x, y) \in E(G)$,

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \quad (1)$$

Example 3.2. d is ultrametric on \mathbb{N} , and we defined $d : \mathbb{N} \times \mathbb{N} \rightarrow [0, +\infty)$ by

$$d(x, y) = \begin{cases} 0, & x = y \\ \max\left\{1 + \frac{1}{x}, 1 + \frac{1}{y}\right\}, & x \neq y. \end{cases}$$

Let $T : \mathbb{N} \rightarrow \mathbb{N}$ be a mapping such that

$$Tx = \begin{cases} x + k, & x \text{ odd} \\ 1, & x \text{ even}, \end{cases}$$

where $k \in \mathbb{Z}^+$. Then, the conditions of Definition 3.1 hold. Indeed,

Case I. x and y are any odd integers ($x \neq y$, $x < y$);

$$\begin{aligned}
d(Tx, Ty) &< \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\
&= \max\left\{1 + \frac{1}{x}, 1 + \frac{1}{y}\right\} \\
&= 1 + \frac{1}{x}.
\end{aligned}$$

Case II. x and y are any even integers ($x \neq y$, $x < y$);

$$\begin{aligned}
d(Tx, Ty) &< \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\
&= \max\left\{1 + \frac{1}{x}, 2\right\} \\
&= 1 + \frac{1}{x}.
\end{aligned}$$

Case III. x is any odd integer and y is any even integer ($x \neq y$, $x < y$);

$$\begin{aligned}
d(Tx, Ty) &< \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\
&= \max\left\{1 + \frac{1}{x}, 1 + \frac{1}{y}\right\} \\
&= 1 + \frac{1}{x}.
\end{aligned}$$

Let us recall the notion of the G - T -invariant before proceeding with our main theorem.

Definition 3.3. [35] Assume that (X, d) is an ultrametric space endowed with a graph G and a self-mapping T on X . We say that a $B(x, r)$ is G - T -invariant if for any $u \in B(x, r)$ such that $(u, x) \in E(G)$, then $Tu \in B(x, r)$.

Theorem 3.4. Let (X, d) be an ultrametric space endowed with a graph G and a G -strongly contractive self-mapping T on X fulfills the subsequent statements:

- (i) there exists an $x_0 \in X$ such that $d(x_0, Tx_0) < 1$,
- (ii) if $x \in X$ is such that $d(x, Tx) < 1$, then there exists a path in \bar{G} between x and Tx with vertices in $B(x, d(x, Tx))$,
- (iii) if $B(x_n, d(x_n, Tx_n))$ is a sequence of nonincreasing closed balls in X , and for each $n \geq 1$, there exists a path in \bar{G} between x_n and x_{n+1} with vertices in $B(x_n, d(x_n, Tx_n))$, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and a $z \in \bigcap_{k=1}^{\infty} B(x_{n_k}, r_{n_k})$ such that for each $k \geq 1$, there exists a path in \bar{G} between x_{n_k} and z with vertices in $B(x_{n_k}, d(x_{n_k}, Tx_{n_k}))$.

In that case, T owns a fixed point in each closed ball of the form $B(x, d(x, Tx))$, where

$$x \in X_0 = \{z \in X : d(z, Tz) < 1\}.$$

Proof. Presume that $x \in X_0$, $r = d(x, Tx)$, and $u \in B(x, r)$ such that $(u, x) \in E(G)$. Then,

$$\begin{aligned}
d(u, x) &\leq d(u, Tu) \\
&\leq \max\{d(u, x), d(x, Tx), d(Tx, Tu)\} \\
&< \max\left\{d(u, x), d(x, Tx), d(u, x), d(x, Tx),\right. \\
&\quad \left.d(u, Tu), d(x, Tu), d(u, Tx)\right\} \\
&< \max\{d(u, x), d(x, Tx)\} \\
&= d(x, Tx).
\end{aligned}$$

In addition,

$$B(x, d(x, Tx)) \cap B(u, d(u, Tu)) \neq \emptyset.$$

Thus,

$$B(u, d(u, Tu)) \subset B(x, d(x, Tx)),$$

so $Tu \in B(x, d(x, Tx))$. This means that $B(x, d(x, Tx))$ is G - T -invariant for all $x \in X_0$. Presume that $x_0 \in X_0$ is a fixed element. Put $x_1 = x_0$, $r_1 = d(x_1, Tx_1)$; if $r_1 = 0$, then x_1 is a fixed point of T such that $x_1 = Tx_1$ and the proof is completed. Otherwise, put

$$E_1 = \left\{ x \in B(x_1, r_1) \mid \begin{array}{l} \text{there is a path in } \bar{G} \text{ between } x \text{ and } x_1 \\ \text{with vertices in } B(x_1, r_1) \end{array} \right\}.$$

