Research Article

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On the existence and Ulam-Hyers stability for implicit fractional differential equation via fractional integral-type boundary conditions

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Abstract: This study investigates the existence of solutions for implicit fractional differential equations with fractional-order integral boundary conditions. We create the required conditions to ensure unique solution and Ulam-Hyers-Rassias stability. We also give examples to highlight the major findings.

Keywords: implicit fractional-order differential equation, existence results, Green's function, boundary value problems, Ulam-Hyers stability

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1 Introduction

Fractional calculus is a powerful tool in applied mathematics, which provides a way to examine various ranges of problems in numerous scientific and technical domains. Fractional derivatives have greatly advanced thanks to Riemann-Liouville, Caputo, Hilfer, Hadamard, and others. You can find ongoing results on fractional calculus and fractional differential equations in [1–6]. The study of partial fractional differential equations and, moreover, ordinary differential equations has advanced significantly in recent years. For additional exploration, one might consult the monographs by Abbas et al. [7], Baleanu et al. [8], Kilbas et al. [9], and Lakshmikantham et al. [10], and many researchers have published some extremely useful findings in this area [11–20].

Recent studies on boundary value problem have been conducted by a number of mathematicians. More precisely, Chasreechai and Tariboon [21] studied the following classes of integral boundary condition problems:

$$\nu''(t) + \chi g(t)h(\nu(\mathfrak{r})) = 0, \quad t \in (0,1), \nu(0) = \alpha_1\int_0^{\eta}\nu(\varsigma)\mathrm{d}\varsigma, \quad \nu(1) = \alpha_2\int_0^{\eta}\nu(\varsigma)\mathrm{d}\varsigma.$$

Hu and Wang [22] have shown that the nonlinear fractional differential equation has a solution with integral boundary conditions:

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$$D^{\zeta}v(t) = g(t, v(t), D^{\delta}v(t)), \quad t \in (0,1), 1 < \zeta \le 2, 0 < \delta \le 1,$$
$$v(0) = v_{\circ}, v(1) = \int_0^1 h(\varsigma)v(\varsigma)\mathrm{d}\varsigma.$$

The boundary value issue provided by Murad and Hadid [23] is as follows:

$$D^{\zeta}v(t) = g(t, v(t), D^{\delta}v(t)), \quad t \in (0,1), \quad 1 < \zeta \le 2, \quad 0 < \delta \le 1,$$
$$v(0) = 0, \quad v(1) = I_0^{\gamma}v(t),$$

with D^{ζ} and D^{δ} being the fractional derivative of Riemann-Liouville.

During a lecture at Wisconsin University in 1940, Ulam first brought up the subject of stability in functional equations. He asked, "Under what circumstances does the existence of an additive mapping that is close to an essentially additive mapping hold?" (see [24] for more information). In 1941, Hyers offered the first response to Ulam's query, focusing on the situation involving Banach spaces [25]. This type of stability is referred to as the Ulam-Hyers stability. In 1978, Rassias significantly expanded the Ulam-Hyers stability by including variables [26]. The concept of stability in functional equations appears when an inequality is employed as a perturbation instead of the original equation. As a result, the difference between the solutions of the inequality and those of the functional equation that is being presented revolves on the issue stability in functional equations. The Ulam-Hyers stability and Ulam-Hyers-Rassias stability in different types of functional equations have received a lot of attention, as addressed in the monographs by [25,27,28].

Motivated by these results, in the aforementioned articles, we focus our attention on the more general case for the implicit fractional-order differential problem (IFDP):

$$^{c}D^{\alpha}y(\tau) = f\left(\tau, y(\tau), {}^{c}D^{\beta}y(\tau), \int_0^{\tau} k(\tau,\varsigma){}^{c}D^{\alpha}y(\varsigma)\mathrm{d}\varsigma\right), \tag{1}$$

$$y(0) = \frac{1}{\Gamma(\gamma)}\int_0^1 (1-\varsigma)^{\gamma-1}h_1(\varsigma, y(\varsigma))\mathrm{d}\varsigma, \tag{2}$$

$$y'(T) = \frac{1}{\Gamma(\gamma)}\int_0^1 (1-\varsigma)^{\gamma-1}h_2(\varsigma, y(\varsigma))\mathrm{d}\varsigma, \tag{3}$$

where $\tau \in I = [0, T), 1 < \beta < \alpha \le 2$, and $0 < \gamma < 1, {}^{c}D^{\delta}$ is the Caputo fractional derivative order $\delta \in \{\alpha, \beta\}$, with $1 < \beta < \alpha \le 2, h_i : I \times R \to R, (i = 1, 2)$ are continuous functions and $f : I \times R^3 \to R$, and $k : I \times I \to R$ are given functions satisfying some assumptions that will be specified later.

It is important to note that Problems (1)–(3) considered in this article involving this type of integral boundary condition have not been extensively investigated yet, and it extends the study of fractional integro-differential equations (FIDEs), and from this point of view, we believe that the obtained results will contribute to the existing literature.

The contents of this article are divided into four sections. Two parts make up Section 2, which includes the primary findings. In Part 1, it was discussed how the integral equation (2.1) and FIDE (1)–(3) are equivalent. The results of Problems (1)–(3) are in Part 2; one of which is demonstrated using the Banach contraction principle, and the other using Krasnoselskii's fixed point theorem. In Section 3, we will additionally talk about Ulam-Hyers-Russian stability of our issue. Additionally, we provide examples that illustrate our conclusions in Section 4.

2 Existence and Ulam-Hyers stability

Here, we show the existence and Ulam-Hyers stability of mild solutions for the IFDP (1)–(3),via the following hypotheses:

(H_1) Functions $h_i : :I \times R \to R,\ i = 1, 2$ are continuous, such that $\exists$ and constants $0 \le k_i < 1$ meet

$$|h_i(\tau, \mu) - h_i(\tau, \nu)| \le k_i|\mu - \nu|.$$

Note that

$$|h_i(\tau, \mu)| \le H_i + k_i|\mu|, \quad \text{where } H_i = \sup_{\tau\in\mathbb{I}}|h_i(\tau, 0)|, i = 1, 2.$$

(H_2) $f : I \times R^3 \to R$ is continuous, such that there exists $\psi \in C(\mathbb{J}, R_+)$, with norm $||\psi||$, meet

$$|f(\tau, v_1, v_2, v_3) - f(\tau, w_1, w_2, w_3)| \le \psi(\tau)(|v_1 - w_1| + |v_2 - w_2| + |v_3 - w_3|).$$

Note that

$$|f(\tau, v_1, v_2, v_3)| \le ||\psi||(|v_1| + |v_2| + |v_3|) + F, \quad \text{where } F = \sup_{\tau\in I}|f(\tau, 0, 0, 0)|,$$

$\forall \tau \in I$, $v_i, w_i \in R$, $i = 1, 2, 3$.

