

Research Article

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New local fractional Hermite-Hadamard-type and Ostrowski-type inequalities with generalized Mittag-Leffler kernel for generalized h -preinvex functions

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Abstract: In this study, based on two new local fractional integral operators involving generalized Mittag-Leffler kernel, Hermite-Hadamard inequality about these two integral operators for generalized h -preinvex functions is obtained. Subsequently, an integral identity related to these two local fractional integral operators is constructed to obtain some new Ostrowski-type local fractional integral inequalities for generalized h -preinvex functions. Finally, we propose three examples to illustrate the partial results and applications. Meanwhile, we also propose two midpoint-type inequalities involving generalized moments of continuous random variables to show the application of the results.

Keywords: local fractional integral operators, Mittag-Leffler kernel, Hermite-Hadamard-type inequalities, Ostrowski-type inequalities, generalized h -preinvex function

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1 Introduction

Let $\mathcal{G} : \Xi \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $a_1, a_2 \in \Xi$, $a_1 < a_2$, then

$$\mathcal{G}\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \mathcal{G}(\xi) d\xi \leq \frac{\mathcal{G}(a_1) + \mathcal{G}(a_2)}{2}. \quad (1.1)$$

If the function $\mathcal{G}$ is concave, then the aforementioned inequality is reversed. Inequality (1.1) related to convexity is called Hermite-Hadamard's inequality [1,2].

Let $\mathcal{G} : \Xi \rightarrow \mathbb{R}$ be a differentiable function in Ξ° (the interior of Ξ) and let $a_1, a_2 \in \Xi^\circ$, $a_1 < a_2$. If $|\mathcal{G}'(\xi)| \leq L$, for all $\xi \in [a_1, a_2]$, then

$$\left| \mathcal{G}(\xi) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \mathcal{G}(\varsigma) d\varsigma \right| \leq \frac{L}{a_2 - a_1} \left[\frac{(\xi - a_1)^2 + (a_2 - \xi)^2}{2} \right], \quad (1.2)$$

for all $\xi \in [a_1, a_2]$. This inequality is called the Ostrowski inequality [3].

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These two inequalities are related to the mean value of the integral of $\mathcal{G}(\xi)$ and can provide upper and lower bound estimates for the calculation of the integral mean value of the function $\mathcal{G}$. Thus, the two inequalities have very wide application significance in the fields of mathematics and engineering calculation. In order to further promote the study of these two kinds of inequalities, most scholars achieve some innovative results by considering different convexity, and the readers can refer to the literature [4–12].

As a mathematical tool, fractional calculus has obvious advantages over integer calculus in dealing with practical problems in nature and mathematical modeling. Because the order of fractional calculus is not limited to integers but also fractions, the mathematical modeling of fractional calculus is more accurate and reasonable in describing practical problems. For example, for the analysis and discussion of differential equation models and fractional order differential equation models, readers can refer to references [13–15]. Naturally, fractional calculus plays an important role in the study of integral inequality [16–22]. For instance, Asjad et al. [16] used nonsingular fractional integral operators to study the Hermite-Hadamard-type inequalities with exp-convexity. Tariq et al. [17] proposed some Simpson-Mercer-type inequalities including Atangana-Baleanu fractional operators. In [18], Tariq et al. also proposed some new Ostrowski-type inequalities pertaining to conformable fractional operators. Sahoo et al. [19] structured some Hermite-Hadamard-type fractional integral inequalities involving twice-differentiable mappings. In [20], Ahmad et al. presented two new fractional integral operators described by exponential kernel and proved some novel Hermite-Hadamard-type, Dragomir-Agarwal, and Pachpatte-type inequalities for convex functions. Next, Wu et al. [21] and Buduk et al. [22] also constructed some Hermite-Hadamard-type inequality and Ostrowski-type inequality for convex functions using these two integral operators, respectively.

Definition 1.1. [20] Let $\mathcal{G} \in L(a_1, a_2)$. The fractional integrals $I_{a_1}^\delta \mathcal{G}(\xi)$ and $I_{a_2}^\delta \mathcal{G}(\xi)$ of order $\delta \in (0, 1)$ are, respectively, defined by:

$$I_{a_1}^\delta \mathcal{G}(\xi) = \frac{1}{\delta} \int_{a_1}^{\xi} \exp\left(-\frac{1-\delta}{\delta}(\xi-\tau)\right) \mathcal{G}(\tau) d\tau, \quad \xi > a_1, \quad (1.3)$$

and

$$I_{a_2}^\delta \mathcal{G}(\xi) = \frac{1}{\delta} \int_{\xi}^{a_2} \exp\left(-\frac{1-\delta}{\delta}(\tau-\xi)\right) \mathcal{G}(\tau) d\tau, \quad \xi < a_2. \quad (1.4)$$

Fractal phenomenon is almost everywhere in nature, and fractal problem is a kind of non-differentiable function problem in mathematics, which is also known as the “mathematical pathological problem” in the world. Fractional calculus can deal with the classical power law phenomenon with continuity, but it cannot solve the phenomenon of discontinuity everywhere. Around the mathematical ill-posed problem that Newton-Leibniz calculus cannot deal with it, Yang proposed the local fractional derivative and local fractional integral of non-differentiable function. It is also called Yang’s fractal theory [23,24]. At present, the Yang’s fractal theory is widely applied in the fields of engineering mechanics and calculation of differential equations [25–30]. Based on the local fractional calculus theory, new achievements in the study of integral inequalities are emerging one after another [31–37]. For instance, in view of the Yang’s fractal theory, two local fractional integral operators with Mittag-Leffler kernel are proposed by Sun [38]. Subsequently, Sun [39] and Xu et al. [40] used them to study the Hermite-Hadamard-type local fractional integral inequalities for generalized preinvex functions and generalized fractal Jensen-Mercer-type inequalities for generalized h -convex functions, respectively.

Definition 1.2. [38] Let $\mathcal{G} : [a_1, a_2] \rightarrow \mathbb{R}^\delta$ be a function on Yang’s fractal sets and $\mathcal{G}(\xi)$ be a local fractional integrable function. The left-side integral operator $I_{a_1}^\delta \mathcal{G}$ and the right-side integral operator $I_{a_2}^\delta \mathcal{G}$ of order $\delta \in (0, 1)$ are, respectively, described as:

$$\mathbb{I}_{a_1^+}^\delta \mathcal{G}(\xi) = \frac{1}{\delta^\delta \Gamma(1 + \delta)} \int_{a_1}^\xi E_\delta \left(-\frac{1 - \delta}{\delta} (\xi - \tau) \right)^\delta \mathcal{G}(\tau) (d\tau)^\delta, \quad \xi > a_1, \quad (1.5)$$

and

$$\mathbb{I}_{a_2^-}^\delta \mathcal{G}(\xi) = \frac{1}{\delta^\delta \Gamma(1 + \delta)} \int_\xi^{a_2} E_\delta \left(-\frac{1 - \delta}{\delta} (\tau - \xi) \right)^\delta \mathcal{G}(\tau) (d\tau)^\delta, \quad \xi < a_2. \quad (1.6)$$

Remark 1.1. If $\delta = 1$, then

$$\lim_{\delta \rightarrow 1} \mathbb{I}_{a_1^+}^\delta \mathcal{G}(\xi) = \int_{a_1}^\xi \mathcal{G}(\tau) d\tau \quad \text{and} \quad \lim_{\delta \rightarrow 1} \mathbb{I}_{a_2^-}^\delta \mathcal{G}(\xi) = \int_\xi^{a_2} \mathcal{G}(\tau) d\tau.$$

Theorem 1.1. [38] Let $\mathcal{G} : [a_1, a_2] \rightarrow \mathbb{R}^\delta$ be a positive function with $0 \leq a_1 < a_2$, and $\mathcal{G}(\xi) \in \mathcal{I}_\xi^{(\delta)}[a_1, a_2]$. Let $\mathcal{G}$ be a generalized h convex function on $[a_1, a_2]$, and we have

$$\begin{aligned} \frac{1^\delta - E_\delta(-\rho)^\delta}{\rho^\delta h^\delta \left(\frac{1}{2} \right)} \mathcal{G} \left(\frac{a_1 + a_2}{2} \right) &\leq \left(\frac{\delta}{a_2 - a_1} \right)^\delta [\mathbb{I}_{a_1^+}^\delta \mathcal{G}(a_2) + \mathbb{I}_{a_2^-}^\delta \mathcal{G}(a_1)] \\ &\leq [\mathcal{G}(a_1) + \mathcal{G}(a_2)] {}_0I_1^{(\delta)} \{ E_\delta(-\rho\tau)^\delta [h^\delta(\tau) + h^\delta(1 - \tau)] \}, \end{aligned} \quad (1.7)$$

where $\rho = \frac{1-\delta}{\delta}(a_2 - a_1)$.

Theorem 1.2. [39] Let $\Omega \subseteq \mathbb{R}$ be an open invex set regarding $\eta : \Omega \times \Omega \rightarrow \mathbb{R}$, and $\mathcal{G} : \Omega \rightarrow \mathbb{R}^\delta$, $0 < \delta < 1$, be a positive function with $\eta(a_2, a_1) > 0$, $a_1, a_2 \in \Omega$, and $\mathcal{G}(\xi) \in \mathcal{I}_\xi^{(\delta)}[a_1, a_1 + \eta(a_2, a_1)]$. If $\mathcal{G}$ is a generalized preinvex function on Ω , and $\eta(\cdot, \cdot)$ meets Condition C, then

$$\begin{aligned} \mathcal{G} \left(a_1 + \frac{1}{2} \eta(a_2, a_1) \right) &\leq \frac{(1 - \delta)^\delta}{2^\delta [1^\delta - E_\delta(-\rho)^\delta]} [\mathbb{I}_{(a_1 + \eta(a_2, a_1))^-}^\delta \mathcal{G}(a_1) + \mathbb{I}_{a_1^+}^\delta \mathcal{G}(a_1 + \eta(a_2, a_1))] \\ &\leq \frac{\mathcal{G}(a_2) + \mathcal{G}(a_1)}{2^\delta}, \end{aligned} \quad (1.8)$$

where $\rho = \frac{1-\delta}{\delta} \eta(a_2, a_1)$.

To the authors' knowledge, there are currently not any studies on Ostrowski-type inequalities using integral operators (1.5) and (1.6). Inspired by the aforementioned references, with the local fractional integral operators (1.5) and (1.6) as auxiliary tools, we will construct the local fractional Hermite-Hadamard-type inequality and some novel Ostrowski-type inequalities for generalized h -preinvex functions on Yang's fractal sets. Some known results in the references or new conclusions can be obtained by taking some specific values for the parameters in the main results. In order to illustrate the correctness and application significance of the outcomes, we will also propose several examples and two midpoint-type inequalities related to moments of continuous random variables.

2 Preliminaries

This section will introduce the basic knowledge and some definitions of Yang's fractal theory. These preliminary knowledge is the theoretical basis for deriving the main results in the next section.

Using Yang's idea [23,24], recall Yang's fractal sets Ξ^δ , $0 < \delta \leq 1$, where the set Ξ is the base set of fractional set.

The δ -type integers $\mathbb{Z}^\delta$ is

$$\mathbb{Z}^\delta = \{0^\delta, \pm 1^\delta, \pm 2^\delta, \pm 3^\delta, \dots\}.$$

The δ -type rational numbers $\mathbb{Q}^\delta$ is

$$\mathbb{Q}^\delta = \left\{ q^\delta = \left(\frac{c}{d} \right)^\delta : c, d \in \mathbb{Z}, d \neq 0 \right\}.$$

The δ -type irrational numbers ∂^δ is

$$\partial^\delta = \left\{ \varrho^\delta \neq \left(\frac{c}{d} \right)^\delta : c, d \in \mathbb{Z}, d \neq 0 \right\}.$$

The δ -type real line numbers $\mathbb{R}^\delta$ is

$$\mathbb{R}^\delta = \mathbb{Q}^\delta \cup \partial^\delta.$$

Recall the following operation properties on $\mathbb{R}^\delta$. Note that δ represents the fractal dimension, not an exponential symbol.