It is evident that x_1 and Tx_1 are elements of E_1 . Set

$$\mu_1 = \inf\{d(x, Tx) : x \in E_1\}.$$

If $r_1 = \mu_1$, then x_1 is a fixed point of T ; otherwise, owing to the fact that $d(x_1, Tx_1) < 1$ and (ii), there exists a path $(x_1 = y_0, \dots, y_n = Tx_1)$ in $B(x_1, r_1)$ from x_1 to Tx_1 . Since $B(x_1, r_1)$ is G - T -invariant, it ensures that $T^2x_1 \in B(x_1, r_1)$, and we attain

$$\begin{aligned} \mu_1 &\leq d(Tx_1, T^2x_1) \\ &< \max\{d(Tx_1, Ty_1), d(Ty_1, Ty_2), \dots, d(Ty_{n-1}, T^2x_1)\} \\ &< \max\{d(x_1, y_1), d(y_1, y_2), \dots, d(y_{n-1}, Tx_1)\} \\ &\leq d(x_1, Tx_1) \\ &= r_1, \end{aligned}$$

which causes a contradiction. Hence, finally, let $\mu_1 < r_1$. Assume that $\{\varepsilon_n\}_{n \in \mathbb{N}}$ is a sequence of positive numbers such that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Choose an element $x_2 \in B(x_1, r_1)$ such that there exists a path in \bar{G} between x_1 and x_2 and

$$r_2 = d(x_2, Tx_2) < \min\{r_1, \mu_1 + \varepsilon_1\}.$$

In a similar way, if $r_2 = \mu_2$, then x_2 is a fixed point of T ; otherwise, $x_3 \in B(x_2, r_2)$ exists such that there exists a path in \bar{G} between x_2 and x_3 and

$$r_3 = d(x_3, Tx_3) < \min\{r_2, \mu_2 + \varepsilon_2\}.$$

Define $x_n \in X$. Let

$$E_n = \left\{ x \in B(x_n, r_n) \mid \begin{array}{l} \text{there is a path in } \bar{G} \text{ between } x \text{ and } x_n \\ \text{with vertices in } B(x_n, r_n) \end{array} \right\}$$

and

$$\mu_n = \inf\{d(x, Tx) : x \in E_n\}.$$

If $r_n = 0$ or $r_n = \mu_n$, using the same way for $n = 1$, the proof is completed. Otherwise, choose element $x_{n+1} \in B(x_n, r_n)$ such that there exists a path between x_n and x_{n+1} and

$$r_{n+1} = d(x_{n+1}, Tx_{n+1}) < \min\{r_n, \mu_n + \varepsilon_n\}.$$

If this processing ends after a finite number of steps, then the proof is completed. Otherwise, we obtain a nonincreasing sequence of nontrivial closed $\{B(x_n, d(x_n, Tx_n))\}$. Because $\{r_n\}_{n \in \mathbb{N}}$ is nonincreasing, $r = \lim_{n \rightarrow \infty} r_n$ exists and $\{\mu_n\}$ is nondecreasing and bounded above, thus $\mu = \lim_{n \rightarrow \infty} \mu_n$. Therefore, by (iii), there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ and $\hat{z} \in \bigcap_{k=1}^\infty B(x_{n_k}, r_{n_k})$ such that for each $k \in \mathbb{N}$, a path exists in \bar{G} between x_{n_k} and \hat{z} with vertices in $B(x_{n_k}, d(x_{n_k}, Tx_{n_k}))$. $T\hat{z} \in B(x_{n_k}, r_{n_k})$ for all $k \geq 1$ because $B(x_{n_k}, r_{n_k})$ is G - T -invariant for all $k \geq 1$. Thus,

$$d(\hat{z}, T\hat{z}) < \max\{d(\hat{z}, x_{n_k}), d(x_{n_k}, T\hat{z})\} \leq r_{n_k}$$

