

Research Article

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On the asymptotics of eigenvalues for a Sturm-Liouville problem with symmetric single-well potential

<https://doi.org/10.1515/dema-2023-0129>

received March 20, 2023; accepted November 14, 2023

Abstract: In this article, Sturm-Liouville problem with one boundary condition including an eigenparameter is considered, and the asymptotic expansion of its eigenparameter is calculated. The problem also has a symmetric single-well potential, which is an important function in quantum mechanics.

Keywords: Sturm-Liouville problem, symmetric single-well potential, eigenvalue parameter in the boundary condition, asymptotic expansions

MSC 2020: 34B30, 34L15

1 Introduction

In this study, we deal with the following equation:

$$y''(t) + [\lambda - q(t)]y(t) = 0, \quad t \in [0, a], \quad (1)$$

where λ is a real parameter and $q(t)$ is a real-valued, continuous function. This equation is known as the regular Sturm-Liouville equation. We consider the equation with the following boundary value conditions:

$$a_1 y(0) - a_2 y'(0) = \lambda [a'_1 y(0) - a'_2 y'(0)] \quad (2)$$

and

$$y(a) \cos \beta + y'(a) \sin \beta = 0, \quad (3)$$

where a_1, a_2, a'_1 , and a'_2 are real constants and $\beta \in [0, \pi)$. Equations (1)–(3) is different from the usual regular Sturm-Liouville problem because the eigenvalue parameter λ is held in the boundary condition at zero. Such problems often arise from physical problems, quantum mechanics, and geophysics. Fulton gives more than a hundred references in [1] and [2] (see also [3]), so his works serve as a historical guide. It is also shown by Walter [4] that this problem is a self-adjoint problem if the relation

$$\delta = \begin{vmatrix} a'_1 & a_1 \\ a'_2 & a_2 \end{vmatrix} > 0 \quad (4)$$

holds. There are a lot of studies with Sturm-Liouville problems where the spectral parameter appears in the boundary conditions, see, e.g., [5–12].

Besides, equation (1) is equal to one-dimensional Schrödinger equation, and especially in recent years, since quantum mechanics has gained importance, there have been a lot of studies on the eigenvalues of Hill's

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equation and Schrödinger's operator with symmetric single-well potential, such as anharmonic oscillator. The eigenvalues of these equations represent excitation energy, and eigenfunctions are named as wavefunctions in physics. A symmetric single-well potential on $[0, a]$ is defined as symmetric with respect to the midpoint $a/2$ and nonincreasing on $[0, a/2]$. The eigenvalue problems with symmetric single-well potential can be found in [13–18].

The purpose of this article is to obtain asymptotic approximations for eigenvalues λ_n of equations (1)–(3) with symmetric single-well potential $q(t)$ such that condition (4) is satisfied. We remark that a symmetric single-well potential on $[0, a]$ means that a continuous function $q(t)$ is symmetric on $[0, a]$ and nonincreasing on $[0, a/2]$, so we have $q(t) = q(a - t)$ mathematically. We assume, without loss of generality, that $q(t)$ has a mean value of zero.

2 The method

First, we note that $q'(t)$ exists since a monotone function on an interval I is differentiable almost everywhere on I [19]. After then, let us consider the following lemma:

Lemma 2.1. *The eigenvalues λ_n of equations (1)–(3) satisfy as $\lambda \rightarrow \infty$*

(i) $a'_2 \neq 0$ and $\beta \neq 0$

$$\begin{aligned} (n+1)\pi &= \int_0^a [r_2(t, \lambda) + \rho_2(t, \lambda)] dt \\ &\quad - \tan^{-1} \left\{ \frac{a_1 - a_2[r_1(0, \lambda) + \rho_1(0, \lambda)] - \lambda[a'_1 - a'_2(r_1(0, \lambda) + \rho_1(0, \lambda))]}{(a_2 - \lambda a'_2)[r_2(0, \lambda) + \rho_2(0, \lambda)]} \right\} \\ &\quad - \tan^{-1} \left\{ \frac{\cos \beta + [r_1(a, \lambda) + \rho_1(a, \lambda)] \sin \beta}{\sin \beta [r_2(a, \lambda) + \rho_2(a, \lambda)]} \right\} + O(\lambda^{-3/2}), \end{aligned}$$