If $\theta^\delta, \vartheta^\delta, \iota^\delta \in \mathbb{R}^\delta$, then

- (1) $\theta^\delta + \vartheta^\delta \in \mathbb{R}^\delta, \theta^\delta \vartheta^\delta \in \mathbb{R}^\delta$.
- (2) $\theta^\delta + \vartheta^\delta = \vartheta^\delta + \theta^\delta = (\theta + \vartheta)^\delta = (\vartheta + \theta)^\delta$.
- (3) $\theta^\delta + (\vartheta^\delta + \iota^\delta) = (\theta + \vartheta)^\delta + \iota^\delta$.
- (4) $\theta^\delta \vartheta^\delta = \vartheta^\delta \theta^\delta = (\theta \vartheta)^\delta = (\vartheta \theta)^\delta$.
- (5) $\theta^\delta (\vartheta^\delta \iota^\delta) = (\theta^\delta \vartheta^\delta) \iota^\delta$.
- (6) $\theta^\delta (\vartheta^\delta + \iota^\delta) = \theta^\delta \vartheta^\delta + \theta^\delta \iota^\delta$.
- (7) $\theta^\delta + 0^\delta = 0^\delta + \theta^\delta = \theta^\delta$ and $\theta^\delta 1^\delta = 1^\delta \theta^\delta = \theta^\delta$.
- (8) $(\theta - \vartheta)^\delta = \theta^\delta - \vartheta^\delta$.
- (9) For each $\theta^\delta \in \mathbb{R}^\delta$, its inverse element $(-\theta)^\delta$ may be written as $-\theta^\delta$; for each $\vartheta^\delta \in \mathbb{R}^\delta \setminus 0^\delta$, its inverse element $(\frac{1}{\vartheta})^\delta$ may be written as $1^\delta / \vartheta^\delta$ but not as $1 / \vartheta^\delta$.

Definition 2.1. [23,24] A non-differentiable function $\mathcal{G} : \mathbb{R} \rightarrow \mathbb{R}^\delta, \xi \rightarrow \mathcal{G}(\xi)$ is called local fractional continuous at ξ_0 , if for any $\varepsilon > 0$, there exists $\varepsilon > 0$, such that

$$|\mathcal{G}(\xi) - \mathcal{G}(\xi_0)| < \varepsilon^\delta$$

holds for $|\xi - \xi_0| < \varepsilon$. If $\mathcal{G}(\xi)$ is local fractional continuous on (a_1, a_2) , then it is denoted by $\mathcal{G}(\xi) \in C_\delta(a_1, a_2)$.

Definition 2.2. [23,24] The local fractional derivative of $\mathcal{G}(\xi)$ of order δ at $\xi = \xi_0$ is defined by:

$$\mathcal{G}^{(\delta)}(\xi_0) = \left. \frac{d^\delta \mathcal{G}(\xi)}{d\xi^\delta} \right|_{\xi=\xi_0} = \lim_{\xi \rightarrow \xi_0} \frac{\Gamma(\delta + 1)(\mathcal{G}(\xi) - \mathcal{G}(\xi_0))}{(\xi - \xi_0)^\delta}.$$

$D_\delta(a_1, a_2)$ represents the δ -local fractional derivative set.

Definition 2.3. [23,24] Let $\mathcal{G}(\xi) \in C_\delta[a_1, a_2]$. The local fractional integral of $\mathcal{G}(\xi)$ of order δ is defined by:

$${}_a \mathcal{I}_a^{(\delta)} \mathcal{G}(\xi) = \frac{1}{\Gamma(\delta + 1)} \int_{a_1}^{a_2} \mathcal{G}(\chi) (d\chi)^\delta = \frac{1}{\Gamma(\delta + 1)} \lim_{\Delta\chi \rightarrow 0} \sum_{j=0}^{N-1} \mathcal{G}(\chi_j) (\Delta\chi_j)^\delta,$$

where $a_1 = \chi_0 < \chi_1 < \dots < \chi_{N-1} < \chi_N = a_2$, $[\chi_j, \chi_{j+1}]$ is a partition of the interval $[a_1, a_2]$, $\Delta\chi_j = \chi_{j+1} - \chi_j$, $\Delta\chi = \max\{\Delta\chi_0, \Delta\chi_1, \dots, \Delta\chi_{N-1}\}$.

We denote $\mathcal{G}(\xi) \in \mathcal{I}_\xi^{(\delta)}[a_1, a_2]$ if there exists ${}_a \mathcal{I}_\xi^{(\delta)} \mathcal{G}(\xi)$ for any $\xi \in [a_1, a_2]$.

Definition 2.4. [23] Mittag-Leffler function on Yang's fractal sets is defined by:

$$E_\delta(\xi^\delta) = \sum_{\lambda=0}^{\infty} \frac{\xi^{\lambda\delta}}{\Gamma(1 + \lambda\delta)}, \quad \xi \in \mathbb{R},$$

which is also called the generalized Mittag-Leffler function.

Lemma 2.1. [23] *Integration and derivative for the generalized Mittag-Leffler function are as follows:*

$$\begin{aligned} {}_{a_1}I_{a_2}^{(\delta)} E_\delta(\xi^\delta) &= E_\delta(a_2^\delta) - E_\delta(a_1^\delta); \\ \frac{d^\delta E_\delta(\kappa \xi^\delta)}{d\xi^\delta} &= \kappa E_\delta(\kappa \xi^\delta), \quad \text{where } \kappa \text{ is a constant.} \end{aligned}$$

Lemma 2.2. [23,24]

(1) *(Integration is anti-differentiation) If $\mathcal{G}(\xi) = \mathcal{F}^{(\delta)}(\xi) \in C_\delta[a_1, a_2]$, then one can derive that*

$${}_{a_1}I_{a_2}^{(\delta)} \mathcal{G}(\xi) = \mathcal{F}(a_2) - \mathcal{F}(a_1).$$

(2) *(Integration by parts) If $\mathcal{G}(\xi), \mathcal{H}(\xi) \in D_\delta[a_1, a_2]$, and $\mathcal{G}^{(\delta)}(\xi), \mathcal{H}^{(\delta)}(\xi) \in C_\delta[a_1, a_2]$, then one can derive that*

$${}_{a_1}I_{a_2}^{(\delta)} \mathcal{G}(\xi) \mathcal{H}^{(\delta)}(\xi) = \mathcal{G}(\xi) \mathcal{H}(\xi)|_{a_1}^{a_2} - {}_{a_1}I_{a_2}^{(\delta)} \mathcal{G}^{(\delta)}(\xi) \mathcal{H}(\xi).$$

Lemma 2.3. [23,24] *Derivative and integration of $\xi^{\lambda\delta}$ satisfy*

$$\begin{aligned} \frac{d^\delta \xi^{\lambda\delta}}{d\xi^\delta} &= \frac{\Gamma(1 + \lambda\delta)}{\Gamma(1 + (\lambda - 1)\delta)} \xi^{(\lambda-1)\delta}, \\ \frac{1}{\Gamma(\delta + 1)} \int_{a_1}^{a_2} \xi^{\lambda\delta} (d\xi)^\delta &= \frac{\Gamma(1 + \lambda\delta)}{\Gamma(1 + (\lambda + 1)\delta)} (a_2^{(\lambda+1)\delta} - a_1^{(\lambda+1)\delta}), \end{aligned}$$

especially,

$${}_{a_1}I_{a_2}^{(\delta)} 1^\delta = \frac{(a_2 - a_1)^\delta}{\Gamma(1 + \delta)}.$$

Lemma 2.4. [23] *(Hölder-Yang's inequality) If $\Phi, \Psi \in C_\delta[a_1, a_2]$, $p, q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, then one can derive that*

$$\frac{1}{\Gamma(\delta + 1)} \int_{a_1}^{a_2} |\Phi(\xi) \Psi(\xi)| (d\xi)^\delta \leq \left(\frac{1}{\Gamma(\delta + 1)} \int_{a_1}^{a_2} |\Phi(\xi)|^p (d\xi)^\delta \right)^{1/p} \left(\frac{1}{\Gamma(\delta + 1)} \int_{a_1}^{a_2} |\Psi(\xi)|^q (d\xi)^\delta \right)^{1/q}.$$

Next, we will recall the definitions of generalized h -preinvex functions on Yang's fractal sets.

Definition 2.5. [41] Let $\Omega \subseteq \mathbb{R}$. If the set Ω satisfies

$$\xi_1, \xi_2 \in \Omega, 0 \leq \iota \leq 1 \Rightarrow \xi_2 + \iota\eta(\xi_1, \xi_2) \in \Omega,$$

then Ω is called the invex set regarding $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

Note that if $\eta(\xi_1, \xi_2) = \xi_1 - \xi_2$, then the invex set derives convex set.

Condition C. [42]: Let $\Omega \subset \mathbb{R}$ be an invex set, for any $\xi_1, \xi_2 \in \Omega$, $\iota \in [0, 1]$, the function $\eta(\cdot, \cdot)$ satisfies

$$\begin{aligned} \eta(\xi_2, \xi_2 + \iota\eta(\xi_1, \xi_2)) &= -\iota\eta(\xi_1, \xi_2), \\ \eta(\xi_1, \xi_2 + \iota\eta(\xi_1, \xi_2)) &= (1 - \iota)\eta(\xi_1, \xi_2). \end{aligned}$$

Remark 2.1. By Condition C, we are easy to obtain

$$\eta(\xi_2 + \iota_2\eta(\xi_1, \xi_2), \xi_2 + \iota_1\eta(\xi_1, \xi_2)) = (\iota_2 - \iota_1)\eta(\xi_1, \xi_2),$$

where $\xi_1, \xi_2 \in \Omega$, $\iota_1, \iota_2 \in [0, 1]$.

The generalized h -preinvex function on Yang's fractal sets proposed by Sun [43] is described as follows.

Definition 2.6. [43] Let $h : (0, 1) \rightarrow \mathbb{R}$ be a nonnegative function, $h^\delta \neq 0^\delta$, and let Ω be an invex set regarding $\eta(\cdot, \cdot)$. If for all $\xi_1, \xi_2 \in \Omega$ and $\iota \in (0, 1)$, we have

$$\mathcal{G}(\xi_2 + \iota\eta(\xi_1, \xi_2)) \leq h^\delta(\iota)\mathcal{G}(\xi_1) + h^\delta(1 - \iota)\mathcal{G}(\xi_2), \quad (2.1)$$

then we say that $\mathcal{G} : \Omega \rightarrow \mathbb{R}^\delta (0 < \delta \leq 1)$ is a generalized h -preinvex function regarding $\eta(\cdot, \cdot)$.

If Inequality (2.1) is reversed, then $\mathcal{G}$ is called the generalized h -preconcave regarding $\eta(\cdot, \cdot)$.

Remark 2.2. If $\delta = 1$, then the generalized h -preinvex derives the classical h -preinvex; if $h^\delta(\iota) = \iota^\delta$, then the generalized h -preinvex derives the generalized preinvex; if $\eta(\xi_1, \xi_2) = \xi_1 - \xi_2$, then the generalized h -preinvex derives the generalized h -convex.

3 Main results

First, we will derive the Hermite-Hadamard's local fractional integral inequalities regarding integral operators (1.5) and (1.6) for the generalized h -preinvex function. Hereinafter, we set $\rho = \frac{1-\delta}{\delta}\eta(a_2, a_1)$.