for all $k \geq 1$. Therefore,

$$\mu_{n_k} \leq d(\dot{z}, T\dot{z}) \leq r \leq r_{n_k+1} \leq \mu_{n_k} + \varepsilon_{n_k}$$

for all $k \geq 1$. Letting $k \rightarrow \infty$, we achieve $d(\dot{z}, T\dot{z}) = r = \mu$. Moreover, if $x \in B(\dot{z}, d(\dot{z}, T\dot{z}))$, then for each $k \in \mathbb{N}$,

$$d(x, \dot{z}) \leq d(\dot{z}, T\dot{z}) \leq r_{n_k}$$

for all $k \geq 1$. Thus,

$$d(x, x_{n_k}) < \max\{d(x, \dot{z}), d(\dot{z}, x_{n_k})\} \leq r_{n_k}$$

for all $k \geq 1$. Consequently, $x \in B(x_{n_k}, r_{n_k})$ for all $k \geq 1$. Let $x \in B(\dot{z}, d(\dot{z}, T\dot{z}))$, and there exists a path between x and \dot{z} . Therefore, a path exists in $B(x_{n_k}, r_{n_k})$ between x_{n_k} and x for all $k \geq 1$. In consequence, $\mu_{n_k} \leq d(x, Tx)$ for all $k \geq 1$. Thereby, for each $k \in \mathbb{N}$, $\mu_{n_k} \leq r_{n_k}$. Thus,

$$\inf\{d(x, Tx) : x \in B(\dot{z}, d(\dot{z}, T\dot{z}))\} = d(\dot{z}, T\dot{z}) = r.$$

For all $k \geq 1$, we have

$$d(\dot{z}, T\dot{z}) \leq r_{n_k} < 1,$$

and thus, it follows that by (ii), there exists a path in $B(\dot{z}, d(\dot{z}, T\dot{z}))$ from \dot{z} to $T\dot{z}$. We assert that $r = 0$. Assume on the contrary that $r > 0$ and assume $(\dot{z} = y_0, y_1, \dots, y_N = T\dot{z})$ is a path in $B(\dot{z}, d(\dot{z}, T\dot{z}))$ between \dot{z} and $T\dot{z}$. $T^2\dot{z} \in B(\dot{z}, d(\dot{z}, T\dot{z}))$ because $B(\dot{z}, d(\dot{z}, T\dot{z}))$ is G - T -invariant, and consequently, we obtain

$$\begin{aligned} d(T\dot{z}, T^2\dot{z}) &\leq \max\{d(T\dot{z}, Ty_1), \dots, d(Ty_{N-1}, T^2\dot{z})\} \\ &< \max\{d(\dot{z}, y_1), \dots, d(y_{N-1}, T\dot{z})\} \\ &\leq d(\dot{z}, T\dot{z}), \end{aligned}$$

which causes a contradiction. Therefore, $r = 0$ and $\dot{z} = T\dot{z}$. As a result, the proof is completed. \square

Example 3.5. Let $\mathcal{X} = \{0, 1, 2, 3, \dots\}$ and

$$d(x, y) = \begin{cases} 0, & x = y \\ x + y, & x \neq y. \end{cases}$$

Then, d be an ultra pseudo metric on \mathcal{X} with Proposition 2.9 because the defined d is a pseudo metric on \mathcal{X} . Consider the graph given by $V(G) = \mathcal{X}$ and $E(G) = \mathcal{X} \times \mathcal{X} \setminus \{(0, 1), (1, 0)\}$. Let $T : \mathcal{X} \rightarrow \mathcal{X}$ mapping be defined as follows:

$$Tx = \begin{cases} \{x\}, & x = 0, x = 1 \\ \{0, 1, 2, \dots, x-1\}, & x \geq 2. \end{cases}$$

The mapping T satisfies all the conditions of Theorem 3.4, and even when the graph structure on \mathcal{X} is removed, T has no fixed point. Indeed,

Case I. Let $y = 0$ and $x > 1$,

$$\begin{aligned} d(Tx, Ty) &< \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &= \max\{x, 2x-1\} \\ &= 2x-1. \end{aligned}$$

Case II. Let $y = 1$ and $x > 1$,

$$\begin{aligned} d(Tx, Ty) &< \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &= \max\{x+1, 2x-1\}. \end{aligned}$$

Case III. Let $x, y > 1$,

$$\begin{aligned} d(Tx, Ty) &< \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &= \max\{x+y-1, 2x-1\} \\ &= 2x-1. \end{aligned}$$

The fixed points of T are 0 and 1. On the other hand, if we assume that there is no graph on X , we see that the contraction condition is not satisfied. Indeed,

$$\begin{aligned} d(Tx, Ty) &< \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &= \max\{1, 0\} \\ &= 1 \end{aligned}$$

is obtained for $x = 0$ and $y = 1$.