(ii) $a'_2 \neq 0$ and $\beta = 0$

$$\begin{aligned} \frac{(2n+3)\pi}{2} &= \int_0^a [r_2(t, \lambda) + \rho_2(t, \lambda)] dt \\ &\quad - \tan^{-1} \left\{ \frac{a_1 - a_2[r_1(0, \lambda) + \rho_1(0, \lambda)] - \lambda[a'_1 - a'_2(r_1(0, \lambda) + \rho_1(0, \lambda))]}{(a_2 - \lambda a'_2)[r_2(0, \lambda) + \rho_2(0, \lambda)]} \right\} + O(\lambda^{-3/2}), \end{aligned}$$

where $r_1(t, \lambda) + \rho_1(t, \lambda)$ and $r_2(t, \lambda) + \rho_2(t, \lambda)$ are defined as follows:

$$\begin{aligned} r_1(t, \lambda) + \rho_1(t, \lambda) &= \frac{1}{2} \lambda^{-1/2} \int_0^t q'(x) \sin 2\lambda^{1/2}(t-x) dx \\ &\quad - \frac{1}{2} \lambda^{-1} \int_0^t q'(x) \left[\int_x^t q(s) ds \right] \cos 2\lambda^{1/2}(t-x) dx \\ &\quad + \frac{1}{4} \lambda^{-1} \int_0^t q^2(x) \cos 2\lambda^{1/2}(t-x) dx + O(\lambda^{-3/2}), \end{aligned} \tag{5}$$

and

$$\begin{aligned}
r_2(t, \lambda) + \rho_2(t, \lambda) &= \lambda^{1/2} - \frac{1}{2}\lambda^{-1/2}q(t) + \frac{1}{2}\lambda^{-1/2} \int_0^t q'(x) \cos 2\lambda^{1/2}(t-x) dx \\
&\quad + \frac{1}{2}\lambda^{-1} \int_0^t q'(x) \left[\int_x^t q(s) ds \right] \sin 2\lambda^{1/2}(t-x) dx \\
&\quad - \frac{1}{4}\lambda^{-1} \int_0^t q^2(x) \sin 2\lambda^{1/2}(t-x) dx + O(\lambda^{-3/2}).
\end{aligned} \tag{6}$$

Lemma 2.2. *The eigenvalues λ_n of equations (1)–(3) satisfy as $\lambda \rightarrow \infty$*

(i) $a'_2 = 0$ and $\beta \neq 0$

$$\begin{aligned}
(n+1)\pi &= \int_0^a [r_2(t, \lambda) + \rho_2(t, \lambda)] dt - \cot^{-1} \left[\frac{a_2[r_2(0, \lambda) + \rho_2(0, \lambda)]}{a_1 - a_2[r_1(0, \lambda) + \rho_1(0, \lambda)] - \lambda a'_1} \right] \\
&\quad - \tan^{-1} \left[\frac{\cos \beta + [r_1(a, \lambda) + \rho_1(a, \lambda)] \sin \beta}{\sin \beta [r_2(a, \lambda) + \rho_2(a, \lambda)]} \right] + O(\lambda^{-3/2}),
\end{aligned}$$

(ii) $a'_2 = 0$ and $\beta = 0$

$$\frac{(2n+3)\pi}{2} = \int_0^a [r_2(t, \lambda) + \rho_2(t, \lambda)] dt - \cot^{-1} \left[\frac{a_2[r_2(0, \lambda) + \rho_2(0, \lambda)]}{a_1 - a_2[r_1(0, \lambda) + \rho_1(0, \lambda)] - \lambda a'_1} \right] + O(\lambda^{-3/2}).$$

Proof. If we rearrange the main theorems of [6] in line with our aim for $N = 2$, we prove the lemmas, easily. $\square$

Lemmas 2.1 and 2.2 are important in literature, because they give good approximations of the asymptotic estimates of the eigenvalues with the continuous potential such that its derivative exists and is integrable. We note that our method is based on Lemmas 2.1 and 2.2.