(H_3) $k(\tau, \varsigma)$ is continuous for all $(\tau, \varsigma) \in I \times I$, and $\exists$ and constant $K > 0$ meet

$$\max_{\tau,\varsigma\in[0,T]}|k(\tau, \varsigma)| = K.$$

Lemma 1. *The mild solution of IFDP* (1)–(3) *is the solution of the integral equation*:

$$y(\tau) = h(\tau, y(\tau)) + \int_0^T G(\tau, \varsigma)u(\varsigma)\mathrm{d}\varsigma. \tag{4}$$

Here, u provides the following integral equation:

$$u(\tau) = f\left(\tau, h(\tau) + \int_0^T G(\tau, \varsigma)u(\varsigma)\mathrm{d}\varsigma, I^{\alpha-\beta}u(\tau), \int_0^\tau k(\tau, \varsigma)u(\varsigma)\mathrm{d}\varsigma\right),$$

where $G(\tau, \varsigma)$ is the Green function described by:

$$G(\tau, \varsigma) = \begin{cases} \dfrac{(\tau - \varsigma)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{\tau(T - \varsigma)^{\alpha-2}}{\Gamma(\alpha - 1)}, & 0 \le \varsigma \le \tau < T \\ \dfrac{-\tau(T - \varsigma)^{\alpha-2}}{\Gamma(\alpha - 1)}, & 0 \le \tau \le \varsigma < T, \end{cases} \tag{5}$$

$$G_\circ \coloneqq \max\left\{\int_0^T G(\tau, \varsigma)\mathrm{d}\varsigma, (\tau, \varsigma) \in I \times I\right\},$$

and

$$h(\tau, y(\tau)) = \frac{1}{\Gamma(\gamma)}\int_0^1 (1 - \varsigma)^{\gamma-1}h_1(\varsigma, y(\varsigma))\mathrm{d}\varsigma + \frac{\tau}{\Gamma(\gamma)}\int_0^1 (1 - \varsigma)^{\gamma-1}h_2(\varsigma, y(\varsigma))\mathrm{d}\varsigma. \tag{6}$$

Proof. Depending on the caputo fractional derivative, we obtain $D^\beta y(\tau) = I^{\alpha-\beta}D^\alpha y(\tau)$ for $\tau \in I$.

Consequently, if y is a solution of IFDP (1)–(3), then

$${}^cD^\alpha y(\tau) = f\left(\tau, y(\tau), I^{\alpha-\beta c}D^\alpha y(\tau), \int_0^\tau k(\tau, \varsigma){}^cD^\alpha y(\varsigma)\mathrm{d}\varsigma\right).$$

Let ${}^{c}D^{\alpha}y(\tau) = v(\tau)$, then

$$v(\tau) = f\left(\tau, y(\tau), I^{\alpha-\beta}v(\tau), \int_0^{\tau} k(\tau, \varsigma)v(\varsigma)\mathrm{d}\varsigma\right) \tag{7}$$

and

$$y(\tau) = a_{\circ} + a_1\tau + \frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau - \varsigma)^{\alpha-1}v(\varsigma)\mathrm{d}\varsigma. \tag{8}$$

By differentiating (8), we obtain

$$y'(\tau) = a_1 + \frac{1}{\Gamma(\alpha - 1)}\int_0^{\tau}(\tau - \varsigma)^{\alpha-2}v(\varsigma)\mathrm{d}\varsigma.$$

By virtue of (2) and (3), we obtain

$$a_{\circ} = \int_0^1 \frac{(1 - \varsigma)^{\gamma-1}}{\Gamma(\gamma)} h_1(\varsigma, y(\varsigma))\mathrm{d}\varsigma,$$

$$a_1 = \frac{1}{\Gamma(\gamma)}\int_0^1 (1 - \varsigma)^{\gamma-1}h_2(\varsigma, y(\varsigma))\mathrm{d}\varsigma - \frac{1}{\Gamma(\alpha - 1)}\int_0^T (T - \varsigma)^{\alpha-2}v(\varsigma)\mathrm{d}\varsigma.$$

Hence, the solution of (1)–(3) is as follows:

$$\begin{aligned} y(\tau) = {} & \frac{1}{\Gamma(\gamma)}\int_0^1 (1 - \varsigma)^{\gamma-1}h_1(\varsigma, y(\varsigma))\mathrm{d}\varsigma + \frac{\tau}{\Gamma(\gamma)}\int_0^1 (1 - \varsigma)^{\gamma-1}h_2(\varsigma, y(\varsigma))\mathrm{d}\varsigma \\ & - \frac{\tau}{\Gamma(\alpha - 1)}\int_0^T (T - \varsigma)^{\alpha-2}v(\varsigma)\mathrm{d}\varsigma + \frac{1}{\Gamma(\alpha)}\int_0^{\tau}(T - \varsigma)^{\alpha-1}v(\varsigma)\mathrm{d}\varsigma. \end{aligned}$$

Consequently, put

$$h(\tau, y(\tau)) = \frac{1}{\Gamma(\gamma)}\int_0^1 (1 - \varsigma)^{\gamma-1}h_1(\varsigma, y(\varsigma))\mathrm{d}\varsigma + \frac{\tau}{\Gamma(\gamma)}\int_0^1 (1 - \varsigma)^{\gamma-1}h_2(\varsigma, y(\varsigma))\mathrm{d}\varsigma,$$

and, we know that $\int_0^T = \int_0^{\tau} + \int_{\tau}^T$, we obtain equation (4). □

Definition 1. We refer to a function $v \in C(I, R)$ that solves integral equation (4) as a mild solution of IFDP (1)–(3), where v is the solution of integral equation:

$$v(\tau) = f\left(\tau, h(\tau) + \int_0^T G(\tau, \varsigma)v(\varsigma)\mathrm{d}\varsigma, I^{\alpha-\beta}v(\tau), \int_0^{\tau} k(\tau, \varsigma)v(\varsigma)\mathrm{d}\varsigma\right),$$

for all $\tau \in I$.