Corollary 3.6. *Theorem 3.4 holds if the condition 3.1 of Definition 3.1 is replaced by*

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty)\},$$

$\forall x, y \in X, x \neq y$ with $(x, y) \in E(G)$.

Corollary 3.7. *Theorem 3.4 holds if the condition 3.1 of Definition 3.1 is replaced by*

$$d(Tx, Ty) < d(x, y)$$

$\forall x, y \in X, x \neq y$ with $(x, y) \in E(G)$.

Corollary 3.8. *Let (X, d) be a spherically complete ultrametric space, and T is a self-mapping on X if T is a mapping such that*

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

$\forall x, y \in X, x \neq y$. Then, T admits a unique fixed point in X .

3.1 An application to dynamic programming

In this section, we implement our findings to solve the subsequent functional equations applied in dynamic programming

$$q(x) = \max_{y \in D} \{f(x, y) + G(x, y, q(\eta(x, y)))\}, x \in W, \quad (1)$$

where $f: W \times D \rightarrow \mathbb{R}$ and $G: W \times D \times \mathbb{R} \rightarrow \mathbb{R}$ are bounded, $\eta: W \times D \rightarrow W$ and X and Y are Banach spaces such that $W \subset X$ and $D \subset Y$.

Equations of the type (1) find their application in optimization theory, computer programming, and dynamic programming. As an application, dynamic programming has been researched by several researchers. Klim and Wardowski [36] extended their concept of F -contraction mapping to nonlinear F -contractions, proving a fixed point theorem through dynamic processes. Asif et al. [37], working on the common fixed point problem in F -metric spaces (F -MS), proved a unique common solution to the functional equations widely used in computer programming and optimization theory. Dynamic programming is organized into two sections: state space and decision space. Suppose that W and D are state space and decision space, respectively.

Let $B(W)$ define the set of all bounded real-valued functions on W . The pair $(B(W), \|\cdot\|)$, where

$$\|h\| = \max_{x \in W} |h(x)|, h \in B(W),$$

is a Banach space and d is the metric defined as follows:

$$d(h, k) = \max_{x \in W} |h(x) - k(x)|.$$

By Proposition 2.9, $d(h, k)$ is an ultra pseudo metric.

We aim to indicate the existence of a solution of equation (1), and we consider the operator $T : B(W) \rightarrow B(W)$ as follows:

$$(Th)(x) = \max_{y \in D} \{f(x, y) + G(x, y, h(\eta(x, y)))\} \quad (2)$$

for all $h \in B(W)$ and $x \in W$. It is evident that the operator T is well-defined since f and G are bounded.

We now establish the subsequent theorem.

Theorem 3.9. *Let $T : B(W) \rightarrow B(W)$ be an operator defined by equation (2), and suppose that the following statements are met:*

- (i) *f and G are bounded,*
- (ii) *$\forall h, k \in B(W), \forall x \in W, \forall y \in D,$*

$$|G(x, y, h(x)) - G(x, y, k(x))| < M(h(x), k(x)),$$

where

$$M(h(x), k(x)) = \max \left\{ \begin{array}{l} d(h(x), k(x)), d(h(x), Th(x)), d(k(x), Tk(x)), \\ d(h(x), Tk(x)), d(k(x), Th(x)) \end{array} \right\},$$

then the functional equation (2) has a bounded solution.