3 Main results

In this article, we obtain the following asymptotic approximations for eigenvalues λ_n of equations (1)–(3) with symmetric single-well potential $q(t)$ by using Lemmas 2.1 and 2.2:

Theorem 3.1. *Let $q(t)$ be symmetric in equation (1). Then, the eigenvalues λ_n of (1)–(3) satisfy as $n \rightarrow \infty$*

(i) $a'_2 \neq 0$ and $\beta \neq 0$

$$\lambda_n^{1/2} = \frac{(n+1)\pi}{a} + \frac{1}{(n+1)\pi} \left[\frac{a'_1}{a'_2} + \cot \beta \right] - \frac{a}{2(n+1)^2\pi^2} \int_0^{a/2} q'(x) \sin \left(\frac{2(n+1)\pi}{a} x \right) dx + O(n^{-3}),$$

(ii) $a'_2 \neq 0$ and $\beta = 0$

$$\lambda_n^{1/2} = \frac{(2n+3)\pi}{2a} + \frac{2}{(2n+3)\pi} \frac{a'_1}{a'_2} + O(n^{-3}),$$

(iii) $a'_2 = 0$ and $\beta \neq 0$

$$\lambda_n^{1/2} = \frac{(2n+3)\pi}{2a} + \frac{2}{(2n+3)\pi} \left[\frac{a_2}{a'_1} + \cot \beta \right] + \frac{2a}{(2n+3)^2\pi^2} \int_0^{a/2} q'(x) \sin \left(\frac{(2n+3)\pi}{a} x \right) dx + O(n^{-3}),$$

(iv) $a'_2 = 0$ and $\beta = 0$

$$\lambda_n^{1/2} = \frac{(n+2)\pi}{a} + \frac{1}{(n+2)\pi} \frac{a_2}{a'_1} + \frac{a}{2(n+2)^2\pi^2} \int_0^{a/2} q'(x) \sin \left(\frac{2(n+2)\pi}{a} x \right) dx + O(n^{-3}).$$

Proof. (i) We calculate the terms in Lemmas 2.1(i). First, from equations (5) and (6), we find that

$$r_1(0, \lambda) + \rho_1(0, \lambda) = O(\lambda^{-3/2}),$$

$$r_2(0, \lambda) + \rho_2(0, \lambda) = \lambda^{1/2} - \frac{1}{2}\lambda^{-1/2}q(0) + O(\lambda^{-3/2}),$$

$$\begin{aligned} r_1(a, \lambda) + \rho_1(a, \lambda) &= \frac{1}{2}\lambda^{-1/2} \int_0^a q'(x) \sin 2\lambda^{1/2}(a-x) dx \\ &\quad - \frac{1}{2}\lambda^{-1} \int_0^a q'(x) \left[\int_x^a q(s) ds \right] \cos 2\lambda^{1/2}(a-x) dx \\ &\quad + \frac{1}{4}\lambda^{-1} \int_0^a q^2(x) \cos 2\lambda^{1/2}(a-x) dx + O(\lambda^{-3/2}), \end{aligned}$$

and

$$\begin{aligned} r_2(a, \lambda) + \rho_2(a, \lambda) &= \lambda^{1/2} - \frac{1}{2}\lambda^{-1/2}q(a) + \frac{1}{2}\lambda^{-1/2} \int_0^a q'(x) \cos 2\lambda^{1/2}(a-x) dx \\ &\quad + \frac{1}{2}\lambda^{-1} \int_0^a q'(x) \left[\int_x^a q(s) ds \right] \sin 2\lambda^{1/2}(a-x) dx \\ &\quad - \frac{1}{4}\lambda^{-1} \int_0^a q^2(x) \sin 2\lambda^{1/2}(a-x) dx + O(\lambda^{-3/2}). \end{aligned}$$