Theorem 3.1. Let $\Omega \subseteq \mathbb{R}$ be an open invex set regarding $\eta : \Omega \times \Omega \rightarrow \mathbb{R}$, and $\mathcal{G} : \Omega \rightarrow \mathbb{R}^\delta (0 < \delta < 1)$ be a generalized h -preinvex function on Ω with $\eta(a_2, a_1) > 0$, $a_1, a_2 \in \Omega$ for $\mathcal{G}(x) \in I_x^{(\delta)}[a_1, a_1 + \eta(a_2, a_1)]$. If $\eta(\cdot, \cdot)$ satisfies Condition C, then the following local fractional integral inequalities hold

$$\begin{aligned} \mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) &\leq \frac{(1-\delta)^\delta h^\delta\left(\frac{1}{2}\right)}{1^\delta - E_\delta\left(-\frac{\rho}{2}\right)^\delta} \left[\mathbb{I}_{\left(a_1 + \frac{\eta(a_2, a_1)}{2}\right)^\delta}^\delta \mathcal{G}(a_1) + \mathbb{I}_{\left(a_1 + \frac{\eta(a_2, a_1)}{2}\right)^\delta}^\delta \mathcal{G}(a_1 + \eta(a_2, a_1)) \right] \\ &\leq \frac{[\mathcal{G}(a_2) + \mathcal{G}(a_1)]\rho^\delta h^\delta\left(\frac{1}{2}\right)}{1^\delta - E_\delta\left(-\frac{\rho}{2}\right)^\delta} \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{1}{2}} [h^\delta(\tau) + h^\delta(1-\tau)] E_\delta(-\rho\tau)^\delta (d\tau)^\delta. \end{aligned} \quad (3.1)$$

Proof. Since $\Omega \subseteq \mathbb{R}$ is an open invex set regarding η and $a_1, a_2 \in \Omega$, we have $a_1 + \eta(a_2, a_1) \in \Omega$. Owing to the generalized h -preinvexity of $\mathcal{G}$ on Ω , we obtain

$$\mathcal{G}\left(v + \frac{1}{2}\eta(\omega, v)\right) \leq h^\delta\left(\frac{1}{2}\right)[\mathcal{G}(v) + \mathcal{G}(\omega)], \quad (3.2)$$

for $v, \omega \in \Omega$.

Using the variable substitutions with $\omega = a_1 + \tau\eta(a_2, a_1)$ and $v = a_1 + (1-\tau)\eta(a_2, a_1)$, we have

$$\begin{aligned} &\mathcal{G}\left(a_1 + (1-\tau)\eta(a_2, a_1) + \frac{1}{2}\eta(a_1 + \tau\eta(a_2, a_1), a_1 + (1-\tau)\eta(a_2, a_1))\right) \\ &\leq h^\delta\left(\frac{1}{2}\right)[\mathcal{G}(a_1 + \tau\eta(a_2, a_1)) + \mathcal{G}(a_1 + (1-\tau)\eta(a_2, a_1))]. \end{aligned} \quad (3.3)$$

By Condition C, we obtain

$$\eta(a_1 + \tau\eta(a_2, a_1), a_1 + (1 - \tau)\eta(a_2, a_1)) = (2\tau - 1)\eta(a_2, a_1). \quad (3.4)$$

By substituting equation (3.4) to Inequality (3.3), we obtain

$$\mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) \leq h^\delta \left(\frac{1}{2}\right) [\mathcal{G}(a_1 + \tau\eta(a_2, a_1)) + \mathcal{G}(a_1 + (1 - \tau)\eta(a_2, a_1))]. \quad (3.5)$$

Multiplying both sides of the aforementioned inequality by $E_\delta(-\rho\tau)^\delta$, and local fractional integrating the resulting inequality regarding t over $[0, \frac{1}{2}]$, we obtain

$$\begin{aligned} & \frac{\mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right)}{h^\delta\left(\frac{1}{2}\right)} \frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{1}{2}} E_\delta(-\rho\tau)^\delta (d\tau)^\delta \\ & \leq \frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{1}{2}} E_\delta(-\rho\tau)^\delta \mathcal{G}(a_1 + \tau\eta(a_2, a_1)) (d\tau)^\delta + \frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{1}{2}} E_\delta(-\rho\tau)^\delta \mathcal{G}(a_1 + (1 - \tau)\eta(a_2, a_1)) (d\tau)^\delta. \end{aligned} \quad (3.6)$$

From the aforementioned inequality, we deduce that

$$\begin{aligned} & \frac{\mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right)}{h^\delta\left(\frac{1}{2}\right)} \frac{1^\delta - E_\delta\left(-\frac{\rho}{2}\right)^\delta}{\rho^\delta} \\ & \leq \frac{\delta^\delta}{\eta^\delta(a_2, a_1)} \left[\frac{1}{\delta^\delta \Gamma(1 + \delta)} \int_{a_1}^{a_1 + \frac{\eta(a_2, a_1)}{2}} E_\delta\left(-\frac{1 - \delta}{\delta}(\omega - a_1)\right)^\delta \mathcal{G}(\omega) (d\omega)^\delta \right. \\ & \quad \left. + \frac{1}{\delta^\delta \Gamma(1 + \delta)} \int_{a_1 + \frac{\eta(a_2, a_1)}{2}}^{a_1 + \eta(a_2, a_1)} E_\delta\left(-\frac{1 - \delta}{\delta}(a_1 + \eta(a_2, a_1) - v)\right)^\delta \mathcal{G}(v) (dv)^\delta \right] \\ & = \frac{\delta^\delta}{\eta^\delta(a_2, a_1)} \left[\mathbb{I}_{(a_1 + \frac{\eta(a_2, a_1)}{2})^-}^\delta \mathcal{G}(a_1) + \mathbb{I}_{(a_1 + \frac{\eta(a_2, a_1)}{2})^+}^\delta \mathcal{G}(a_1 + \eta(a_2, a_1)) \right]. \end{aligned} \quad (3.7)$$

Thus, the left-side inequality of (3.1) holds.

For the right-side inequality, using the generalized h -preinvexity of $\mathcal{G}$, we have

$$\mathcal{G}(a_1 + \tau\eta(a_2, a_1)) \leq h^\delta(\tau)\mathcal{G}(a_2) + h^\delta(1 - \tau)\mathcal{G}(a_1)$$

and

$$\mathcal{G}(a_1 + (1 - \tau)\eta(a_2, a_1)) \leq h^\delta(1 - \tau)\mathcal{G}(a_2) + h^\delta(\tau)\mathcal{G}(a_1).$$

Adding the aforementioned two inequalities, we obtain

$$\mathcal{G}(a_1 + \tau\eta(a_2, a_1)) + \mathcal{G}(a_1 + (1 - \tau)\eta(a_2, a_1)) \leq [h^\delta(\tau) + h^\delta(1 - \tau)][\mathcal{G}(a_2) + \mathcal{G}(a_1)]. \quad (3.8)$$

Multiplying both sides of equation (3.8) by $E_\delta(-\rho\tau)^\delta$, and local fractional integrating the resulting inequality regarding τ over $[0, \frac{1}{2}]$, we deduce that

$$\begin{aligned} & \frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{1}{2}} E_\delta(-\rho\tau)^\delta \mathcal{G}(a_1 + \tau\eta(a_2, a_1)) (d\tau)^\delta + \frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{1}{2}} E_\delta(-\rho\tau)^\delta \mathcal{G}(a_1 + (1 - \tau)\eta(a_2, a_1)) (d\tau)^\delta \\ & \leq [\mathcal{G}(a_2) + \mathcal{G}(a_1)] \frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{1}{2}} [h^\delta(\tau) + h^\delta(1 - \tau)] E_\delta(-\rho\tau)^\delta (d\tau)^\delta. \end{aligned} \quad (3.9)$$

According to equations (3.6) and (3.7), the aforementioned inequality becomes

$$\begin{aligned} & \frac{\delta^\delta}{\eta^\delta(a_2, a_1)} \left[\mathbb{I}_{\left(a_1 + \frac{\eta(a_2, a_1)}{2}\right)^-}^\delta \mathcal{G}(a_1) + \mathbb{I}_{\left(a_1 + \frac{\eta(a_2, a_1)}{2}\right)^+}^\delta \mathcal{G}(a_1 + \eta(a_2, a_1)) \right] \\ & \leq [\mathcal{G}(a_2) + \mathcal{G}(a_1)] \frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{1}{2}} [h^\delta(\tau) + h^\delta(1 - \tau)] E_\delta(-\rho\tau)^\delta (d\tau)^\delta. \end{aligned} \quad (3.10)$$

By equations (3.7) and (3.10), we obtain

$$\begin{aligned} & \frac{\mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right)}{h^\delta\left(\frac{1}{2}\right)} \frac{1^\delta - E_\delta\left(-\frac{\rho}{2}\right)^\delta}{\rho^\delta} \\ & \leq \frac{\delta^\delta}{\eta^\delta(a_2, a_1)} \left[\mathbb{I}_{\left(a_1 + \frac{\eta(a_2, a_1)}{2}\right)^-}^\delta \mathcal{G}(a_1) + \mathbb{I}_{\left(a_1 + \frac{\eta(a_2, a_1)}{2}\right)^+}^\delta \mathcal{G}(a_1 + \eta(a_2, a_1)) \right] \\ & \leq [\mathcal{G}(a_2) + \mathcal{G}(a_1)] \frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{1}{2}} [h^\delta(\tau) + h^\delta(1 - \tau)] E_\delta(-\rho\tau)^\delta (d\tau)^\delta. \end{aligned} \quad (3.11)$$

This completes the proof. $\square$

Remark 3.1. For $\delta \rightarrow 1$, we have

$$\lim_{\delta \rightarrow 1} \frac{(1 - \delta)^\delta}{1^\delta - E_\delta\left(-\frac{\rho}{2}\right)^\delta} = \frac{2}{\eta(a_2, a_1)}$$

and

$$\lim_{\delta \rightarrow 1} \frac{\rho^\delta}{1^\delta - E_\delta\left(-\frac{\rho}{2}\right)^\delta} = 2.$$

Thus, for $\delta \rightarrow 1$, by equation (3.1), we have the following inequalities for h -preinvex functions:

$$\mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) \leq \frac{2h\left(\frac{1}{2}\right)}{\eta(a_2, a_1)} \int_{a_1}^{a_1 + \eta(a_2, a_1)} \mathcal{G}(x) dx \leq 2h\left(\frac{1}{2}\right) [\mathcal{G}(a_2) + \mathcal{G}(a_1)] \int_0^{\frac{1}{2}} [h(\tau) + h(1 - \tau)] d\tau. \quad (3.12)$$

Remark 3.2. In equation (3.12), if $h(\tau) = \tau$, we obtain the the following Hermite-Hadamard-Noor-type inequality [44]:

$$\mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) \leq \frac{1}{\eta(a_2, a_1)} \int_{a_1}^{a_1 + \eta(a_2, a_1)} \mathcal{G}(x) dx \leq \frac{\mathcal{G}(a_2) + \mathcal{G}(a_1)}{2}. \quad (3.13)$$

Corollary 3.1. If we set $\eta(a_2, a_1) = a_2 - a_1$ in Theorem 3.1 with $a_2 > a_1$, then we obtain the following local fractional inequalities for the generalized h -convex functions:

$$\begin{aligned}
\mathcal{G}\left(\frac{a_1 + a_2}{2}\right) &\leq \frac{(1 - \delta)^\delta h^\delta\left(\frac{1}{2}\right)}{1^\delta - E_\delta\left(-\frac{\rho}{2}\right)^\delta} \left[\mathbb{I}_{\left(\frac{a_1+a_2}{2}\right)^-}^\delta \mathcal{G}(a_1) + \mathbb{I}_{\left(\frac{a_1+a_2}{2}\right)^+}^\delta \mathcal{G}(a_2) \right] \\
&\leq \frac{[\mathcal{G}(a_2) + \mathcal{G}(a_1)] \rho^\delta h^\delta\left(\frac{1}{2}\right)}{1^\delta - E_\delta\left(-\frac{\rho}{2}\right)^\delta} \frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{1}{2}} [h^\delta(\tau) + h^\delta(1 - \tau)] E_\delta(-\rho\tau)^\delta (d\tau)^\delta.
\end{aligned} \tag{3.14}$$

Corollary 3.2. If we set $h^\delta(\tau) = \tau^\delta$ in Theorem 3.1, then we obtain the following local fractional inequalities for generalized preinvex functions:

$$\begin{aligned}
\mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) &\leq \frac{(1 - \delta)^\delta}{2^\delta \left[1^\delta - E_\delta\left(-\frac{\rho}{2}\right)^\delta\right]} \left[\mathbb{I}_{\left(a_1 + \frac{\eta(a_2, a_1)}{2}\right)^-}^\delta \mathcal{G}(a_1) + \mathbb{I}_{\left(a_1 + \frac{\eta(a_2, a_1)}{2}\right)^+}^\delta \mathcal{G}(a_1 + \eta(a_2, a_1)) \right] \\
&\leq \frac{\mathcal{G}(a_2) + \mathcal{G}(a_1)}{2^\delta}.
\end{aligned} \tag{3.15}$$

Proof. By $h^\delta(\tau) = \tau^\delta$, we obtain

$$\frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{1}{2}} [h^\delta(\tau) + h^\delta(1 - \tau)] E_\delta(-\rho\tau)^\delta (d\tau)^\delta = \frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{1}{2}} E_\delta(-\rho\tau)^\delta (d\tau)^\delta = \frac{1^\delta - E_\delta\left(-\frac{\rho}{2}\right)^\delta}{\rho^\delta}. \tag{3.16}$$

Substituting equation (3.16) into equation (3.1), by calculation, it is known that the result is true.