Proof. Let $\lambda \in \mathbb{R}^+$ be an arbitrary, $x \in W$, and $h \in B(W)$. Without loss of generality, we suppose that $Th \neq h$. Then, $y_1, y_2 \in D$ exist such that

$$\begin{aligned} (Th)(x) &< f(x, y_1) + G(x, y_1, h(\eta(x, y_1))) + \lambda, \\ (Tk)(x) &< f(x, y_2) + G(x, y_2, k(\eta(x, y_2))) + \lambda, \\ (Th)(x) &\geq f(x, y_2) + G(x, y_2, h(\eta(x, y_2))), \\ (Tk)(x) &\geq f(x, y_1) + G(x, y_1, k(\eta(x, y_1))). \end{aligned}$$

Then, we achieve

$$\begin{aligned} (Th)(x) - (Tk)(x) &< G(x, y_1, h(\eta(x, y_1))) - G(x, y_1, k(\eta(x, y_1))) + \lambda \\ &\leq |G(x, y_1, h(\eta(x, y_1))) - G(x, y_1, k(\eta(x, y_1)))| + \lambda \\ &< M(h(x), k(x)) + \lambda \end{aligned}$$

and

$$\begin{aligned} (Tk)(x) - (Th)(x) &< G(x, y_2, k(\eta(x, y_2))) - G(x, y_2, h(\eta(x, y_2))) + \lambda \\ &\leq |G(x, y_2, k(\eta(x, y_2))) - G(x, y_2, h(\eta(x, y_2)))| + \lambda \\ &< M(h(x), k(x)) + \lambda \end{aligned}$$

for all $\lambda > 0$. Therefore,

$$|(Th)(x) - (Tk)(x)| < M(h(x), k(x)) + \lambda$$

for all $\lambda > 0$. Hence,

$$d((Th)(x), (Tk)(x)) < M(h(x), k(x)).$$

Consequently, Corollary 3.8 implies the existence of a bounded solution of equation (2). \square

4 Conclusion

Ultimately, in the context of ultrametric space, we bring on and carry forward the results by exploiting the space's spherical completeness with a graph for single-valued G -strongly contractive mappings. We emphasize the significance of employing a distinct type of completeness. We specifically aim to extend the application of relevant fixed-point solutions to dynamic programming, which will assist in advancing the literature in ultrametric spaces.

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References

- [1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.
- [2] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **60** (1968), 71–76.
- [3] S. Reich, *Some problems and results in fixed point theory*, Contemporary Math. **21** (1983), 179–187.
- [4] G. E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of Reich*, Can. Math. Bull. **16** (1973), 201–206.
- [5] V. Berinde, *Approximating fixed points of weak ϕ -contractions using the Picard iteration*, Fixed Point Theory **4** (2003), no. 2, 131–142.
- [6] E. Karapinar and B. Samet, *Generalized α - ψ -contractive type mappings and related fixed point theorems with applications*, Abstract Appl. Anal. **2012** (2012), Article ID 793486, 17 pages.
- [7] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **2012** (2012), no. 94, 6 pages.
- [8] E. Karapinar, A. Fulga and R. P. Agarwal, *A survey: F-contractions with related fixed point results*, J. Fixed Point Theory Appl. **22** (2020), 1–58.
- [9] A. Petruşel, *On Frigon-Granas-type multifunctions*, Nonlinear Anal. Forum **7** (2002), no. 1, 113–121.
- [10] J. Ginişwamy and P. G. Maheshwari, *Fixed point theorems under F-contraction in ultrametric space*, Adv. Fixed Point Theory **7** (2017), no. 1, 144–154.
- [11] F. Echenique, *A short and constructive proof of Tarski's fixed-point theorem*, Int. J. Game Theory **33** (2005), no. 2, 215–218.
- [12] R. Espínola and W. A. Kirk, *Fixed point theorems in R-trees with applications to graph theory*, Topology Appl. **153** (2006), no. 7, 1046–1055.
- [13] J. Jachymski, *The contraction principle for mappings on a complete metric space with a graph*, Proc. Amer. Math. Soc. **136** (2008), no. 1, 1359–1373.
- [14] I. Beg, A. R. Butt, and S. Radojević, *The contraction principle for set-valued mapping on a metric space with a graph*, Comput. Math. Appl. **60** (2010), 1214–1219.
- [15] A. Sultana and V. Vetrivel, *Fixed points of Mizoguchi-Takahashi contraction on a metric space with a graph and applications*, J. Math. Anal. Appl. **417** (2014), no. 1, 336–344.
- [16] T. Sistani and M. Kazemipour, *Fixed point theorems for α - ψ -contractions on metric spaces with a graph*, J. Adv. Math. Stud. **7** (2014), no. 1, 65–79.
- [17] C. Chifu, G. Petruşel, and M-F. Bota, *Fixed points and strict fixed points for multivalued contractions of Reich type on metric spaces endowed with a graph*, Fixed Point Theory Appl. **203** (2013), 12.
- [18] T. Dinevari and M. Frigon, *Fixed point results for multivalued contractions on a metric space with a graph*, J. Math. Anal. Appl. **405** (2013), no. 2, 507–517.