Therefore, if we define

$$\Omega = \frac{a_1 - \lambda a'_1 + O(\lambda^{-1/2})}{-\lambda^{3/2}a'_2 + \lambda^{1/2}a_2 + \frac{1}{2}\lambda^{1/2}a'_2q(0) - \frac{1}{2}\lambda^{-1/2}a_2q(0) + O(\lambda^{-1/2})} \quad (7)$$

and

$$\eta = \frac{\cos \beta + \sin \beta \left[\frac{1}{2}\lambda^{-1/2}A_2 - \frac{1}{2}\lambda^{-1}B_1 + \frac{1}{4}\lambda^{-1}C_1 \right] + O(\lambda^{-3/2})}{\sin \beta \left[\lambda^{1/2} - \frac{1}{2}\lambda^{-1/2}q(a) + \frac{1}{2}\lambda^{-1/2}A_1 + \frac{1}{2}\lambda^{-1}B_2 - \frac{1}{4}\lambda^{-1}C_2 \right] + O(\lambda^{-3/2})}, \quad (8)$$

where

$$\begin{aligned} A_1 &:= \int_0^a q'(x) \cos(2\lambda^{1/2}(a-x)) dx, \\ A_2 &:= \int_0^a q'(x) \sin(2\lambda^{1/2}(a-x)) dx, \\ B_1 &:= \int_0^a q'(x) \left[\int_x^a q(s) ds \right] \cos(2\lambda^{1/2}(a-x)) dx, \\ B_2 &:= \int_0^a q'(x) \left[\int_x^a q(s) ds \right] \sin(2\lambda^{1/2}(a-x)) dx, \\ C_1 &:= \int_0^a q^2(x) \cos(2\lambda^{1/2}(a-x)) dx, \\ C_2 &:= \int_0^a q^2(x) \sin(2\lambda^{1/2}(a-x)) dx, \end{aligned} \quad (9)$$

we can rearrange Lemmas 2.1(i) as follows:

$$(n+1)\pi = \int_0^a [r_2(t, \lambda) + \rho_2(t, \lambda)] dt - \tan^{-1}(\Omega) - \tan^{-1}(\eta) + O(\lambda^{-3/2}). \quad (10)$$

We will calculate the asymptotic eigenvalues from the last equation. By using series expansion for Ω , we obtain

$$\begin{aligned} \Omega &= \frac{-\lambda a_1' + a_1 + O(\lambda^{-1/2})}{-\lambda^{3/2} a_2' \left[1 - \frac{a_2}{a_2'} \lambda^{-1} - \frac{1}{2} \lambda^{-1} q(0) + O(\lambda^{-2}) \right]} \\ &= \left\{ \lambda^{-1/2} \frac{a_1'}{a_2'} - \lambda^{-3/2} \frac{a_1}{a_2'} + O(\lambda^{-2}) \right\} \left\{ 1 + \lambda^{-1} \frac{a_2}{a_2'} + \frac{1}{2} \lambda^{-1} q(0) + O(\lambda^{-2}) \right\} \\ &= \lambda^{-1/2} \frac{a_1'}{a_2'} + O(\lambda^{-3/2}). \end{aligned}$$

From the last equation and inverse tangent expansion $\tan^{-1}(\Omega) = \Omega - \frac{\Omega^3}{3} + \dots$, we find

$$\tan^{-1}(\Omega) = \lambda^{-1/2} \frac{a_1'}{a_2'} + O(\lambda^{-3/2}). \quad (11)$$