Lemma 2. *The Lipschitzian function $h : I \times R \to R$ has a Lipschitz constant c, with*

$$||h(\tau, x) - h(\tau, y)|| \le c||x - y||,$$

where $c = \frac{k_1 + k_2T}{\Gamma(\gamma + 1)}$.

Proof. We obtain for any for all $x, y \in X$ and $\tau \in I$:

$$\begin{aligned}|h(\tau, x(\tau)) - h(\tau, y(\tau))| &\leq \Bigg| \frac{1}{\Gamma(\gamma)} \int_0^1 (1-\varsigma)^{\gamma-1} h_1(\varsigma, x(\varsigma)) \mathrm{d}\varsigma + \frac{\tau}{\Gamma(\gamma)} \int_0^1 (1-\varsigma)^{\gamma-1} h_2(\varsigma, x(\varsigma)) \mathrm{d}\varsigma \\ &\quad - \frac{1}{\Gamma(\gamma)} \int_0^1 (1-\varsigma)^{\gamma-1} h_1(\varsigma, y(\varsigma)) \mathrm{d}\varsigma - \frac{\tau}{\Gamma(\gamma)} \int_0^1 (1-\varsigma)^{\gamma-1} h_2(\varsigma, y(\varsigma)) \mathrm{d}\varsigma \Bigg| \\ &\leq \frac{1}{\Gamma(\gamma)} \int_0^1 (1-\varsigma)^{\gamma-1} |h_1(\varsigma, x(\varsigma)) - h_1(\varsigma, y(\varsigma))| \mathrm{d}\varsigma \\ &\quad + \frac{\tau}{\Gamma(\gamma)} \int_0^1 (1-\varsigma)^{\gamma-1} |h_2(\varsigma, x) - h_2(\varsigma, y)| \mathrm{d}\varsigma \\ &\leq \frac{1}{\Gamma(\gamma+1)} \|h_1(\varsigma, x) - h_1(\varsigma, y)\| + \frac{\tau}{\Gamma(\gamma+1)} \|h_2(\varsigma, x) - h_2(\varsigma, y)\| \\ &\leq \frac{k_1}{\Gamma(\gamma+1)} \|x - y\| + \frac{k_2 \tau}{\Gamma(\gamma+1)} \|x - y\| \\ &\leq \frac{k_1 + k_2 T}{\Gamma(\gamma+1)} \|x - y\|.\end{aligned}$$

Thus,

$$\|h(\tau, x) - h(\tau, y)\| \leq c\|x - y\|,$$

where $c = \frac{k_1 + k_2 T}{\Gamma(\gamma+1)}$. □

Based on Krasnoselskii's [26] fixed point theorem, we will prove the first existence result for IFDP (1)–(3).

Theorem 1. *There is at least one mild solution on I for IFDP* (1)–(3) *under the assumptions* (H_1) *through* (H_3):

$$\frac{k_1 + tk_2}{\Gamma(\gamma+1)} + \frac{G_\circ \|\psi\|}{1 - \aleph} < 1.$$

Proof. Define the operator $A : C(I, R) \to C(I, R)$ by:

$$Ay(\tau) = h(\tau, y(\tau)) + \int_0^T G(\tau, \varsigma) v(\varsigma) \mathrm{d}\varsigma, \tag{9}$$

where

$$v(\tau) = f\left(\tau, y(\tau), I^{\alpha-\beta} v(\tau), \int_0^\tau k(\tau, \varsigma) v(\varsigma) \mathrm{d}\varsigma\right),$$

where G and h are the functions defined by (5) and (6), respectively. Define the ball

$$B_\varrho = \{y \in C(I, R) : \|y\| \leq \varrho\},$$

with

$$\varrho \geq \frac{\frac{H_1 + TH_2}{\Gamma(\gamma+1)} + \frac{G_\circ F}{1 - \aleph}}{1 - \left(\frac{k_1 + Tk_2}{\Gamma(\gamma+1)} + \frac{G_\circ \|\psi\|}{1 - \aleph}\right)}.$$

Additionally, we define the operators A_1 and A_2 on B_ϱ by:

$$A_1 y(\tau) = h(\tau, y(\tau)),$$
$$A_2 y(\tau) = \int_0^T G(\tau, \varsigma) v(\varsigma) d\varsigma.$$

Taking into account that A_1 and A_2 are set on B_ϱ, and $\forall y \in C(I, R)$,

$$Ay(\tau) = A_1 y(\tau) + A_2 y(\tau), \quad \tau \in I.$$

There are various claims that the proof will provide:

Step 1: Let $y_1, y_2 \in B_\varrho$ and $\tau \in I$, then

$$\begin{aligned} |A_1 y_1(\tau) + A_2 y_2(\tau)| &\leq |A_1 y_1(\tau)| + |A_2 y_2(\tau)| \\ &\leq |h(\tau, y_1(\tau))| + \int_0^T |G(\tau, \varsigma)||v(\varsigma)| d\varsigma, \end{aligned} \tag{10}$$

where $v(\tau) = f\left(\tau, y_2(\tau), I^{\alpha-\beta} v(\tau), \int_0^\tau k(\tau, \varsigma) v(\varsigma) d\varsigma\right)$,

$$\begin{aligned} |v(\tau)| &= \left| f\left(\tau, y_2(\tau), I^{\alpha-\beta} v(\tau), \int_0^\tau k(\tau, \varsigma) v(\varsigma) d\varsigma\right) \right| \\ &\leq F + |\psi(\tau)||y_2(\tau)| + |\psi(\tau)| \int_0^\tau \frac{(\tau - \varsigma)^{\alpha-\beta-1}}{\Gamma(\alpha - \beta)} |v(\varsigma)| d\varsigma + |\psi(\tau)| \int_0^\tau |k(\tau, \varsigma)||v(\varsigma)| d\varsigma. \end{aligned}$$

Taking supremum for $\tau \in I$, we have

$$\|v\| \leq F + \|\psi\| \|y_2\| + \|\psi\| \frac{T^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)} \|v\| + \|\psi\| K \|v\| T.$$

Thus,

$$\|v\| \leq \frac{F + \|\psi\| \varrho}{1 - \left(\frac{\|\psi\| \, T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \|\psi\| KT\right)}.$$