Now, we will give a lemma as an auxiliary tool to derive the main results. $\square$

Lemma 3.1. Let $\Omega \subseteq R$ be an open invex set regarding $\eta : \Omega \times \Omega \rightarrow R$ and $\mathcal{G} : \Omega \rightarrow \mathbb{R}^\delta (0 < \delta < 1)$ be a function with $\eta(a_2, a_1) > 0$, $a_1, a_2 \in \Omega$ and $a_1 < a_2$. If $\mathcal{G}^{(\delta)}(x) \in \mathcal{I}_x^{(\delta)}[a_1, a_1 + \eta(a_2, a_1)]$, then for all $x \in [a_1, a_1 + \eta(a_2, a_1)]$, the following local fractional integral identity holds

$$\begin{aligned}
&\frac{(1 - \delta)^\delta}{2^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(x - a_1)\right)^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(a_1 + \eta(a_2, a_1) - x)\right)^\delta} (\mathbb{I}_x^\delta \mathcal{G}(a_1) + \mathbb{I}_x^\delta \mathcal{G}(a_1 + \eta(a_2, a_1))) - \mathcal{G}(x) \\
&= \frac{\eta^\delta(a_2, a_1)}{2^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(x - a_1)\right)^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(a_1 + \eta(a_2, a_1) - x)\right)^\delta} \\
&\times \left[\frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (E_\delta(-\rho\tau)^\delta - 1^\delta) \mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1)) (d\tau)^\delta \right. \\
&\quad \left. + \frac{1}{\Gamma(1 + \delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1 - \tau))^\delta) \mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1)) (d\tau)^\delta \right].
\end{aligned} \tag{3.17}$$

Proof. Since $\Omega \subseteq R$ is an open invex set regarding η and $a_1, a_2 \in \Omega$, we have $a_1 + \eta(a_2, a_1) \in \Omega$. Using local fractional integration by parts and letting $a_1 + \tau\eta(a_2, a_1) = u$, $\tau \in [0, 1]$, we obtain

$$\begin{aligned}
& \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (E_\delta(-\rho\tau)^\delta - 1^\delta) \mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1)) (d\tau)^\delta \\
&= \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (E_\delta(-\rho\tau)^\delta) \mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1)) (d\tau)^\delta - \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} \mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1)) (d\tau)^\delta \\
&= \left(\frac{1}{\eta(a_2, a_1)} \right)^\delta \left[E_\delta(-\rho\tau)^\delta \mathcal{G}(a_1 + \tau\eta(a_2, a_1)) \Big|_0^{\frac{x-a_1}{\eta(a_2, a_1)}} - \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} \mathcal{G}(a_1 \right. \\
&\quad \left. + \tau\eta(a_2, a_1)) (E_\delta(-\rho\tau)^\delta)^{(\delta)} (d\tau)^\delta \right] - \left(\frac{1}{\eta(a_2, a_1)} \right)^\delta \mathcal{G}(a_1 + \tau\eta(a_2, a_1)) \Big|_0^{\frac{x-a_1}{\eta(a_2, a_1)}} \\
&= \left(\frac{1}{\eta(a_2, a_1)} \right)^\delta \left[E_\delta \left(-\frac{1-\delta}{\delta} (x-a_1) \right)^\delta \mathcal{G}(x) - \mathcal{G}(a_1) + \left(\frac{1-\delta}{\delta} \right)^\delta \frac{1}{\Gamma(1+\delta)} \int_{a_1}^x \mathcal{G}(u) E_\delta \left(-\frac{1-\delta}{\delta} (u-a_1) \right)^\delta (du)^\delta \right] \\
&\quad - \left(\frac{1}{\eta(a_2, a_1)} \right)^\delta [\mathcal{G}(x) - \mathcal{G}(a_1)] \\
&= \left(\frac{1}{\eta(a_2, a_1)} \right)^\delta \left[E_\delta \left(-\frac{1-\delta}{\delta} (x-a_1) \right)^\delta \mathcal{G}(x) - \mathcal{G}(x) + (1-\delta) \mathbb{I}_x^\delta \mathcal{G}(a_1) \right].
\end{aligned} \tag{3.18}$$

Similarly, we have

$$\begin{aligned}
& \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) \mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1)) (d\tau)^\delta \\
&= \left(\frac{1}{\eta(a_2, a_1)} \right)^\delta \left[-\mathcal{G}(x) + E_\delta \left(-\frac{1-\delta}{\delta} (a_1 + \eta(a_2, a_1) - x) \right)^\delta \mathcal{G}(x) + (1-\delta) \mathbb{I}_x^{\delta,+} \mathcal{G}(a_1 + \eta(a_2, a_1)) \right].
\end{aligned} \tag{3.19}$$

Adding equations (3.18) and (3.19), we obtain

$$\begin{aligned}
& \left(\frac{1}{\eta(a_2, a_1)} \right)^\delta [(1-\delta)^\delta (\mathbb{I}_x^\delta \mathcal{G}(a_1) + \mathbb{I}_x^{\delta,+} \mathcal{G}(a_1 + \eta(a_2, a_1)))] \\
&\quad - \left[2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x-a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_1 + \eta(a_2, a_1) - x) \right)^\delta \right] \mathcal{G}(x) \\
&= \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (E_\delta(-\rho\tau)^\delta - 1^\delta) \mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1)) (d\tau)^\delta \\
&\quad + \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) \mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1)) (d\tau)^\delta.
\end{aligned} \tag{3.20}$$

This completes the proof. $\square$

Remark 3.3. For $\delta \rightarrow 1$, we have

$$\lim_{\delta \rightarrow 1} \frac{(1 - \delta)^\delta}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x - a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_1 + \eta(a_2, a_1) - x) \right)^\delta} = \frac{1}{\eta(a_2, a_1)},$$

$$\lim_{\delta \rightarrow 1} \frac{E_\delta(-\rho\tau)^\delta - 1^\delta}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x - a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_1 + \eta(a_2, a_1) - x) \right)^\delta} = -\tau,$$

and

$$\lim_{\delta \rightarrow 1} \frac{1^\delta - E_\delta(-\rho(1 - \tau))^\delta}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x - a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_1 + \eta(a_2, a_1) - x) \right)^\delta} = 1 - \tau.$$

Thus, for $\delta \rightarrow 1$, Identity (3.17) becomes

$$\begin{aligned} & \frac{1}{\eta(a_2, a_1)} \int_{a_1}^{a_1 + \eta(a_2, a_1)} \mathcal{G}(u) du - \mathcal{G}(x) \\ &= \eta(a_2, a_1) \left[\int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1 - \tau) \mathcal{G}'(a_1 + \tau \eta(a_2, a_1)) d\tau - \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} \tau \mathcal{G}'(a_1 + \tau \eta(a_2, a_1)) d\tau \right]. \end{aligned}$$

Using Lemma 3.1, we can obtain the following Ostrowski-type local fractional integral inequalities.

Theorem 3.2. Let $\Omega \subseteq \mathbb{R}$ be an open invex set regarding $\eta : \Omega \times \Omega \rightarrow \mathbb{R}$ and $\mathcal{G} : \Omega \rightarrow \mathbb{R}^\delta$ ($0 < \delta < 1$) be a function with $\eta(a_2, a_1) > 0$, $a_1, a_2 \in \Omega$, and $\mathcal{G}^{(\delta)}(x) \in \mathcal{I}_x^{(\delta)}[a_1, a_1 + \eta(a_2, a_1)]$. If $|\mathcal{G}^{(\delta)}|$ is a generalized h -preinvex function on Ω , then the following Ostrowski-type local fractional integral inequality holds

$$\begin{aligned} & \left| \frac{(1 - \delta)^\delta}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x - a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_1 + \eta(a_2, a_1) - x) \right)^\delta} (\mathbb{I}_x^\delta \mathcal{G}(a_1) + \mathbb{I}_x^\delta \mathcal{G}(a_1 + \eta(a_2, a_1))) - \mathcal{G}(x) \right| \\ & \leq \frac{\eta^\delta(a_2, a_1)}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x - a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_1 + \eta(a_2, a_1) - x) \right)^\delta} \\ & \quad \times \left[\left(\frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta) h^\delta(\tau) (d\tau)^\delta \right. \right. \\ & \quad \left. \left. + \frac{1}{\Gamma(1 + \delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1 - \tau))^\delta) h^\delta(\tau) (d\tau)^\delta \right) |\mathcal{G}^{(\delta)}(a_2)| \right. \\ & \quad \left. + \left(\frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta) h^\delta(1 - \tau) (d\tau)^\delta \right. \right. \\ & \quad \left. \left. + \frac{1}{\Gamma(1 + \delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1 - \tau))^\delta) h^\delta(1 - \tau) (d\tau)^\delta \right) |\mathcal{G}^{(\delta)}(a_1)| \right]. \end{aligned} \quad (3.21)$$

Proof. Since $\Omega \subseteq R$ is an open invex set regarding η and $a_1, a_2 \in \Omega$, we have $a_1 + \eta(a_2, a_1) \in \Omega$. Since $|\mathcal{G}^{(\delta)}|$ is a generalized h -preinvex function on Ω , we obtain

$$|\mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1))| \leq h^\delta(\tau)|\mathcal{G}^{(\delta)}(a_2)| + h^\delta(1 - \tau)|\mathcal{G}^{(\delta)}(a_1)|, \quad (3.22)$$

for $\tau \in [0, 1]$. Using Lemma 3.1, by the properties of modules, it follows that

$$\begin{aligned} & \left| \frac{(1 - \delta)^\delta}{2^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(x - a_1)\right)^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(a_1 + \eta(a_2, a_1) - x)\right)^\delta} (\mathbb{I}_x^\delta \mathcal{G}(a_1) + \mathbb{I}_x^\delta \mathcal{G}(a_1 + \eta(a_2, a_1))) - \mathcal{G}(x) \right| \\ & \leq \frac{\eta^\delta(a_2, a_1)}{2^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(x - a_1)\right)^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(a_1 + \eta(a_2, a_1) - x)\right)^\delta} \\ & \quad \times \left[\frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta) |\mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1))| (d\tau)^\delta \right. \\ & \quad \left. + \frac{1}{\Gamma(1 + \delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1 - \tau))^\delta) |\mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1))| (d\tau)^\delta \right] \\ & \leq \frac{\eta^\delta(a_2, a_1)}{2^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(x - a_1)\right)^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(a_1 + \eta(a_2, a_1) - x)\right)^\delta} \\ & \quad \times \left[\frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta) (h^\delta(\tau)|\mathcal{G}^{(\delta)}(a_2)| + h^\delta(1 - \tau)|\mathcal{G}^{(\delta)}(a_1)|) (d\tau)^\delta \right. \\ & \quad \left. + \frac{1}{\Gamma(1 + \delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1 - \tau))^\delta) (h^\delta(\tau)|\mathcal{G}^{(\delta)}(a_2)| + h^\delta(1 - \tau)|\mathcal{G}^{(\delta)}(a_1)|) (d\tau)^\delta \right] \quad (3.23) \\ & = \frac{\eta^\delta(a_2, a_1)}{2^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(x - a_1)\right)^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(a_1 + \eta(a_2, a_1) - x)\right)^\delta} \\ & \quad \times \left[\left(\frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta) h^\delta(\tau) (d\tau)^\delta \right. \right. \\ & \quad \left. \left. + \frac{1}{\Gamma(1 + \delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1 - \tau))^\delta) h^\delta(\tau) (d\tau)^\delta \right) |\mathcal{G}^{(\delta)}(a_2)| \right. \\ & \quad \left. + \left(\frac{1}{\Gamma(1 + \delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta) h^\delta(1 - \tau) (d\tau)^\delta \right. \right. \\ & \quad \left. \left. + \frac{1}{\Gamma(1 + \delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1 - \tau))^\delta) h^\delta(1 - \tau) (d\tau)^\delta \right) |\mathcal{G}^{(\delta)}(a_1)| \right]. \end{aligned}$$