Similarly, we derive series expansion for η , and find that

$$\begin{aligned} \eta &= \frac{\cot \beta + \frac{1}{2} \lambda^{-1/2} A_2 - \frac{1}{2} \lambda^{-1} B_1 + \frac{1}{4} \lambda^{-1} C_1 + O(\lambda^{-3/2})}{\lambda^{1/2} \left[1 - \frac{1}{2} \lambda^{-1} q(a) + \frac{1}{2} \lambda^{-1} A_1 + \frac{1}{2} \lambda^{-3/2} B_2 - \frac{1}{4} \lambda^{-3/2} C_2 + O(\lambda^{-2}) \right]} \\ &= \left\{ \lambda^{-1/2} \cot \beta + \frac{1}{2} \lambda^{-1} A_2 - \frac{1}{2} \lambda^{-3/2} B_1 + \frac{1}{4} \lambda^{-3/2} C_1 + O(\lambda^{-2}) \right\} \\ &\quad \times \left\{ 1 + \frac{1}{2} \lambda^{-1} q(a) - \frac{1}{2} \lambda^{-1} A_1 - \frac{1}{2} \lambda^{-3/2} B_2 + \frac{1}{4} \lambda^{-3/2} C_2 + O(\lambda^{-2}) \right\} \\ &= \lambda^{-1/2} \cot \beta + \frac{1}{2} \lambda^{-1} A_2 + O(\lambda^{-3/2}). \end{aligned}$$

From the last equality, equation (9), and inverse tangent expansion, we obtain that

$$\tan^{-1}(\eta) = \lambda^{-1/2} \cot \beta + \frac{1}{2} \lambda^{-1} \int_0^a q'(x) \sin 2\lambda^{1/2}(a-x) dx + O(\lambda^{-3/2}).$$

We study with symmetric single-well potential $q(t)$. In this case, we have

$$\int_{a/2}^a q'(x) \sin 2\lambda^{1/2}(a-x) dx = - \int_{a/2}^0 q'(a-u) \sin 2\lambda^{1/2} u du = \int_{a/2}^0 q'(u) \sin 2\lambda^{1/2} u du.$$

The last equality holds because $q(t)$ is symmetric and $q'(t)$ exists, so $q'(t) = -q'(a-t)$. Then,

$$\begin{aligned} \int_0^a q'(x) \sin 2\lambda^{1/2}(a-x) dx &= \int_0^{a/2} q'(x) \sin 2\lambda^{1/2}(a-x) dx + \int_{a/2}^a q'(x) \sin 2\lambda^{1/2}(a-x) dx \\ &= \int_0^{a/2} q'(x) \sin 2\lambda^{1/2}(a-x) dx - \int_0^{a/2} q'(x) \sin 2\lambda^{1/2} x dx \\ &= \sin 2\lambda^{1/2} a \int_0^{a/2} q'(x) \cos 2\lambda^{1/2} x dx - \cos 2\lambda^{1/2} a \int_0^{a/2} q'(x) \sin 2\lambda^{1/2} x dx \\ &\quad - \int_0^{a/2} q'(x) \sin 2\lambda^{1/2} x dx \end{aligned}$$

so

$$\int_0^a q'(x) \sin 2\lambda^{1/2}(a-x) dx = \sin 2\lambda^{1/2} a \int_0^{a/2} q'(x) \cos 2\lambda^{1/2} x dx - [1 + \cos 2\lambda^{1/2} a] \int_0^{a/2} q'(x) \sin 2\lambda^{1/2} x dx. \quad (12)$$

If we use the last equation in $\tan^{-1}(\eta)$, we see that

$$\begin{aligned} \tan^{-1}(\eta) &= \lambda^{-1/2} \cot \beta + \frac{1}{2} \lambda^{-1} \sin 2\lambda^{1/2} a \int_0^{a/2} q'(x) \cos 2\lambda^{1/2} x dx \\ &\quad - \frac{1}{2} \lambda^{-1} [1 + \cos 2\lambda^{1/2} a] \int_0^{a/2} q'(x) \sin 2\lambda^{1/2} x dx. \end{aligned} \quad (13)$$