And

$$\begin{aligned} |h(\tau, y_1(\tau))| &\leq \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \varsigma)^{\gamma-1} |h_1(\varsigma, y_1(\varsigma))| d\varsigma + \frac{\tau}{\Gamma(\gamma)} \int_0^1 (1 - \varsigma)^{\gamma-1} |h_2(\varsigma, y_1(\varsigma))| d\varsigma \\ &\leq \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \varsigma)^{\gamma-1} [H_1 + k_1 |y_1(\varsigma)|] d\varsigma + \frac{\tau}{\Gamma(\gamma)} \int_0^1 (1 - \varsigma)^{\gamma-1} [H_2 + k_2 |y_2(\varsigma)|] d\varsigma \\ &\leq \frac{[H_1 + k_1 \|y_1\|]}{\Gamma(\gamma + 1)} + \frac{\tau [H_2 + k_2 \|y_2\|]}{\Gamma(\gamma + 1)} \\ &\leq \frac{H_1 + \tau H_2}{\Gamma(\gamma + 1)} + \frac{\varrho [k_1 + \tau k_2]}{\Gamma(\gamma + 1)}. \end{aligned}$$

Therefore, (10) indicates that, for each $\tau \in I$,

$$\begin{aligned} |A_1 y_1(\tau) + A_2 y_2(\tau)| &\leq \frac{H_1 + TH_2}{\Gamma(\gamma + 1)} + \frac{\varrho [k_1 + Tk_2]}{\Gamma(\gamma + 1)} + \frac{G_\circ (F + \|\psi\| \varrho)}{1 - \aleph}, \\ &\leq \varrho, \end{aligned}$$

where $\aleph = \left(\frac{\|\psi\| \, T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \|\psi\| KT\right)$. Taking supremum over $\tau \in I$, we have

$$\|A_1 y_1 + A_2 y_2\| \leq \varrho.$$

This proves that $A_1y_1 + A_2y_2 \in B_\varrho$ for every $y_1, y_2 \in B_\varrho$.

Step 2: Lemma 2 makes it evident that A_1 is a contraction mapping for $c < 1$.

Step 3: First, establish the continuity of operator A_2.

Let $\{y_n\}_{n\in N}$ be a sequence where $y_n \to y$ as $n \to \infty$ in $C(I, R)$.

Then, for each $\tau \in I$,

$$|A_2y_n - A_2y| \le \int_0^T |G(\tau, \varsigma)||v_n(\varsigma) - v(\varsigma)|\mathrm{d}\varsigma, \tag{11}$$

where $v_n, v \in C(I, R)$, with

$$v_n(\tau) = f\left(\tau, y_n(\tau), I^{\alpha-\beta}v_n(\tau), \int_0^\tau k(\tau, \varsigma)v_n(\varsigma)\mathrm{d}\varsigma\right),$$

$$v(\tau) = f\left(\tau, y(\tau), I^{\alpha-\beta}v(\tau), \int_0^\tau k(\tau, \varsigma)v(\varsigma)\mathrm{d}\varsigma\right),$$

so that by (H_2), we obtain

$$\begin{aligned}|v_n(\tau) - v(\tau)| &= \left| f\left(\tau, y_n(\tau), I^{\alpha-\beta}v_n(\tau), \int_0^\tau k(\tau, \varsigma)v_n(\varsigma)\mathrm{d}\varsigma\right) - f\left(\tau, y(\tau), I^{\alpha-\beta}v(\tau), \int_0^\tau k(\tau, \varsigma)v(\varsigma)\mathrm{d}\varsigma\right)\right| \\ &\le \psi(\tau)\left(|y_n(\tau) - y(\tau)| + \int_0^\tau \frac{(\tau-\varsigma)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|v_n(\varsigma) - v(\varsigma)|\mathrm{d}\varsigma + \int_0^\tau k(\tau, \varsigma)|v_n(\varsigma) - v(\varsigma)|\mathrm{d}\varsigma\right) \\ &\le \|\psi\|\left(\|y_n - y\| + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|v_n - v\| + KT\|v_n - v\|\right).\end{aligned}$$

Thus,

$$\|v_n - v\| \le \frac{\|\psi\|}{1 - \|\psi\|\left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)}\|y_n - y\|.$$

Since $y_n \to y$, then, we have $v_n(\tau) \to v(\tau)$ as $n \to \infty$ for each $\tau \in I$, let $\varepsilon > 0$ be shall ensure, for each $\tau \in I$, we obtain $|v_n(\tau)| \le \varepsilon$ and $|v(\tau)| \le \varepsilon$. Then,

$$|G(\tau, \varsigma)||v_n(\varsigma) - v(\varsigma)| \le |G(\tau, \varsigma)|[|v_n(\varsigma)| + |v(\varsigma)|] \le 2\varepsilon|G(\tau, \varsigma)|.$$

The function $\varsigma \to 2\varepsilon|G(\tau, \varsigma)|$ is integrable on I, for each $\tau \in I$. When using the Lebesgue dominated convergence theorem, and (11), it implies

$$\|A_2y_n - A_2y\| \to 0, \quad \text{as } n \to \infty.$$

Consequently, A_2 is continuous.

Furthermore, it is simple to confirm that

$$\|A_2y\| \le G_0\left(\frac{F + \|\psi\|\varrho}{1 - \aleph}\right)T \le \varrho,$$

according to the definitions of $\aleph$ and ϱ. This demonstrates that A_2 is uniformly bounded on B_ϱ.

In the end, we demonstrate that A_2 maps bounded sets into equicontinuous sets of $C(I, R)$, i.e., B_ϱ is equicontinuous.

Next, suppose that $\forall\varepsilon > 0$, $\exists\delta > 0$, and $\tau_1, \tau_2 \in I$, $\tau_1 < \tau_2$, $|\tau_2 - \tau_1| < \delta$. Then, we obtain

$$|A_2y(\tau_2) - A_2y(\tau_1)| \leq \int_0^T |G(\tau_2,\varsigma) - G(\tau_1,\varsigma)||v(\varsigma)|d\varsigma$$
$$\leq ||v|| \int_0^T |G(\tau_2,\varsigma) - G(\tau_1,\varsigma)|d\varsigma$$
$$\leq \frac{F + ||\psi||\varrho}{1 - \aleph} \int_0^T |G(\tau_2,\varsigma) - G(\tau_1,\varsigma)|d\varsigma.$$

As $\tau_1 \to \tau_2$, the right-hand side of the aforementioned inequality tends to zero and is independent of y.