This completes the proof. $\square$

Remark 3.4. For $\delta \rightarrow 1$ in Theorem 3.2, by Remark 3.3, we have

$$\begin{aligned} & \left| \frac{1}{\eta(a_2, a_1)} \int_{a_1}^{a_1 + \eta(a_2, a_1)} \mathcal{G}(u) du - \mathcal{G}(x) \right| \\ & \leq \eta(a_2, a_1) \left[\left| \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} \tau h(\tau) d\tau + \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1-\tau) h(\tau) d\tau \right| |\mathcal{G}'(a_2)| \right. \\ & \quad \left. + \left| \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} \tau h(1-\tau) d\tau + \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1-\tau) h(1-\tau) d\tau \right| |\mathcal{G}'(a_1)| \right]. \end{aligned} \quad (3.24)$$

Remark 3.5. In equation (3.24), if $h(\tau) = \tau$, we obtain the the following inequality:

$$\begin{aligned} & \left| \frac{1}{\eta(a_2, a_1)} \int_{a_1}^{a_1 + \eta(a_2, a_1)} \mathcal{G}(u) du - \mathcal{G}(x) \right| \\ & \leq \eta(a_2, a_1) \left[\left| \left(\frac{2}{3} \left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^3 - \frac{1}{2} \left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^2 + \frac{1}{6} \right) |\mathcal{G}'(a_2)| \right. \right. \\ & \quad \left. \left. + \left(\frac{1}{2} \left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^2 - \frac{1}{3} \left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^3 + \frac{1}{3} \left(\frac{\eta(a_2, a_1) + a_1 - x}{\eta(a_2, a_1)} \right)^3 \right) |\mathcal{G}'(a_1)| \right| \right]. \end{aligned} \quad (3.25)$$

Furthermore, taking $x = a_1 + \frac{1}{2}\eta(a_2, a_1)$ in equation (3.25), we obtain the midpoint-type inequality:

$$\left| \frac{1}{\eta(a_2, a_1)} \int_{a_1}^{a_1 + \eta(a_2, a_1)} \mathcal{G}(u) du - \mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) \right| \leq \frac{\eta(a_2, a_1)}{8} [|\mathcal{G}'(a_2)| + |\mathcal{G}'(a_1)|], \quad (3.26)$$

which is Theorem 5 proved by Sarikaya et al. [45].

Corollary 3.3. If we set $\eta(a_2, a_1) = a_2 - a_1$ in Theorem 3.2 with $a_2 > a_1$, then we obtain the following Ostrowski-type local fractional inequality for generalized h -convex functions:

$$\begin{aligned}
& \left| \frac{(1-\delta)^\delta}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x-a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_2-x) \right)^\delta} (\mathbb{I}_x^\delta \mathcal{G}(a_1) + \mathbb{I}_x^\delta \mathcal{G}(a_2)) - \mathcal{G}(x) \right| \\
& \leq \frac{(a_2-a_1)^\delta}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x-a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_2-x) \right)^\delta} \\
& \quad \times \left[\left(\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{a_2-a_1}} (1^\delta - E_\delta(-\rho\tau)^\delta) h^\delta(\tau) (d\tau)^\delta \right. \right. \\
& \quad \left. \left. + \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{a_2-a_1}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) h^\delta(\tau) (d\tau)^\delta \right) |\mathcal{G}^{(\delta)}(a_2)| \right. \\
& \quad \left. + \left(\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{a_2-a_1}} (1^\delta - E_\delta(-\rho\tau)^\delta) h^\delta(1-\tau) (d\tau)^\delta \right. \right. \\
& \quad \left. \left. + \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{a_2-a_1}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) h^\delta(1-\tau) (d\tau)^\delta \right) |\mathcal{G}^{(\delta)}(a_1)| \right], \tag{3.27}
\end{aligned}$$

where $\rho = \frac{1-\delta}{\delta}(a_2 - a_1)$.

Corollary 3.4. *If we set $h^\delta(\tau) = \tau^\delta$ in Theorem 3.2, then we obtain the following Ostrowski-type local fractional inequality for generalized preinvex functions:*

$$\begin{aligned}
& \left| \frac{(1-\delta)^\delta}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x-a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_1 + \eta(a_2, a_1) - x) \right)^\delta} (\mathbb{I}_x^\delta \mathcal{G}(a_1) + \mathbb{I}_x^\delta \mathcal{G}(a_1 + \eta(a_2, a_1))) - \mathcal{G}(x) \right| \\
& \leq \frac{\eta^\delta(a_2, a_1)}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x-a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_1 + \eta(a_2, a_1) - x) \right)^\delta} \\
& \quad \times [(A_1(\rho, E(\tau), \tau, x) + A_2(\rho, E(1-\tau), \tau, x)) |\mathcal{G}^{(\delta)}(a_2)| \\
& \quad + (A_1(\rho, E(\tau), x) - A_1(\rho, E(\tau), \tau, x) + A_2(\rho, E(1-\tau), x) - A_2(\rho, E(1-\tau), \tau, x)) |\mathcal{G}^{(\delta)}(a_1)|], \tag{3.28}
\end{aligned}$$

where

$$\begin{aligned}
A_1(\rho, E(\tau), x) &= \frac{1^\delta}{\Gamma(1+\delta)} \left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta + \left(\frac{1}{\rho} \right)^\delta \left[E_\delta \left(-\rho \frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta - 1^\delta \right]; \\
A_1(\rho, E(\tau), \tau, x) &= \frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)} \left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^{2\delta} + \left(\frac{1}{\rho} \right)^\delta \left[\left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta + \frac{\Gamma(1+\delta)}{\rho^\delta} \right] E_\delta \left(-\rho \frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta - \frac{\Gamma(1+\delta)}{\rho^\delta}; \\
A_2(\rho, E(1-\tau), x) &= \frac{1^\delta}{\Gamma(1+\delta)} \left(1 - \frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta + \left(\frac{1}{\rho} \right)^\delta \left[E_\delta \left(-\rho \left(1 - \frac{x-a_1}{\eta(a_2, a_1)} \right) \right)^\delta - 1^\delta \right];
\end{aligned}$$

$$A_2(\rho, E(1-\tau), \tau, x) = \frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)} \left[1^\delta - \left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^{2\delta} \right] - \left(\frac{1}{\rho} \right)^\delta \left[1^\delta - \frac{\Gamma(1+\delta)}{\rho^\delta} \right] \\ + \left(\frac{1}{\rho} \right)^\delta \left[\left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta - \frac{\Gamma(1+\delta)}{\rho^\delta} \right] E_\delta \left(-\rho \left(1 - \frac{x-a_1}{\eta(a_2, a_1)} \right) \right)^\delta.$$

Proof. By $h^\delta(\tau) = \tau^\delta$, from Theorem 3.2, we obtain

$$\left| \frac{(1-\delta)^\delta}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x-a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_1 + \eta(a_2, a_1) - x) \right)^\delta} (\mathbb{I}_x^\delta \mathcal{G}(a_1) + \mathbb{I}_{x^+}^\delta \mathcal{G}(a_1 + \eta(a_2, a_1))) - \mathcal{G}(x) \right| \\ \leq \frac{\eta^\delta(a_2, a_1)}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x-a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_1 + \eta(a_2, a_1) - x) \right)^\delta} \\ \times \left[\left(\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta) \tau^\delta (d\tau)^\delta \right. \right. \\ \left. \left. + \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) \tau^\delta (d\tau)^\delta \right) |\mathcal{G}^{(\delta)}(a_2)| \right. \\ \left. + \left(\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta) (1-\tau)^\delta (d\tau)^\delta \right. \right. \\ \left. \left. + \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) (1-\tau)^\delta (d\tau)^\delta \right) |\mathcal{G}^{(\delta)}(a_1)| \right]. \quad (3.29)$$

By calculation, we obtain

$$\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta) (d\tau)^\delta \\ = \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} 1^\delta (d\tau)^\delta - \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} E_\delta(-\rho\tau)^\delta (d\tau)^\delta \\ = \frac{1^\delta}{\Gamma(1+\delta)} \left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta + \left(\frac{1}{\rho} \right)^\delta \left[E_\delta \left(-\rho \frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta - 1^\delta \right] \\ = A_1(\rho, E(\tau), x); \quad (3.30)$$

$$\begin{aligned}
& \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta) \tau^\delta (d\tau)^\delta \\
&= \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} \tau^\delta (d\tau)^\delta - \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} \tau^\delta E_\delta(-\rho\tau)^\delta (d\tau)^\delta \\
&= \frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)} \left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^{2\delta} + \left(\frac{1}{\rho} \right)^\delta \left[\left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta + \frac{\Gamma(1+\delta)}{\rho^\delta} E_\delta \left(-\rho \frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta - \frac{\Gamma(1+\delta)}{\rho^\delta} \right] \\
&= A_1(\rho, E(\tau), \tau, x);
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
& \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) (d\tau)^\delta \\
&= \frac{1^\delta}{\Gamma(1+\delta)} \left(1 - \frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta + \left(\frac{1}{\rho} \right)^\delta \left[E_\delta \left(-\rho \left(1 - \frac{x-a_1}{\eta(a_2, a_1)} \right) \right)^\delta - 1^\delta \right] \\
&= A_2(\rho, E(1-\tau), x);
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
& \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) \tau^\delta (d\tau)^\delta \\
&= \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 \tau^\delta (d\tau)^\delta - \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 \tau^\delta E_\delta(-\rho(1-\tau))^\delta (d\tau)^\delta \\
&= \frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)} \left[1^\delta - \left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^{2\delta} \right] - \left(\frac{1}{\rho} \right)^\delta \left[1^\delta - \frac{\Gamma(1+\delta)}{\rho^\delta} \right] \\
&\quad + \left(\frac{1}{\rho} \right)^\delta \left[\left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta - \frac{\Gamma(1+\delta)}{\rho^\delta} E_\delta \left(-\rho \left(1 - \frac{x-a_1}{\eta(a_2, a_1)} \right) \right)^\delta \right] \\
&= A_2(\rho, E(1-\tau), \tau, x).
\end{aligned} \tag{3.33}$$

Thus, we obtain

$$\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta) (1-\tau)^\delta (d\tau)^\delta = A_1(\rho, E(\tau), x) - A_1(\rho, E(\tau), \tau, x) \tag{3.34}$$

and

$$\frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) (1-\tau)^\delta (d\tau)^\delta = A_2(\rho, E(1-\tau), x) - A_2(\rho, E(1-\tau), \tau, x). \tag{3.35}$$

Substituting equations (3.31) and (3.33)–(3.35) into equation (3.29), we achieve the desired result. This completes the proof. $\square$