- [19] J. de Groot, *Non-Archimedean metrics in topology*, Proc. Am. Math. Soc. **7** (1956), no. 6, 948–956.
- [20] A. J. Lemin and V. A. Lemin, *On a universal ultrametric space*, Topol. Appl. **103** (2000), no. 3, 339–345.
- [21] A. J. Lemin, *On ultrametrization of general metric spaces*, Proc. Am. Math. Soc. **131** (2003), no. 3, 979–989.
- [22] S. Priess-Crampe and P. Ribenboim, *Generalized ultrametric spaces I*, Abh. Math. Sem. Univ. Hamburg **66** (1996), 55–73.
- [23] S. Priess-Crampe and P. Ribenboim, *Generalized ultrametric spaces II*, Abh. Math. Sem. Univ. Hamburg **67** (1997), 19–31.
- [24] P. Ribenboim, *The new theory of ultrametric spaces*, Period. Math. Hung. **32** (1996), no. 1–2, 103–111.
- [25] L. Gajić, *On ultrametric spaces*, Novi Sad J. Math. **31** (2001), no. 2, 69–71.
- [26] L. Gajić, *A multivalued fixed point theorem in ultrametric spaces*, Matematički Vesnik **54** (2002), no. 3–4, 89–91.
- [27] K. P. R. Rao, G. N. V. Kishore, and T. Ranga Rao, *Some coincidence point theorems in ultrametric spaces*, Int. J. Math. Anal. **1** (2007), no. 18, 897–902.
- [28] Q. Zhang and Y. Song, *Fixed point theory for generalized ϕ -weak contractions*, Appl. Math. Lett. **22** (2009), 75–78.
- [29] D. R. Kumar and M. Pitchaimani, *A generalization of set-valued Presic-Reich type contractions in ultrametric spaces with applications*, J. Fixed Point Theory Appl. **19** (2017), 1871–1887.
- [30] A. C. M. Van Rooij, *Non-Archimedean Functional Analysis*, Marcel Dekker, New York, 1978.
- [31] Anonymous Lecture 2. SF2956 Topological Data Analysis, Pseudo Metrics, 05.03.2023, <https://kth.instructure.com/courses/7193/files/1209457/download?wrap=1>.
- [32] Anonymous Lecture 4. SF2956 Topological Data Analysis, Ultra Pseudo Metrics, 05.03.2023, <https://kth.instructure.com/courses/7193/files/1209457/download?wrap=1>.
- [33] A. M. Robert, *A Course in p-adic Analysis*, vol. 198 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.
- [34] R. Johnsonbaugh, *Discrete Mathematics*, Prentice-Hall, Inc., New Jersey, 1997.
- [35] H. Mamghaderi and H. P. Masiha, *Fixed point theorem in ultrametric space*, Appl. Anal. Discrete Math. **12** (2018), no. 2, 336–346.
- [36] D. Klim and D. Wardowski, *Fixed points of dynamic processes of set-valued F-contractions and application to functional equations*, Fixed Point Theory and Applications **2015** (2015), no. 1, 1–9.
- [37] A. Asif, S. U. Khan, T. Abdeljawad, M. Arshad, and A. Ali, *3D dynamic programming approach to functional equations with applications*, J. Function Spaces **2020** (2020), 9 pages.
- [38] M. Younis, H. Ahmad, L. Chen, and M. Han, *Computation and convergence of fixed points in graphical spaces with an application to elastic beam deformations*, J. Geometry Phys. **192** (2023), 1–13, doi: <https://doi.org/10.1016/j.geomphys.2023.104955>.
- [39] M. Younis, D. Singh, L. Chen, and M. Metwali, *A study on the solutions of notable engineering models*, Math. Model. Anal. **27** (2022), no. 3, 492–509, DOI: <https://doi.org/10.3846/mma.2022.15276>.
- [40] M. Younis and D. Bahuguna, *A unique approach to graph-based metric spaces with an application to rocket ascension*, Comput. Appl. Math. **42** (2023), no. 1, 19 pages.