Consequently,

$$|A_2y(\tau_2) - A_2y(\tau_1)| \to 0, \quad \forall |\tau_2 - \tau_1| \to 0.$$

As a result, $\{Ay\}$ is equi-continuous on B_ϱ according to the Arzela-Ascoli theorem [29], and A is a compact operator; we conclude that $A : C(I,R) \to C(I,R)$ is continuous and compact. As a result, all assumptions of Krasnoselskii's fixed point theorem are met and demonstrate that $A_1 + A_2$ has a fixed point on B_ϱ. Hence, there is a mild solution to the IFDP (1)–(3). □

Our second conclusion establishes the existence of a unique solution to IFDPs (1)–(3) using Banach's fixed point theorem.

Theorem 2. *Suppose that hypotheses of Theorem* 1 *meet, with*

$$c + \frac{G_\circ ||\psi||}{1 - ||\psi||\left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)} < 1. \tag{12}$$

Then, I has a unique mild solution on I provided by IFDP (1)–(3).

Proof. From Theorem 1, it is evident that IFDPs (1)–(3) have at least one solution. Therefore, all that remains is for us to demonstrate that the operator A mentioned in (13) is a contraction.

Now, select $x, y \in C(I,R)$. Then, for $\tau \in I$, we obtain

$$Ax(\tau) - Ay(\tau) = h(\tau, x(\tau)) + \int_0^T G(\tau,\varsigma)u(\varsigma)d\varsigma - h(\tau, y(\tau)) - \int_0^T G(\tau,\varsigma)v(\varsigma)d\varsigma, \tag{13}$$

where $u, v \in C(I,R)$ corresponds to:

$$u(\tau) = f\left(\tau, x(\tau), I^{\alpha-\beta}u(\tau), \int_0^\tau k(\tau,\varsigma)u(\varsigma)d\varsigma\right),$$
$$v(\tau) = f\left(\tau, y(\tau), I^{\alpha-\beta}v(\tau), \int_0^\tau k(\tau,\varsigma)v(\varsigma)d\varsigma\right).$$

Then, for $\tau \in I$,

$$|Ax(\tau) - Ay(\tau)| \leq |h(\tau, x(\tau)) - h(\tau, y(\tau))| + \int_0^T G(\tau,\varsigma)|u(\varsigma) - v(\varsigma)|d\varsigma. \tag{14}$$

However, by the assumption (H_2), we obtain

$$|u(\tau) - v(\tau)| = \left| f\left(\tau, x(\tau), I^{\alpha-\beta}u(\tau), \int_0^\tau k(\tau,\varsigma)u(\varsigma)\mathrm{d}\varsigma\right) - f\left(\tau, y(\tau), I^{\alpha-\beta}v(\tau), \int_0^\tau k(\tau,\varsigma)v(\varsigma)\mathrm{d}\varsigma\right) \right|$$
$$\leq |\psi(\tau)| \left(|x(\tau) - y(\tau)| + \int_0^\tau \frac{(\tau-\varsigma)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |u(\varsigma) - v(\varsigma)|\mathrm{d}\varsigma + \int_0^\tau k(\tau,\varsigma)|u(\varsigma) - v(\varsigma)|\mathrm{d}\varsigma \right)$$
$$\leq \|\psi\| \left(\|x - y\| + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \|u - v\| + K\|u - v\|T \right).$$

Thus,

$$\|u - v\| \leq \frac{\|\psi\|}{1 - \|\psi\| \left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right)} \|x - y\|.$$

Considering (14) and using Lemma 2, we obtain

$$|Ax(\tau) - Ay(\tau)| \leq c\|x - y\| + \frac{G_\circ \|\psi\|}{1 - \|\psi\| \left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right)} \|x - y\|$$
$$\leq \left(c + \frac{G_\circ \|\psi\|}{1 - \|\psi\| \left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right)} \right) \|x - y\|.$$

For $\tau \in I$, we obtain

$$\|Ax - Ay\| \leq \left(c + \frac{G_\circ \|\psi\|}{1 - \|\psi\| \left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right)} \right) \|x - y\|.$$

By $\left(c + \frac{G_\circ \| \psi\|}{1 - \| \psi \| \left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right)} \right) < 1$, we deduce that A is a contraction. As a result, according to Banach's contraction principle, A has a unique fixed point on I and that is a mild solution of IFDPs (1)–(3). □

We now take into account the Ulam stability for IFDPs (1)–(3). Suppose $\varepsilon > 0$ and $\Phi : I \to R_+$ be a continuous function. We take into account the following inequalities:

$$\left| {}^cD^\alpha y(\tau) - f\left(\tau, y(\tau), {}^cD^\beta y(\tau), \int_0^\tau k(\tau,\varsigma){}^cD^\alpha y(\varsigma)\mathrm{d}\varsigma\right) \right| \leq \varepsilon, \quad \tau \in I, \tag{15}$$

$$\left| {}^cD^\alpha y(\tau) - f\left(\tau, y(\tau), {}^cD^\beta y(\tau), \int_0^\tau k(\tau,\varsigma){}^cD^\alpha y(\varsigma)\mathrm{d}\varsigma\right) \right| \leq \Phi(\tau), \quad \tau \in I, \tag{16}$$

$$\left| {}^cD^\alpha y(\tau) - f\left(\tau, y(\tau), {}^cD^\beta y(\tau), \int_0^\tau k(\tau,\varsigma){}^cD^\alpha y(\varsigma)\mathrm{d}\varsigma\right) \right| \leq \varepsilon\Phi(\tau), \quad \tau \in I. \tag{17}$$

Definition 2. [30] The IFDPs (1)–(3) are Ulam-Hyers stable if there exists a real number $c_f > 0$ such that there exists a solution $x \in C(I, R)$ of (1)–(3) for:

$$|y(\tau) - x(\tau)| \leq \varepsilon c_f, t \in I,$$

for each solution $y \in C(I, R)$ of Inequality (15).

Definition 3. [30] The IFDPs (1)–(3) are generalized to be Ulam-Hyers stable if there is $c_f \in C(R_+, R_+)$ with $c_f(0) = 0$ so that there is a solution $x \in C(I, R)$ of (1)–(3) with:

$$|y(\tau) - x(\tau)| \leq c_f(\varepsilon), \quad \tau \in I,$$

for each $\varepsilon > 0$ and for each solution $y \in C(I, R)$ of Inequality (15).