Now, we should compute $\int_0^a [r_2(t, \lambda) + \rho_2(t, \lambda)] dt$ to calculate equation (10). From equation (6), we have

$$\int_0^a [r_2(t, \lambda) + \rho_2(t, \lambda)] dt = \lambda^{1/2} \int_0^a 1 dt - \frac{1}{2} \lambda^{-1/2} \int_0^a q(t) dt + \frac{1}{2} \lambda^{-1/2} I_1 + \frac{1}{2} \lambda^{-1} I_2 - \frac{1}{4} \lambda^{-1} I_3 + O(\lambda^{-3/2}), \quad (14)$$

where

$$\begin{aligned} I_1 &:= \int_0^a \left\{ \int_0^t q'(x) \cos 2\lambda^{1/2}(t-x) dx \right\} dt, \\ I_2 &:= \int_0^a \left\{ \int_0^t q'(x) \left[\int_x^t q(s) ds \right] \sin 2\lambda^{1/2}(t-x) dx \right\} dt, \end{aligned}$$

and

$$I_3 := \int_0^a \left\{ \int_0^t q^2(x) \sin 2\lambda^{1/2}(t-x) dx \right\} dt.$$

The second term on the right side of equation (14) is zero, because $q(t)$ has a mean value of zero. Now, we apply Leibniz formula for integrals I_1 , I_2 , and I_3 as follows:

$$I_1 = \frac{1}{2\lambda^{1/2}} \left\{ \int_0^t q'(x) \sin 2\lambda^{1/2}(t-x) dx \right\} \Bigg|_{t=0}^a = \frac{1}{2} \lambda^{-1/2} \int_0^a q'(x) \sin 2\lambda^{1/2}(a-x) dx, \quad (15)$$

$$\begin{aligned} I_2 &= \left\{ -\frac{1}{2\lambda^{1/2}} \int_0^t q'(x) \left[\int_x^t q(s) ds \right] \cos 2\lambda^{1/2}(t-x) dx \right\} \Bigg|_{t=0}^a \\ &\quad + \left\{ \frac{1}{2\lambda^{1/2}} \int_0^t q(t) q'(x) \cos 2\lambda^{1/2}(t-x) dx \right\} \Bigg|_{t=0}^a \\ &= -\frac{1}{2} \lambda^{-1/2} \int_0^a q'(x) \left[\int_x^a q(s) ds \right] \cos 2\lambda^{1/2}(a-x) dx + \frac{1}{2} \lambda^{-1/2} q(a) \int_0^a q'(x) \cos 2\lambda^{1/2}(a-x) dx, \end{aligned} \quad (16)$$

and

$$I_3 = \left\{ -\frac{1}{2\lambda^{1/2}} \int_0^t q^2(x) \cos 2\lambda^{1/2}(t-x) dx \right\} \Bigg|_{t=0}^a + \left\{ \frac{1}{2\lambda^{1/2}} q^2(t) \right\} \Bigg|_{t=0}^a = -\frac{1}{2} \lambda^{-1/2} \int_0^a q^2(x) \cos 2\lambda^{1/2}(a-x) dx. \quad (17)$$

The last equality holds because $q(t) = q(a - t)$. Therefore, the fourth and fifth terms on the right side of (14) obtain into error term $O(\lambda^{-3/2})$ because of equations (16) and (17). So by using equations (12) and (15), we rewrite equation (14) as follows:

$$\begin{aligned} \int_0^a [r_2(t, \lambda) + \rho_2(t, \lambda)] dt &= \lambda^{1/2} a + \frac{1}{4} \lambda^{-1} \sin 2\lambda^{1/2} a \int_0^{a/2} q'(x) \cos 2\lambda^{1/2} x dx \\ &\quad - \frac{1}{4} \lambda^{-1} [1 + \cos 2\lambda^{1/2} a] \int_0^{a/2} q'(x) \sin 2\lambda^{1/2} x dx + O(\lambda^{-3/2}). \end{aligned} \quad (18)$$

Finally, substituting equations (11), (13), and (18) into equation (10) and using reversion, we prove the theorem.