Corollary 3.5. *If we set $x = a_1 + \frac{1}{2}\eta(a_2, a_1)$ in Theorem 3.2, then we obtain the following midpoint-type local fractional inequality:*

$$\begin{aligned}
& \left| \frac{(1-\delta)^\delta}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)^\delta} \left(\mathbb{I}_{(a_1 + \frac{1}{2}\eta(a_2, a_1))^-}^\delta \mathcal{G}(a_1) + \mathbb{I}_{(a_1 + \frac{1}{2}\eta(a_2, a_1))^+}^\delta \mathcal{G}(a_1 + \eta(a_2, a_1)) \right) - \mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) \right| \\
& \leq \frac{\eta^\delta(a_2, a_1)}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)^\delta} \left[\left| \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{1}{2}} (1^\delta - E_\delta(-\rho\tau)^\delta) h^\delta(\tau) (d\tau)^\delta \right. \right. \\
& \quad \left. \left. + \frac{1}{\Gamma(1+\delta)} \int_{\frac{1}{2}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) h^\delta(\tau) (d\tau)^\delta \right| |\mathcal{G}^{(\delta)}(a_2)| \right. \\
& \quad \left. + \left| \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{1}{2}} (1^\delta - E_\delta(-\rho\tau)^\delta) h^\delta(1-\tau) (d\tau)^\delta + \frac{1}{\Gamma(1+\delta)} \int_{\frac{1}{2}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) h^\delta(1 \right. \right. \\
& \quad \left. \left. - \tau) (d\tau)^\delta \right| |\mathcal{G}^{(\delta)}(a_1)| \right]. \tag{3.36}
\end{aligned}$$

Theorem 3.3. Let $\Omega \subseteq \mathbb{R}$ be an open invex set regarding $\eta : \Omega \times \Omega \rightarrow \mathbb{R}$ and $\mathcal{G} : \Omega \rightarrow \mathbb{R}^\delta$ ($0 < \delta < 1$) be a function with $\eta(a_2, a_1) > 0$, $a_1, a_2 \in \Omega$, and $\mathcal{G}^{(\delta)}(x) \in I_x^{(\delta)}[a_1, a_1 + \eta(a_2, a_1)]$. If $|\mathcal{G}^{(\delta)}|^q$ is a generalized h -preinvex function on Ω with $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, then the following Ostrowski-type local fractional integral inequality holds

$$\begin{aligned}
& \left| \frac{(1-\delta)^\delta}{2^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(x - a_1)\right)^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(a_1 + \eta(a_2, a_1) - x)\right)^\delta} (\mathbb{I}_x^\delta \mathcal{G}(a_1) + \mathbb{I}_{x^+}^\delta \mathcal{G}(a_1 + \eta(a_2, a_1))) - \mathcal{G}(x) \right| \\
& \leq \frac{\eta^\delta(a_2, a_1)}{2^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(x - a_1)\right)^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(a_1 + \eta(a_2, a_1) - x)\right)^\delta} \\
& \quad \times \left[\left(\frac{\left(\frac{x - a_1}{\eta(a_2, a_1)}\right)^\delta}{\Gamma(1+\delta)} + \frac{E_\delta(-p\rho \frac{x - a_1}{\eta(a_2, a_1)})^\delta - 1^\delta}{(p\rho)^\delta} \right)^{\frac{1}{p}} \right. \\
& \quad \times \left. \left(\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x - a_1}{\eta(a_2, a_1)}} (h^\delta(\tau) |\mathcal{G}^{(\delta)}(a_2)|^q + h^\delta(1-\tau) |\mathcal{G}^{(\delta)}(a_1)|^q) (d\tau)^\delta \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left(\frac{\left(1 - \frac{x - a_1}{\eta(a_2, a_1)}\right)^\delta}{\Gamma(1+\delta)} + \frac{E_\delta\left(-p\rho\left(1 - \frac{x - a_1}{\eta(a_2, a_1)}\right)\right)^\delta - 1^\delta}{(p\rho)^\delta} \right)^{\frac{1}{p}} \right. \\
& \quad \left. \times \left(\frac{1}{\Gamma(1+\delta)} \int_{\frac{x - a_1}{\eta(a_2, a_1)}}^1 (h^\delta(\tau) |\mathcal{G}^{(\delta)}(a_2)|^q + h^\delta(1-\tau) |\mathcal{G}^{(\delta)}(a_1)|^q) (d\tau)^\delta \right)^{\frac{1}{q}} \right]. \tag{3.37}
\end{aligned}$$

Proof. Since $\Omega \subseteq R$ is an open invex set regarding η and $a_1, a_2 \in \Omega$, we have $a_1 + \eta(a_2, a_1) \in \Omega$. Using Lemma 3.1, by the properties of modules, it follows that

$$\begin{aligned} & \left| \frac{(1-\delta)^\delta}{2^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(x-a_1)\right)^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(a_1+\eta(a_2, a_1)-x)\right)^\delta} (\mathbb{I}_x^\delta \mathcal{G}(a_1) + \mathbb{I}_x^\delta \mathcal{G}(a_1 + \eta(a_2, a_1))) - \mathcal{G}(x) \right| \\ & \leq \frac{\eta^\delta(a_2, a_1)}{2^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(x-a_1)\right)^\delta - E_\delta\left(-\frac{1-\delta}{\delta}(a_1+\eta(a_2, a_1)-x)\right)^\delta} \\ & \quad \times \left[\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta) |\mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1))| (d\tau)^\delta \right. \\ & \quad \left. + \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) |\mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1))| (d\tau)^\delta \right]. \end{aligned} \quad (3.38)$$

By calculation, we have

$$\begin{aligned} & \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta)^p (d\tau)^\delta \\ & \leq \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^{p\delta} - E_\delta(-p\rho\tau)^\delta) (d\tau)^\delta \\ & = \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} 1^\delta (d\tau)^\delta - \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} E_\delta(-p\rho\tau)^\delta (d\tau)^\delta \\ & = \frac{\left(\frac{x-a_1}{\eta(a_2, a_1)}\right)^\delta}{\Gamma(1+\delta)} + \frac{E_\delta(-p\rho\frac{x-a_1}{\eta(a_2, a_1)})^\delta - 1^\delta}{(p\rho)^\delta}, \end{aligned} \quad (3.39)$$

which uses the fact that $(m-n)^p \leq m^p - n^p$ for any $m > n \geq 0$ and $p \geq 1$.

Similarly, we have

$$\begin{aligned} & \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta)^p (d\tau)^\delta \\ & \leq \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^{p\delta} - E_\delta(-p\rho(1-\tau))^\delta) (d\tau)^\delta \\ & = \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 1^\delta (d\tau)^\delta - \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 E_\delta(-p\rho(1-\tau))^\delta (d\tau)^\delta \\ & = \frac{\left(1 - \frac{x-a_1}{\eta(a_2, a_1)}\right)^\delta}{\Gamma(1+\delta)} + \frac{E_\delta\left(-p\rho\left(1 - \frac{x-a_1}{\eta(a_2, a_1)}\right)\right)^\delta - 1^\delta}{(p\rho)^\delta}. \end{aligned} \quad (3.40)$$

Thus, using the Hölder-Yang's inequality, by the h -preinvexity of $|\mathcal{G}^{(\delta)}|^q$, combining equations (3.39) and (3.40), we obtain

$$\begin{aligned}
 & \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta) |\mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1))| (d\tau)^\delta \\
 & \leq \left(\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (1^\delta - E_\delta(-\rho\tau)^\delta)^p (d\tau)^\delta \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} |\mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1))|^q (d\tau)^\delta \right)^{\frac{1}{q}} \\
 & \leq \left(\frac{\left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta}{\Gamma(1+\delta)} + \frac{\left(E_\delta(-p\rho \frac{x-a_1}{\eta(a_2, a_1)})^\delta - 1^\delta \right)^{\frac{1}{p}}}{(p\rho)^\delta} \right) \\
 & \quad \times \left(\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (h^\delta(\tau) |\mathcal{G}^{(\delta)}(a_2)|^q + h^\delta(1-\tau) |\mathcal{G}^{(\delta)}(a_1)|^q) (d\tau)^\delta \right)^{\frac{1}{q}}, \tag{3.41}
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) |\mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1))| (d\tau)^\delta \\
 & \leq \left(\frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta)^p (d\tau)^\delta \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 |\mathcal{G}^{(\delta)}(a_1 + \tau\eta(a_2, a_1))|^q (d\tau)^\delta \right)^{\frac{1}{q}} \\
 & \leq \left(\frac{\left(1 - \frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta}{\Gamma(1+\delta)} + \frac{\left(E_\delta\left(-p\rho\left(1 - \frac{x-a_1}{\eta(a_2, a_1)}\right)\right)^\delta - 1^\delta \right)^{\frac{1}{p}}}{(p\rho)^\delta} \right) \\
 & \quad \times \left(\frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (h^\delta(\tau) |\mathcal{G}^{(\delta)}(a_2)|^q + h^\delta(1-\tau) |\mathcal{G}^{(\delta)}(a_1)|^q) (d\tau)^\delta \right)^{\frac{1}{q}}. \tag{3.42}
 \end{aligned}$$

Substituting equations (3.41) and (3.42) into equation (3.38), we achieve the desired result. This completes the proof. $\square$

Corollary 3.6. *If we set $\eta(a_2, a_1) = a_2 - a_1$ in Theorem 3.3 with $a_2 > a_1$, then we obtain the following Ostrowski-type local fractional inequality for generalized h -convex:*

$$\begin{aligned}
& \left| \frac{(1-\delta)^\delta}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x-a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_2-x) \right)^\delta} (\mathbb{I}_x^\delta \mathcal{G}(a_1) + \mathbb{I}_x^\delta \mathcal{G}(a_2)) - \mathcal{G}(x) \right| \\
& \leq \frac{(a_2-a_1)^\delta}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x-a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_2-x) \right)^\delta} \\
& \quad \times \left[\left(\frac{\left(\frac{x-a_1}{a_2-a_1} \right)^\delta}{\Gamma(1+\delta)} + \frac{E_\delta \left(-p\rho \frac{x-a_1}{a_2-a_1} \right)^\delta - 1^\delta}{(p\rho)^\delta} \right)^{\frac{1}{p}} \right. \\
& \quad \times \left. \left(\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{a_2-a_1}} (h^\delta(\tau) |\mathcal{G}^{(\delta)}(a_2)|^q + h^\delta(1-\tau) |\mathcal{G}^{(\delta)}(a_1)|^q) (d\tau)^\delta \right)^{\frac{1}{q}} \right. \\
& \quad + \left. \left(\frac{\left(\frac{a_2-x}{a_2-a_1} \right)^\delta}{\Gamma(1+\delta)} + \frac{E_\delta \left(-p\rho \frac{a_2-x}{a_2-a_1} \right)^\delta - 1^\delta}{(p\rho)^\delta} \right)^{\frac{1}{p}} \right. \\
& \quad \times \left. \left(\frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{a_2-a_1}}^1 (h^\delta(\tau) |\mathcal{G}^{(\delta)}(a_2)|^q + h^\delta(1-\tau) |\mathcal{G}^{(\delta)}(a_1)|^q) (d\tau)^\delta \right)^{\frac{1}{q}} \right], \tag{3.43}
\end{aligned}$$

where $\rho = \frac{1-\delta}{\delta} (a_2 - a_1)$.