Definition 4. [30] The IFDPs (1)–(3) are Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that there is a solution $x \in C(I, R)$ of (1)–(3) with:

$$|y(\tau) - x(\tau)| \leq \varepsilon c_{f,\Phi}\Phi(\tau), \quad \tau \in I,$$

for each $\varepsilon > 0$ and for each solution $y \in C(I, R)$ of Inequality (17).

Definition 5. [30] The IFDPs (3) is generalized to be Ulam-Hyers-Rassias stable with respect to Φ; if the actual number $c_{f,\Phi} > 0$ exists in such a way that for each solution $y \in C(I, R)$ of Inequality (16), there is a solution $x \in C(I, R)$ of (1)–(3) with the solution $x \in C(I, R)$ of the inequality:

$$|y(\tau) - x(\tau) > | \leq c_{f,\Phi}\Phi(\tau), \quad \tau \in I.$$

2.1 Ulam-Hyers stability

We now give the Ulam-Hyers stable finding that is shown in the following.

Theorem 3. *Suppose that hypothesis of Theorem* 2 *meet. Thus, IFDPs* (1)–(3) *are Ulam-Hyers stable.*

Proof. Take $\varepsilon > 0$ and let the function $z \in C(I, R)$ fulfill Inequality (15), i.e.,

$$\left| {}^cD^\alpha z(\tau) - f\left(\tau, z(\tau), {}^cD^\beta z(\tau), \int_0^\tau k(\tau, \varsigma){}^cD^\alpha z(\varsigma)\mathrm{d}\varsigma\right) \right| \leq \varepsilon, \quad \tau \in I,$$

and permit FIDEs (1)–(3) to have a unique solution $y \in C(I, R)$; consequently, Lemma 1 gives the equivalence between IFDPs (1)–(3) and integral equation:

$$y(\tau) = h(\tau, y(\tau)) + \int_0^T G(\tau, \varsigma)v(\varsigma)\mathrm{d}\varsigma,$$

where

$$v(\tau) = f\left(\tau, h(\tau, y(\tau)) + \frac{1}{\Gamma(\alpha)}\int_0^T G(\tau, \varsigma)v(\varsigma)\mathrm{d}\varsigma, I^{\alpha-\beta}v(\tau), \int_0^\tau k(\tau, \varsigma)v(\varsigma)\mathrm{d}\varsigma\right).$$

Operating by I^α on (15), and by integrating, we obtain

$$\left| z(\tau) - h(\tau, z(\tau)) - \int_0^T G(\tau, \varsigma)u(\varsigma)\mathrm{d}\varsigma \right| \leq \frac{\varepsilon T^\alpha}{\Gamma(\alpha + 1)}. \tag{18}$$

For each $\tau \in I$, we have

$$
\begin{aligned}
|z(\tau) - y(\tau)| &= \left| z(\tau) - h(\tau, y(\tau)) - \int_0^T G(\tau, \varsigma) v(\varsigma) \mathrm{d}\varsigma \right| \\
&\le \left| z(\tau) - h(\tau, z(\tau)) + \int_0^T G(\tau, \varsigma) u(\varsigma) \mathrm{d}\varsigma \right| + \left| h(\tau, z(\tau)) + \int_0^T G(\tau, \varsigma) u(\varsigma) \mathrm{d}\varsigma - h(\tau, y(\tau)) \right. \\
&\quad \left. - \int_0^T G(\tau, \varsigma) v(\varsigma) \mathrm{d}\varsigma \right| \\
&\le \frac{\varepsilon T^\alpha}{\Gamma(\alpha + 1)} + |h(\tau, z(\tau)) - h(\tau, y(\tau))| + \int_0^T G(\tau, \varsigma) |u(\varsigma) - v(\varsigma)| \mathrm{d}\varsigma \\
&\le \frac{\varepsilon T^\alpha}{\Gamma(\alpha + 1)} + c|z(\tau) - y(\tau)| + \int_0^T G(\tau, \varsigma) |u(\varsigma) - v(\varsigma)| \mathrm{d}\varsigma \\
&\le \frac{\varepsilon T^\alpha}{\Gamma(\alpha + 1)} + c\|z - y\| + G_\circ \|u - v\|.
\end{aligned}
$$

In actuality, the demonstration of Theorem 2 gives us:

$$
\|u - v\| \le \frac{\|\psi\|}{1 - \|\psi\|\left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)} \|z - y\|.
$$

Then, $\forall \tau \in I$

$$
\|z - y\| \le \frac{\varepsilon T^\alpha}{\Gamma(\alpha + 1)} + c\|z - y\| + \frac{G_\circ \|\psi\|}{1 - \|\psi\|\left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)} \|z - y\|.
$$

Thus,

$$
\|z - y\| \le \frac{\varepsilon T^\alpha}{\Gamma(\alpha + 1)} \left[1 - \left(c + \frac{G_\circ \|\psi\|}{1 - \|\psi\|\left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)}\right)\right]^{-1} = \varsigma\varepsilon,
$$

for let $\varsigma = \frac{T^\alpha}{\Gamma(\alpha+1)} \left[1 - \left(c + \frac{G_\circ \|\psi\|}{1 - \|\psi\|\left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)}\right)\right]^{-1}$. So, FIDEs (1)–(3) are Ulam-Hyers stable. □

By setting $\Phi(\varepsilon) = \varsigma\varepsilon$, $\Phi(0) = 0$. It can be demonstrated that FIDEs (1)–(3) are generalized to be Ulam-Hyers stable result.

2.2 Ulam-Hyers-Rassias stability

Now, we express the next Ulam-Hyers-Rassias stable result.

Theorem 4. *Assume assumptions* (H_1)–(H_3) *and*
(H_4) *The function* $\Phi \in C(I, R_+)$ *is increasing and there exists* $\lambda_\Phi > 0$ *such that, for each* $\tau \in J$, *we have*

$$
I^\alpha \Phi(\tau) \le \lambda_\Phi \Phi(\tau),
$$

which are satisfied. Then, IFDPs (1)–(3) *are Ulam-Hyers-Rassias stable with respect to* Φ.