(ii) Similar to (i), we can arrange Lemmas 2.1(ii) as follows:

$$\frac{(2n+3)\pi}{2} = \int_0^a [r_2(t, \lambda) + \rho_2(t, \lambda)] dt - \tan^{-1}(\Omega) + O(\lambda^{-3/2}),$$

where Ω is defined by equation (7). Theorem 3.1(ii) follows from the substitution of equations (11) and (18) into the last equation and using reversion.

(iii) We can reformulate Lemmas 2.2(i) as follows:

$$(n+1)\pi = \int_0^a [r_2(t, \lambda) + \rho_2(t, \lambda)] dt - \cot^{-1}(\delta) - \tan^{-1}(\eta) + O(\lambda^{-3/2}), \quad (19)$$

where η is defined by equation (8) and δ is defined by

$$\delta := \frac{\lambda^{1/2} a_2 - \frac{1}{2} \lambda^{-1/2} a_2 q(0) + O(\lambda^{-3/2})}{a_1 - \lambda a_1' + O(\lambda^{-3/2})}. \quad (20)$$

By using series expansion to δ , we obtain

$$\begin{aligned} \delta &= \frac{\lambda^{1/2} a_2 - \frac{1}{2} \lambda^{-1/2} a_2 q(0) + O(\lambda^{-3/2})}{-\lambda a_1' \left[1 - \lambda^{-1} \frac{a_1}{a_1'} + O(\lambda^{-5/2}) \right]} \\ &= \left\{ -\lambda^{-1/2} \frac{a_2}{a_1'} + \frac{1}{2} \lambda^{-3/2} \frac{a_2}{a_1'} q(0) + O(\lambda^{-5/2}) \right\} \\ &\quad \times \left\{ 1 + \lambda^{-1} \frac{a_1}{a_1'} + \lambda^{-2} \frac{a_1^2}{(a_1')^2} + O(\lambda^{-5/2}) \right\} \\ &= -\lambda^{-1/2} \frac{a_2}{a_1'} + O(\lambda^{-3/2}). \end{aligned}$$

From the last equation and inverse cotangent expansion $\cot^{-1}(\delta) = \frac{\pi}{2} - \delta + \frac{\delta^3}{3} + \dots$, we write that

$$\cot^{-1}(\delta) = \frac{\pi}{2} + \lambda^{-1/2} \frac{a_2}{a_1'} + O(\lambda^{-3/2}). \quad (21)$$

Substituting equations (13), (18), and (21) into equation (19), we prove the theorem.

(iv) We can reformulate Lemmas 2.2(ii) as follows:

$$\frac{(2n+3)\pi}{2} = \int_0^a [r_2(t, \lambda) + \rho_2(t, \lambda)] dt - \cot^{-1}(\delta) + O(\lambda^{-3/2}),$$

where δ is defined by equation (20). Theorem 3.1(iv) follows from substitution of equations (18) and (21) into the last equation and using reversion. $\square$

Example 1. Let us consider the following equation. This eigenvalue equation is named as anharmonic oscillator in physics:

$$y''(t) + [\lambda - q(t)]y(t) = 0, \quad t \in [0, \pi),$$

where $q(t) = \frac{1}{4}\left(t - \frac{\pi}{2}\right)^4 + \frac{1}{2}\left(t - \frac{\pi}{2}\right)^2$. This potential is symmetric single-well. Since we accepted $q(t)$ has mean value of zero for the sake of brevity, we should rearrange the anharmonic oscillator potential so that its mean value is zero. Therefore, $q(t)$ can be calculated as follows:

$$q(t) = \frac{1}{4}\left(t - \frac{\pi}{2}\right)^4 + \frac{1}{2}\left(t - \frac{\pi}{2}\right)^2 - \frac{\pi^2}{24} - \frac{\pi^4}{320}.$$

In this case, by calculating the integral terms in Theorem 3.1, we obtain the following as $n \rightarrow \infty$:

(i) if $a'_2 \neq 0$ and $\beta \neq 0$

$$\lambda_n^{1/2} = n + 1 + \frac{1}{(n+1)\pi} \left[\frac{a'_1}{a'_2} + \cot \beta \right] + \frac{\pi}{32(n+1)^5} [(\pi^2 + 4)n(n+2) + \pi^2 - 2] + O(n^{-3}),$$

(ii) if $a'_2 \neq 0$ and $\beta = 0$

$$\lambda_n^{1/2} = \frac{(2n+3)}{2} + \frac{2}{(2n+3)\pi} \frac{a'_1}{a'_2} + O(n^{-3}),$$

(iii) if $a'_2 = 0$ and $\beta \neq 0$

$$\begin{aligned} \lambda_n^{1/2} &= \frac{(2n+3)}{2} + \frac{2}{(2n+3)\pi} \left[\frac{a_2}{a'_1} + \cot \beta \right] + \frac{2}{8(2n+3)^6} \\ &\quad \times \{8(4n^2 + 12n + 3)(-1)^{n+1} - \pi(2n+3)[4(\pi^2 + 4)n(n+3) + 9\pi^2 + 12]\} + O(n^{-3}), \end{aligned}$$

(iv) if $a'_2 = 0$ and $\beta = 0$

$$\lambda_n^{1/2} = n + 2 + \frac{1}{(n+2)\pi} \frac{a_2}{a'_1} - \frac{\pi}{32(n+2)^5} [(\pi^2 + 4)n(n+4) + 4\pi^2 + 10] + O(n^{-3}).$$

Example 2. The Morse potential is a commonly accepted model for the description of a covalent bond like that between hydrogen and oxygen atoms. One can write the Morse potential in a compact form [20]:

$$q(t) = q_0 \left[\frac{1}{2}(-1 + A \cosh at)^2 + B \sinh at \right].$$

[20] studies for $A > 1, B = 0$ so that the potential is symmetric single-well in this situation. Hence, we can accept the Morse potential on $[0, 1]$ as follows:

$$q(t) = \frac{1}{2} \left[-1 + 2 \cosh \left(t - \frac{1}{2} \right) \right]^2.$$

Since we accepted that $q(t)$ has mean value of zero for the sake of brevity, we should rearrange the Morse potential so that its mean value is zero. Therefore, $q(t)$ can be calculated as follows:

$$q(t) = \frac{1}{2} \left[-1 + 2 \cosh \left(t - \frac{1}{2} \right) \right]^2 + \frac{e^{-1} \left[8e \sinh \left(\frac{1}{2} \right) - e^2 - 3e + 1 \right]}{2}.$$

For this potential, by calculating the integral terms in Theorem 3.1, the asymptotic eigenvalues of equations (1)–(3) are given easily.

4 Conclusions

The asymptotic approximations for eigenvalues λ_n of equations (1)–(3) with symmetric single-well potential $q(t)$ are obtained in this work. These approximations for eigenvalues have been obtained with better error

terms than previous works in the literature. When we applied the method from the beginning with a symmetric single potential, we obtained our results in Theorem 3.1 with better error terms than the results obtained by substituting the symmetric potential directly in the results in Lemmas 2.1 and 2.2. This is the importance of our results. Also, as a the result of the Theorem 3.1, we can say that, if we study with symmetric single well potential for (1.1)–(1.3), it is enough to observe the half interval instead of the whole interval of the problem. Because our result Theorem 3.1 can present the asymptotic eigenvalues on the half interval.

Acknowledgement: I would like to express my deepest gratitude to the referees and the handling editor for their valuable comments.

Funding information: The author states that no was funding involved.

Author contributions: The author has accepted responsibility for the entire content of this article and approved its submission.

Conflict of interest: The author states that there is no conflict of interest.

Ethical approval: The conducted research is not related to either human or animal use.

Data availability statement: No data were used to support the study.

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