Corollary 3.7. If we set $h^\delta(\tau) = \tau^\delta$ in Theorem 3.3, then we obtain the following Ostrowski-type local fractional inequality for generalized preinvex functions:

$$\begin{aligned}
& \left| \frac{(1-\delta)^\delta}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x-a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_1 + \eta(a_2, a_1) - x) \right)^\delta} (\mathbb{I}_x^\delta \mathcal{G}(a_1) + \mathbb{I}_x^\delta \mathcal{G}(a_1 + \eta(a_2, a_1))) - \mathcal{G}(x) \right| \\
& \leq \frac{\eta^\delta(a_2, a_1)}{2^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (x-a_1) \right)^\delta - E_\delta \left(-\frac{1-\delta}{\delta} (a_1 + \eta(a_2, a_1) - x) \right)^\delta} \\
& \quad \times \left[\left(\frac{\left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta}{\Gamma(1+\delta)} + \frac{E_\delta \left(-p\rho \frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta - 1^\delta}{(p\rho)^\delta} \right)^{\frac{1}{p}} \right. \\
& \quad \times \left. \left(\frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)} \left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^{2\delta} |\mathcal{G}^{(\delta)}(a_2)|^q + \left(1^\delta - \left(1 - \frac{x-a_1}{\eta(a_2, a_1)} \right)^{2\delta} \right) |\mathcal{G}^{(\delta)}(a_1)|^q \right)^{\frac{1}{q}} \right. \\
& \quad + \left. \left(\frac{\left(1 - \frac{x-a_1}{\eta(a_2, a_1)} \right)^\delta}{\Gamma(1+\delta)} + \frac{E_\delta \left(-p\rho \left(1 - \frac{x-a_1}{\eta(a_2, a_1)} \right) \right)^\delta - 1^\delta}{(p\rho)^\delta} \right)^{\frac{1}{p}} \right. \\
& \quad \times \left. \left(\frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)} \left(1^\delta - \left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^{2\delta} \right) |\mathcal{G}^{(\delta)}(a_2)|^q + \left(1 - \frac{x-a_1}{\eta(a_2, a_1)} \right)^{2\delta} |\mathcal{G}^{(\delta)}(a_1)|^q \right)^{\frac{1}{q}} \right]. \tag{3.44}
\end{aligned}$$

Proof. By $h^\delta(\tau) = \tau^\delta$, we obtain

$$\begin{aligned} & \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (h^\delta(\tau)|\mathcal{G}^{(\delta)}(a_2)|^q + h^\delta(1-\tau)|\mathcal{G}^{(\delta)}(a_1)|^q)(d\tau)^\delta \\ &= \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} (\tau^\delta|\mathcal{G}^{(\delta)}(a_2)|^q + (1-\tau)^\delta|\mathcal{G}^{(\delta)}(a_1)|^q)(d\tau)^\delta \\ &= \frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)} \left[\left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^{2\delta} |\mathcal{G}^{(\delta)}(a_2)|^q + \left(1^\delta - \left(1 - \frac{x-a_1}{\eta(a_2, a_1)} \right)^{2\delta} \right) |\mathcal{G}^{(\delta)}(a_1)|^q \right] \end{aligned} \quad (3.45)$$

and

$$\begin{aligned} & \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (h^\delta(\tau)|\mathcal{G}^{(\delta)}(a_2)|^q + h^\delta(1-\tau)|\mathcal{G}^{(\delta)}(a_1)|^q)(d\tau)^\delta \\ &= \frac{1}{\Gamma(1+\delta)} \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (\tau^\delta|\mathcal{G}^{(\delta)}(a_2)|^q + (1-\tau)^\delta|\mathcal{G}^{(\delta)}(a_1)|^q)(d\tau)^\delta \\ &= \frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)} \left[\left(1^\delta - \left(\frac{x-a_1}{\eta(a_2, a_1)} \right)^{2\delta} \right) |\mathcal{G}^{(\delta)}(a_2)|^q + \left(1 - \frac{x-a_1}{\eta(a_2, a_1)} \right)^{2\delta} |\mathcal{G}^{(\delta)}(a_1)|^q \right]. \end{aligned} \quad (3.46)$$

Substituting equations (3.45) and (3.46) into equation (3.37), we achieve the desired result. This completes the proof. $\square$

Corollary 3.8. If we set $x = a_1 + \frac{1}{2}\eta(a_2, a_1)$ in Theorem 3.3, then we obtain the following midpoint-type local fractional inequality:

$$\begin{aligned} & \left| \frac{(1-\delta)^\delta}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)} \left(\mathbb{I}_{(a_1 + \frac{1}{2}\eta(a_2, a_1))^-}^\delta \mathcal{G}(a_1) + \mathbb{I}_{(a_1 + \frac{1}{2}\eta(a_2, a_1))^+}^\delta \mathcal{G}(a_1 + \eta(a_2, a_1)) \right) - \mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) \right| \\ & \leq \frac{\eta^\delta(a_2, a_1)}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)} \left[\frac{\left(\frac{1}{2}\right)^\delta}{\Gamma(1+\delta)} + \frac{E_\delta\left(-\frac{p\rho}{2}\right)^\delta - 1^\delta}{(p\rho)^\delta} \right]^{\frac{1}{p}} \\ & \quad \times \left[\left(\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{1}{2}} (h^\delta(\tau)|\mathcal{G}^{(\delta)}(a_2)|^q + h^\delta(1-\tau)|\mathcal{G}^{(\delta)}(a_1)|^q)(d\tau)^\delta \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{1}{\Gamma(1+\delta)} \int_{\frac{1}{2}}^1 (h^\delta(\tau)|\mathcal{G}^{(\delta)}(a_2)|^q + h^\delta(1-\tau)|\mathcal{G}^{(\delta)}(a_1)|^q)(d\tau)^\delta \right)^{\frac{1}{q}} \right]. \end{aligned} \quad (3.47)$$

Corollary 3.9. If we set $h^\delta(\tau) = \tau^\delta$ in Corollary 3.8, then we obtain the following midpoint-type local fractional inequality for generalized preinvex functions:

$$\begin{aligned}
& \left| \frac{(1-\delta)^\delta}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)^\delta} \left(\mathbb{I}_{(a_1 + \frac{1}{2}\eta(a_2, a_1))}^\delta \mathcal{G}(a_1) + \mathbb{I}_{(a_1 + \frac{1}{2}\eta(a_2, a_1))^\tau}^\delta \mathcal{G}(a_1 + \eta(a_2, a_1)) \right) - \mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) \right| \\
& \leq \frac{\eta^\delta(a_2, a_1) 4^{\frac{\delta}{p}}}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)^\delta} \left(\frac{\left(\frac{1}{2}\right)^\delta}{\Gamma(1+\delta)} + \frac{E_\delta\left(-\frac{p\rho}{2}\right)^\delta - 1^\delta}{(p\rho)^\delta} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)} \right)^{\frac{1}{q}} (|\mathcal{G}^{(\delta)}(a_2)| + |\mathcal{G}^{(\delta)}(a_1)|).
\end{aligned} \tag{3.48}$$

Proof. Letting $h^\delta(\tau) = \tau^\delta$ in equation (3.47), we have

$$\begin{aligned}
& \left| \frac{(1-\delta)^\delta}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)^\delta} \left(\mathbb{I}_{(a_1 + \frac{1}{2}\eta(a_2, a_1))}^\delta \mathcal{G}(a_1) + \mathbb{I}_{(a_1 + \frac{1}{2}\eta(a_2, a_1))^\tau}^\delta \mathcal{G}(a_1 + \eta(a_2, a_1)) \right) - \mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) \right| \\
& \leq \frac{\eta^\delta(a_2, a_1)}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)^\delta} \left(\frac{\left(\frac{1}{2}\right)^\delta}{\Gamma(1+\delta)} + \frac{E_\delta\left(-\frac{p\rho}{2}\right)^\delta - 1^\delta}{(p\rho)^\delta} \right)^{\frac{1}{p}} \\
& \quad \times \left[\left(\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{1}{2}} (\tau^\delta |\mathcal{G}^{(\delta)}(a_2)|^q + (1-\tau)^\delta |\mathcal{G}^{(\delta)}(a_1)|^q) (d\tau)^\delta \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left(\frac{1}{\Gamma(1+\delta)} \int_{\frac{1}{2}}^1 (\tau^\delta |\mathcal{G}^{(\delta)}(a_2)|^q + (1-\tau)^\delta |\mathcal{G}^{(\delta)}(a_1)|^q) (d\tau)^\delta \right)^{\frac{1}{q}} \right].
\end{aligned} \tag{3.49}$$

By calculation, we obtain

$$\begin{aligned}
& \left(\frac{1}{\Gamma(1+\delta)} \int_0^{\frac{1}{2}} (\tau^\delta |\mathcal{G}^{(\delta)}(a_2)|^q + (1-\tau)^\delta |\mathcal{G}^{(\delta)}(a_1)|^q) (d\tau)^\delta \right)^{\frac{1}{q}} \\
& + \left(\frac{1}{\Gamma(1+\delta)} \int_{\frac{1}{2}}^1 (\tau^\delta |\mathcal{G}^{(\delta)}(a_2)|^q + (1-\tau)^\delta |\mathcal{G}^{(\delta)}(a_1)|^q) (d\tau)^\delta \right)^{\frac{1}{q}} \\
& = \left(\frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)} \right)^{\frac{1}{q}} \left[\left(\left(\frac{1}{4} \right)^\delta |\mathcal{G}^{(\delta)}(a_2)|^q + \left(\frac{3}{4} \right)^\delta |\mathcal{G}^{(\delta)}(a_1)|^q \right)^{\frac{1}{q}} + \left(\left(\frac{3}{4} \right)^\delta |\mathcal{G}^{(\delta)}(a_2)|^q + \left(\frac{1}{4} \right)^\delta |\mathcal{G}^{(\delta)}(a_1)|^q \right)^{\frac{1}{q}} \right].
\end{aligned} \tag{3.50}$$

Let $m_1 = |\mathcal{G}^{(\delta)}(a_2)|^q$, $n_1 = 3^\delta |\mathcal{G}^{(\delta)}(a_1)|^q$, $m_2 = 3^\delta |\mathcal{G}^{(\delta)}(a_2)|^q$, $n_2 = |\mathcal{G}^{(\delta)}(a_1)|^q$, and $0 < \frac{1}{q} < 1$. According to the fact that

$$\sum_{i=1}^k (m_i + n_i)^r \leq \sum_{i=1}^k m_i^r + \sum_{i=1}^k n_i^r,$$

for $0 < r < 1$, $m_1, m_2, \dots, m_k \geq 0$ and $n_1, n_2, \dots, n_k \geq 0$, we obtain

$$\begin{aligned}
& \left(\left(\frac{1}{4} \right)^\delta |\mathcal{G}^{(\delta)}(a_2)|^q + \left(\frac{3}{4} \right)^\delta |\mathcal{G}^{(\delta)}(a_1)|^q \right)^{\frac{1}{q}} + \left(\left(\frac{3}{4} \right)^\delta |\mathcal{G}^{(\delta)}(a_2)|^q + \left(\frac{1}{4} \right)^\delta |\mathcal{G}^{(\delta)}(a_1)|^q \right)^{\frac{1}{q}} \\
& \leq \left(\frac{1}{4} \right)^{\frac{\delta}{q}} (4^\delta |\mathcal{G}^{(\delta)}(a_2)| + 4^\delta |\mathcal{G}^{(\delta)}(a_1)|) \\
& = 4^{\frac{\delta}{p}} (|\mathcal{G}^{(\delta)}(a_2)| + |\mathcal{G}^{(\delta)}(a_1)|).
\end{aligned} \tag{3.51}$$

Substituting equations (3.50) and (3.51) into equation (3.49), we achieve the desired result. This completes the proof. $\square$

The results given in the corollaries and the remarks can be regarded as special cases of the main theorem. Some special cases are proved results in the references, and some special cases are new conclusions.

4 Some examples

Next, several examples are given to illustrate the main results.

Example 1. By equation (3.12) in Remark 3.1, we obtain

$$\mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) \leq \frac{2h\left(\frac{1}{2}\right)}{\eta(a_2, a_1)} \int_{a_1}^{a_1+\eta(a_2, a_1)} \mathcal{G}(x)dx \leq 2h\left(\frac{1}{2}\right)[\mathcal{G}(a_2) + \mathcal{G}(a_1)] \int_0^{\frac{1}{2}} [h(\tau) + h(1-\tau)]d\tau.$$

Let $\mathcal{G}(u) = e^{2u}$, $u \in [a_1, a_1 + \eta(a_2, a_1)]$ with $a_2 > a_1$. If we choose $h(\tau) = e^\tau$, $0 < \tau < 1$, then it is obvious that $\mathcal{G}(u) = e^{2u}$ is an h -preinvex function for $\eta(a_2, a_1) = a_2 - a_1$. Hence, taking $a_1 = 1$ and $a_2 = 3$, we have

$$\begin{aligned}
\mathcal{G}\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) &= e^{a_1+a_2} = e^4 = 54.5982, \\
\frac{2h\left(\frac{1}{2}\right)}{\eta(a_2, a_1)} \int_{a_1}^{a_1+\eta(a_2, a_1)} \mathcal{G}(x)dx &= \frac{e^{\frac{1}{2}}}{a_2 - a_1} (e^{2a_2} - e^{2a_1}) = \frac{e^{\frac{1}{2}}}{2} (e^6 - e^2) = 326.4796, \\
2h\left(\frac{1}{2}\right)[\mathcal{G}(a_2) + \mathcal{G}(a_1)] \int_0^{\frac{1}{2}} [h(\tau) + h(1-\tau)]d\tau &= 2e^{\frac{1}{2}}(e^{2a_2} + e^{2a_1})(e - 1) \\
&= 2e^{\frac{1}{2}}(e^6 + e^2)(e - 1) = 2327.7.
\end{aligned}$$

Obviously, $54.5982 < 326.4796 < 2327.7$, i.e., Inequalities (3.12) hold in this case.