Proof. Let $z \in C(I, R)$ be a solution of Inequality (17), i.e.,

$$\left| {}^{c}D^{\alpha}z(\tau) - f\left(\tau, z(\tau), {}^{c}D^{\beta}z(\tau), \int\limits_{0}^{\tau} k(\tau,\varsigma){}^{c}D^{\alpha}z(\varsigma)\mathrm{d}\varsigma\right)\right| \leq \varepsilon\Phi, \tau \in I.$$

Furthermore, suppose that y is a solution of Problems (1)–(3). As a result, we obtain

$$y(\tau) = h(\tau, y(\tau)) + \int\limits_{0}^{T} G(\tau,\varsigma)v(\varsigma)\mathrm{d}\varsigma,$$

with $v \in C(I, R)$

$$v(\tau) = f\left(\tau, y(\tau), I^{\alpha-\beta}v(\tau), \int\limits_{0}^{\tau} k(\tau,\varsigma)v(\varsigma)\mathrm{d}\varsigma\right).$$

Using I^{α} on both sides of Inequality (17) and integrating after that, we obtain

$$\left| z(\tau) - h(\tau, z(\tau)) - \int\limits_{0}^{T} G(\tau,\varsigma)u(\varsigma)\mathrm{d}\varsigma \right| \leq \frac{\varepsilon}{\Gamma(\alpha)}\int\limits_{0}^{\tau}(\tau-\varsigma)^{\alpha-1}\Phi(\varsigma)\mathrm{d}\varsigma \leq \varepsilon\lambda_{\Phi}\Phi(\tau),$$

with $u \in C(I, R)$

$$u(\tau) = f\left(\tau, z(\tau), I^{\alpha-\beta}u(\tau), \int\limits_{0}^{\tau} k(\tau,\varsigma)u(\varsigma)\mathrm{d}\varsigma\right).$$

For each $\tau \in I$, we have

$$\begin{aligned}
|z(\tau) - y(\tau)| = |z(\tau) - |h(\tau, y(\tau)) + \int\limits_{0}^{T} G(\tau,\varsigma)v(\varsigma)\mathrm{d}\varsigma| \\
&\leq \left| z(\tau) - h(\tau, z(\tau)) + \int\limits_{0}^{T} G(\tau,\varsigma)u(\varsigma)\mathrm{d}\varsigma \right| \\
&\quad + \left| h(\tau, z(\tau)) + \int\limits_{0}^{T} G(\tau,\varsigma)u(\varsigma)\mathrm{d}\varsigma - h(\tau, y(\tau)) - \int\limits_{0}^{T} G(\tau,\varsigma)v(\varsigma)\mathrm{d}\varsigma \right| \\
&\leq \varepsilon\lambda_{\Phi}\Phi(\tau) + |h(\tau, z(\tau)) - h(\tau, y(\tau))| + \int\limits_{0}^{T} G(\tau, s)|u(\varsigma) - v(\varsigma)|\mathrm{d}\varsigma \\
&\leq \varepsilon\lambda_{\Phi}\Phi(\tau) + c|z(\tau) - y(\tau)| + \int\limits_{0}^{T} G(\tau,\varsigma)|u(\varsigma) - v(\varsigma)|\mathrm{d}\varsigma \\
&\leq \varepsilon\lambda_{\Phi}\Phi(\tau) + c||z - y|| + G_{\circ}||u - v||.
\end{aligned}$$

In fact, from the evidence of Theorem 2,

$$||u - v|| \leq \frac{||\psi||}{1 - ||\psi||\left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)}||z - y||.$$

Hence, for each $\tau \in I$,

$$||z - y|| \leq \varepsilon\lambda_{\Phi}\Phi(\tau) + c||z - y|| + \frac{G_{\circ}||\psi||}{1 - ||\psi||\left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)}||z - y||.$$

Thus,

$$\|z-y\| \le \left[1-\left(c+\frac{G_\circ\|\psi\|}{1-\|\psi\|\left(KT+\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)}\right)\right]\varepsilon\lambda_\Phi\Phi(\tau) = c_\Phi\varepsilon\Phi(\tau),$$

where

$$c_\Phi = \left[1-\left(c+\frac{G_\circ\|\psi\|}{1-\|\psi\|\left(KT+\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)}\right)\right]\lambda_\Phi.$$

Therefore, IFDPs (1)–(3) are Ulam-Hyers-Rassias stable considering Φ. □

3 Examples

In this section, we give some examples to support the main results.

Example 1. Take the following IFDP into account:

$$^cD^{\frac{3}{2}}y(\tau) = \frac{2+y(\tau)+{}^cD^{\frac{4}{3}}y(\tau)+\int_0^1 e^{\tau-\varsigma c}D^{\frac{3}{2}}y(\varsigma)\mathrm{d}s}{2e^{\tau+1}\left(1+y(\tau)+{}^cD^{\frac{4}{3}}y(\tau)+\int_0^1 e^{\tau-\varsigma c}D^{\frac{3}{2}}y(\varsigma)\mathrm{d}s\right)}, \quad \tau\in[0,1] \tag{19}$$

$$y(0) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^1(1-\varsigma)^{\frac{1}{2}}\frac{y}{10e^{-\varsigma+2}(1+y)}\mathrm{d}\varsigma, \tag{20}$$

$$y'(T) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^1(1-\varsigma)^{\frac{1}{2}}\frac{\cos y}{30(\varsigma+2)}\mathrm{d}\varsigma. \tag{21}$$

Set

$$f(\tau,u,v,w) = \frac{2+|u|+|v|+|w|}{2e^{\tau+1}(1+|u|+|v|+|w|)}.$$

Note that f is the continuous function. For each $u_i, v_i, w_i \in R(i=1,2)$, and $\tau\in[0,1]$,

$$|f(\tau,u_1,v_1,w_1)-f(\tau,u_2,v_2,w_2)| \le \frac{1}{2e^2}(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|).$$

Thus, (H_2) is verified with $\psi(\tau)=\frac{1}{2e^{\tau+1}}$:

$$|f(\tau,u,v,w)| = \frac{1}{2e^{\tau+1}}(2+|u|+|v|+|w|),$$

where $f(\tau,0,0,0)=\frac{1}{e^{\tau+1}}$, and $\|\psi\|=\frac{1}{2e^2}$.

Set $h_1(\tau,x(\tau))=\frac{\cos x(\tau)}{30(\tau+2)}$ and $h_2(\tau,x(\tau))=\frac{x(\tau)}{10e^{-\tau+2}(1+x)}$, then

$$|h_1(\tau,x(\tau))-h_1(\tau,y(\tau))| \le \left|\frac{\cos x(\tau)}{30(\tau+2)}-\frac{\cos y(\tau)}{30(\tau+2)}\right| \le \frac{1}{30}|x(\tau)-y(\tau)|$$

and

$$|h_2(\tau, x(\tau)) - h_2(\tau, y(\tau))| \leq \left| \frac{x(\tau)}{10e^{-\tau+2}(1+x)} - \frac{y(\tau)}{10e^{-\tau+2}(1+y)} \right| \leq \frac{1}{10e}|x(\tau) - y(\tau)|.$$

Hence, the condition (H_1) is satisfied with $k_1 = \frac{1}{30}$ and $k_2 = \frac{1}{10e}$. From (5), clearly for $\alpha = \frac{3}{2}$, then $G_0 < 1$. Thus, condition

$$\frac{k_1 + \tau k_2}{\Gamma(\gamma+1)} + \frac{G_\circ \|\psi\|}{1 - \aleph} = 0.4103216862 < 1,$$

where $\aleph = \frac{\|\psi\| \, T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \|\psi\| KT = 0.2568803025$, is satisfied with $T = 1$, $\alpha = \frac{3}{2}$, $\beta = \frac{4}{3}$, $F = \frac{1}{e}$, $\|\psi\| = \frac{1}{2e^2} k_1 = \frac{1}{30}$, $k_2 = \frac{1}{10e}$, and $K = e$. We can see that all hypotheses of Theorem 1 are fulfilled. Consequently, IFDPs (22)–(24) have at least one mild solution on I.

Example 2. Take the following IFDP into account:

$$^{c}D^{\frac{4}{3}} = \frac{e^{-\tau}}{e^{\tau}+8}\left(\frac{|y(\tau)|}{1+|y(\tau)|} - \frac{\left|^{c}D^{\frac{5}{4}}y(\tau)\right|}{1+\left|^{c}D^{\frac{5}{4}}y(\tau)\right|} - \frac{\left|\int_0^1 \ln(\tau+s)^{c}D^{\frac{5}{4}}y(\tau)\right|}{1+\left|\int_0^1 \ln(\tau+\varsigma)^{c}D^{\frac{5}{4}}y(\tau)\right|} \right) \tag{22}$$

$$y(0) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^1 (1-\varsigma)^{\frac{1}{2}} \frac{\sin y(\varsigma)}{20} \mathrm{d}\varsigma, \tag{23}$$

$$y'(T) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^1 (1-\varsigma)^{\frac{1}{2}} \frac{e^{-y(\varsigma)}}{30} \mathrm{d}\varsigma. \tag{24}$$

Set

$$f(\tau, u, v, w) = \frac{e^{-\tau}}{e^{\tau}+8}\left(\frac{|u(\tau)|}{1+|u(\tau)|} - \frac{|v(\tau)|}{1+|v(\tau)|} - \frac{|w(\tau)|}{1+|w(\tau)|} \right).$$

Note that f is the continuous function. For each $u_i, v_i, w_i \in R (i = 1, 2)$, and $\tau \in [0, 1]$,

$$\begin{aligned} |f(\tau, u_1, v_1, w_1) - f(\tau, u_2, v_2, w_2)| &\leq \frac{e^{-\tau}}{e^{\tau}+8}(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|) \\ &\leq \frac{1}{9}(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|). \end{aligned}$$

Thus, (H_2) is verified with $\|\psi\| = \frac{1}{9}$. In addition, we obtain

$$|h(\tau, x(\tau)) - h(\tau, y(\tau))| \leq \frac{1}{1.772453} \int_0^1 (1-\varsigma)^{\frac{1}{2}} \frac{|\sin x(\varsigma) - \sin y(\varsigma)|}{20} \mathrm{d}\varsigma + \frac{1}{1.772453} \int_0^1 (1-\varsigma)^{\frac{1}{2}} \frac{|e^{-x(\varsigma)} - e^{-y(\varsigma)}|}{30} \mathrm{d}\varsigma.$$

Also, Lemma 2 holds, which means that h is the Lipschitz function with constant $c = 0.031348808$. Now,

$$G(\tau, \varsigma) = \begin{cases} \dfrac{(\tau-\varsigma)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{\tau(1-\varsigma)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \leq \varsigma \leq \tau < 1, \\ \dfrac{-\tau(1-\varsigma)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \leq \tau \leq \varsigma < 1. \end{cases} \tag{25}$$

Then,

$$\begin{aligned}\int_0^1 G(\tau,\varsigma)\mathrm{d}s &= \int_0^\tau G(\tau,\varsigma)\mathrm{d}\varsigma + \int_\tau^1 G(\tau,\varsigma)\mathrm{d}\varsigma \\ &= \frac{\tau^\alpha}{\Gamma(\alpha+1)} + \frac{\tau(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\tau}{\Gamma(\alpha)} - \frac{(\tau)(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \frac{\tau^\alpha}{\Gamma(\alpha+1)} - \frac{\tau}{\Gamma(\alpha)}.\end{aligned} \tag{26}$$

Clearly, for $\alpha = \frac{4}{3}$, then

$$G_0 = \max_{\tau,s\in[0,1]} \int_0^1 G(\tau,\varsigma)\mathrm{d}\varsigma < 1.$$

We shall check Condition (12). Indeed,

$$c + \frac{\|\psi\| G_\circ}{1 - \|\psi\|\left(KT + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)} = 0.1672284568 < 1,$$

which is satisfied with $\alpha = \frac{4}{3}$, $\beta = \frac{5}{4}$, $c = 0.03134$, $\|\psi\| = \frac{1}{9}$, and $K = \ln(2)$. The IFDPs (22)–(24) have a unique mild solution on I, as a result of Theorem 2.

4 Conclusion

In this study, we initially established the connection between the integration equation (4) and IFDPs (1)–(3) by analyzing the associated Green's function of the problem. Subsequently, utilizing the fixed point theorem of Krasnoselskii and the Banach contraction principle, we demonstrated the existence and uniqueness of mild solutions to boundary value problems of implicit fractional-order differential equations. Furthermore, we investigated the Hyers-Ulam and Hyers-Ulam-Rassias stability of IFDPs (1)–(3). We conclude the article with illustrative examples to demonstrate the relevance of the outcome. Our findings in the given context are novel and significantly add to the literature on this recently discovered area of research. Due to the small number of publications on implicit combined Caputo fractional differential equations, we believe that there are different research alternatives, such as coupled systems and problems with infinite delays.

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