Example 2. By equation (3.24) in Remark 3.4, we obtain

$$\begin{aligned}
& \left| \frac{1}{\eta(a_2, a_1)} \int_{a_1}^{a_1+\eta(a_2, a_1)} \mathcal{G}(u)du - \mathcal{G}(x) \right| \\
& \leq \eta(a_2, a_1) \left[\left(\int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} \tau h(\tau) d\tau + \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1-\tau) h(\tau) d\tau \right) |\mathcal{G}'(a_2)| \right. \\
& \quad \left. + \left(\int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} \tau h(1-\tau) d\tau + \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1-\tau) h(1-\tau) d\tau \right) |\mathcal{G}'(a_1)| \right].
\end{aligned}$$

Let $\mathcal{G}(u) = e^{2u}$, $u \in [a_1, a_1 + \eta(a_2, a_1)]$ with $a_2 > a_1$. If we choose $h(\tau) = \tau^{\frac{1}{3}}$, $0 < \tau < 1$, then it is obvious that $|\mathcal{G}'(u)| = 2e^{2u}$ is an h -preinvex function for $\eta(a_2, a_1) = a_2 - a_1$. Hence, taking $a_1 = 1$, $a_2 = 3$, we have

$$\left| \frac{1}{\eta(a_2, a_1)} \int_{a_1}^{a_1 + \eta(a_2, a_1)} \mathcal{G}(u) du - \mathcal{G}(x) \right| = \left| \frac{e^6 - e^2}{4} - e^{2x} \right|$$

and

$$\begin{aligned} & \eta(a_2, a_1) \left[\left(\int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} \tau h(\tau) d\tau + \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1-\tau) h(\tau) d\tau \right) |\mathcal{G}'(a_2)| \right. \\ & \quad \left. + \left(\int_0^{\frac{x-a_1}{\eta(a_2, a_1)}} \tau h(1-\tau) d\tau + \int_{\frac{x-a_1}{\eta(a_2, a_1)}}^1 (1-\tau) h(1-\tau) d\tau \right) |\mathcal{G}'(a_1)| \right] \\ &= \left[\frac{3 \cdot 2^{2/3}}{7} (x-1)^{7/3} - \frac{3 \cdot 2^{2/3}}{4} (x-1)^{4/3} + \frac{9}{7} \right] e^6 + \left[\frac{3 \cdot 2^{2/3}}{7} (3-x)^{7/3} - 3 \left(\frac{3-x}{2} \right)^{4/3} + \frac{9}{7} \right] e^2. \end{aligned}$$

We use MATLAB software to plot function curves on the left-hand and right-hand sides of Inequality (3.24) for $x \in [1, 3]$, as shown in Figure 1. It is obvious that for $x \in [1, 3]$, the left function value is smaller than the right function value. This also shows that Inequality (3.24) is true in this case.

Example 3. Taking $\mathcal{G}(u) = \frac{\Gamma(1 + \frac{k}{q}\delta)}{\Gamma(1 + (\frac{k}{q} + 1)\delta)} u^{(\frac{k}{q} + 1)\delta}$, $k > 1$, $q > 1$, $u \in [a_1, a_1 + \eta(a_2, a_1)]$, $0 < a_1 < a_2$, it is obvious that $|\mathcal{G}^{(\delta)}(u)|^q = u^{k\delta}$ is a generalized preinvex function for $\eta(a_2, a_1) = a_2 - a_1$. Hence, by Corollary 3.9, we obtain a special midpoint-type inequality:

$$\begin{aligned} & \left| \frac{(1-\delta)^\delta}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)^\delta} \left[\left(\frac{a_1+a_2}{2} \right)^\delta a_1^{(\frac{k}{q}+1)\delta} + \left(\frac{a_1+a_2}{2} \right)^\delta a_2^{(\frac{k}{q}+1)\delta} \right] - \left(\frac{a_1+a_2}{2} \right)^{(\frac{k}{q}+1)\delta} \right| \\ & \leq \frac{\Gamma\left(1 + \left(\frac{k}{q} + 1\right)\delta\right)}{\Gamma\left(1 + \frac{k}{q}\delta\right)} \frac{(a_2 - a_1)^\delta 4^{\frac{\delta}{p}}}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)^\delta} \left[\frac{\left(\frac{1}{2}\right)^\delta}{\Gamma(1+\delta)} + \frac{E_\delta\left(-\frac{\rho}{2}\right)^\delta - 1^\delta}{(p\rho)^\delta} \right]^{\frac{1}{p}} \left(\frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)} \right)^{\frac{1}{q}} \left(a_2^{\frac{k\delta}{q}} + a_1^{\frac{k\delta}{q}} \right), \end{aligned}$$

where $\rho = \frac{1-\delta}{\delta}(a_2 - a_1)$.

5 Applications involving generalized moments

Let X be a continuous random variable with the generalized probability density function $f: \Omega \rightarrow R^\delta$, where Ω is an open invex set regarding $\eta: \Omega \times \Omega \rightarrow R$ and $\eta(a_2, a_1) > 0$, $a_1, a_2 \in \Omega$. We define the generalized n -th central moment about any $u \in R$ of X , $n \geq 0$ as:

$$M_\delta^n(u) = \frac{1}{\Gamma(1+\delta)} \int_{a_1}^{a_1 + \eta(a_2, a_1)} (x-u)^{n\delta} f(x) (dx)^\delta, \quad n = 1, 2, 3, \dots$$

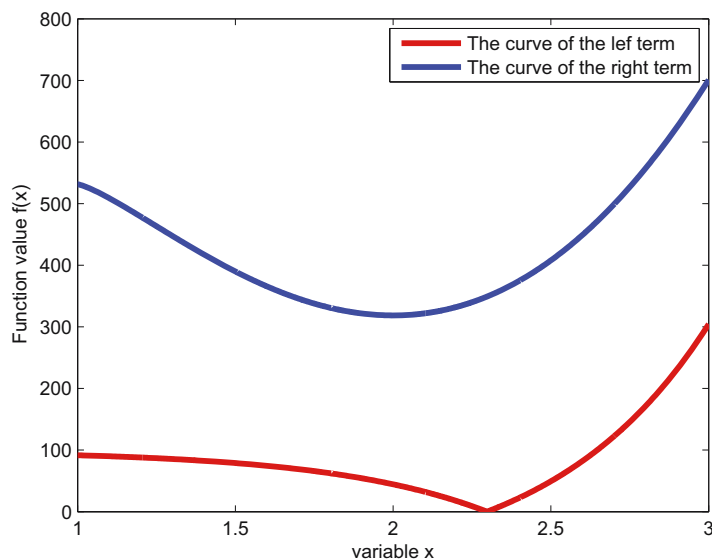


Figure 1: Curve description of Inequality (3.24) for $\mathcal{G}(u) = e^{2u}$ and $h(\tau) = \tau^{\frac{1}{3}}$.

By calculation, we obtain

$$\begin{aligned} (M_\delta^n(u))^{(\delta)} &= -\frac{\Gamma(1+n\delta)\Gamma(1+\delta)}{\Gamma(1+(n-1)\delta)} \frac{1}{\Gamma(1+\delta)} \int_{a_1}^{a_1+\eta(a_2,a_1)} (x-u)^{(n-1)\delta} f(x) (dx)^\delta \\ &= -\frac{\Gamma(1+n\delta)\Gamma(1+\delta)}{\Gamma(1+(n-1)\delta)} M_\delta^{n-1}(u). \end{aligned}$$

Proposition 1. Let $\mathcal{G}(u) = M_\delta^n(u)$. If the conditions of Corollary 3.5 are satisfied, then we obtain the midpoint-type inequalities involving generalized moment as follows:

$$\begin{aligned} &\left| \frac{(1-\delta)^\delta}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)^\delta} \left(\mathbb{I}_{(a_1+\frac{1}{2}\eta(a_2,a_1))^-}^\delta M_\delta^n(a_1) + \mathbb{I}_{(a_1+\frac{1}{2}\eta(a_2,a_1))^+}^\delta M_\delta^n(a_1 + \eta(a_2, a_1)) \right) - M_\delta^n\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) \right| \\ &\leq \frac{\eta^\delta(a_2, a_1)}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)^\delta} \frac{\Gamma(1+n\delta)\Gamma(1+\delta)}{\Gamma(1+(n-1)\delta)} \left[\left| \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{1}{2}} (1^\delta - E_\delta(-\rho\tau)^\delta) h^\delta(\tau) (d\tau)^\delta \right. \right. \\ &\quad \left. \left. + \frac{1}{\Gamma(1+\delta)} \int_{\frac{1}{2}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) h^\delta(\tau) (d\tau)^\delta \right| |M_\delta^{n-1}(a_2)| \right. \\ &\quad \left. + \left| \frac{1}{\Gamma(1+\delta)} \int_0^{\frac{1}{2}} (1^\delta - E_\delta(-\rho\tau)^\delta) h^\delta(1-\tau) (d\tau)^\delta + \frac{1}{\Gamma(1+\delta)} \int_{\frac{1}{2}}^1 (1^\delta - E_\delta(-\rho(1-\tau))^\delta) h^\delta(1-\tau) (d\tau)^\delta \right| |M_\delta^{n-1}(a_1)| \right]. \end{aligned} \quad (5.1)$$

Proposition 2. Let $\mathcal{G}(u) = M_\delta^n(u)$. If the conditions of Corollary 3.9 are satisfied, then we obtain the midpoint-type inequalities involving generalized moment as follows:

$$\left| \frac{(1-\delta)^\delta}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)^\delta} \left(\mathbb{I}_{(a_1 + \frac{1}{2}\eta(a_2, a_1))}^\delta M_\delta^n(a_1) + \mathbb{I}_{(a_1 + \frac{1}{2}\eta(a_2, a_1))^+}^\delta M_\delta^n(a_1 + \eta(a_2, a_1)) \right) - M_\delta^n\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) \right| \leq \frac{\eta^\delta(a_2, a_1) 4^{\frac{\delta}{p}}}{2^\delta - 2^\delta E_\delta\left(-\frac{\rho}{2}\right)^\delta} \left(\frac{\left(\frac{1}{2}\right)^\delta}{\Gamma(1+\delta)} + \frac{E_\delta\left(-\frac{p\rho}{2}\right)^\delta - 1^\delta}{(p\rho)^\delta} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)} \right)^{\frac{1}{q}} \frac{\Gamma(1+n\delta)\Gamma(1+\delta)}{\Gamma(1+(n-1)\delta)} (|M_\delta^{n-1}(a_2)| + |M_\delta^{n-1}(a_1)|). \quad (5.2)$$

6 Conclusion

This study is carried out in Yang's fractal theory. First, based on two new local fractional integral operators involving Mittag-Leffler kernel proposed by Sun [38], we acquire the local fractional Hermite-Hadamard inequality about these two integral operators for generalized h -preinvex functions. Subsequently, an integral identity related to these two local fractional integral operators is constructed, which is used as an auxiliary tool to obtain some new Ostrowski-type local fractional integral inequalities for generalized h -preinvex functions. Finally, we propose three examples to illustrate the partial results and applications. We also propose two midpoint-type inequalities involving generalized moments of continuous random variables, which shows the application of the results. It is worth mentioning that this topic may provide some references for the study of inequalities and the bounded estimation of mathematical problems involving the engineering fields.

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