Research Article

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An inertial shrinking projection self-adaptive algorithm for solving split variational inclusion problems and fixed point problems in Banach spaces

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Abstract: In this article, we study the split variational inclusion and fixed point problems using Bregman weak relatively nonexpansive mappings in the p-uniformly convex smooth Banach spaces. We introduce an inertial shrinking projection self-adaptive iterative scheme for the problem and prove a strong convergence theorem for the sequences generated by our iterative scheme under some mild conditions in real p-uniformly convex smooth Banach spaces. The algorithm is designed to select its step size self-adaptively and does not require the prior estimate of the norm of the bounded linear operator. Finally, we provide some numerical examples to illustrate the performance of our proposed scheme and compare it with other methods in the literature.

Keywords: Bregman distance, split inclusion problem, inertial algorithm, fixed point problem, Banach spaces

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1 Introduction

Let E_1 and E_2 be p-uniformly convex real Banach spaces which are also uniformly smooth. Let $B_1 : E_1 \to 2^{E_1^*}$, $B_2 : E_2 \to 2^{E_2^*}$ be maximal monotone mappings and $A : E_1 \to E_2$ be a bounded linear mapping. In this article, we consider the split variational inclusion problem (SVIP) in real Banach spaces, which is to find $x^* \in E_1$ such that

$$0 \in B_1(x^*) \tag{1}$$

and

$$y^* = Ax^* \in E_2 \ \text{ such that } \ 0 \in B_2(y^*), \tag{2}$$

using the Bregman distance technique. We denote the solution set of (1) and (2) by Γ, that is,

$$\Gamma = \{x^* \in E_1 : 0 \in B_1(x^*) \quad \text{and} \quad y^* = Ax^* \in E_2 \quad \text{such that } \ 0 \in B_2(y^*)\}.$$

The SVIP can be reduced to numerous problems such as convex minimization problems, split variational inequality problems, split zero problems, split equilibrium problems, split feasibility problems for modeling

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the intensity-modulated radiation therapy (IMRT) treatment planning and many constrained optimization problems [1,2]. Moreover, SVIP has also been used for applications in signal processing, data compression, image reconstruction, resolution enhancement, and sensor networks; for further examples, see [3,4,22].

Now, we present the following useful notions for solving the SVIP:

(i) Let E^* be the dual space of E, denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x^*, x \rangle$.

(ii) Let $B : E \to 2^{E^*}$ be a set-valued mapping, then the domain of B is defined as

$$\mathrm{dom}(B) = \{x \in E : Bx \neq \varnothing\},$$

where the graph of B is given as

$$G(B) = \{(x, x^*) \in E \times E^* : x^* \in Bx\}.$$

(iii) A set-valued mapping B is said to be monotone if:

$$\langle x^* - y^*, x - y \rangle \geq 0 \text{ whenever } (x, x^*), (y, y^*) \in G(B)$$

and B is said to be maximal monotone if its graph is not contained in the graph of any other monotone operator on E; thus, the set:

$$B^{-1}(0) = \{\bar{x} \in E : 0 \in B(\bar{x})\}$$

is closed and convex.

(iv) Resolvent of B is the operator $\mathrm{Res}_p^{\lambda B}: E \to 2^E$ defined by

$$\mathrm{Res}_p^{\lambda B} = (J_p^E + \lambda B)^{-1} \circ J_p^E, \quad \lambda > 0. \tag{3}$$

The resolvent operator $\mathrm{Res}_p^{\lambda B}$ is a Bregman firmly nonexpansive operator and $0 \in B(x)$ if and only if $x = \mathrm{Res}_p^{\lambda B}(x)$ [5].

Let f be a given operator and B be a maximal monotone mapping under some continuity assumption on f. Then, the normal cone of some nonempty, closed, and convex set C of a Hilbert space H at a point $u \in C$ that is defined by

$$N_C(u) = \{d \in H : \langle d, y - u \rangle \leq 0, \ \forall y \in C\}.$$

If the set-valued mapping B is defined by the following:

$$B(u) = \begin{cases} f(u) + N_C(u), & \text{if } u \in C \\ \varnothing, & \text{otherwise}, \end{cases} \tag{4}$$

then the variational inclusion problem is equivalent to solving the variational inequalities, which is to find $x^* \in C$ such that $\langle f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C$.

Many authors have proposed various iterative methods for solving the SVIP and split variational inequality problems in real Hilbert spaces. Censor et al. [6] first introduced the following algorithm for solving the split variational inequality problem in real Hilbert spaces, for $x_1 \in H_1$, the sequence $\{x_n\}$ is generated by:

$$x_{n+1} = P_C(I - \lambda f)(x_n + \gamma A^*(P_Q(I - \lambda g) - I)Ax_n), \quad n \geq 1, \tag{5}$$

where $\gamma \in (0, \frac{1}{L})$ and L is the spectral radius of the operator A^*A. The authors proved a weak convergence result for the sequence generated by (5). In 2002, Byrne [3] obtained a weak convergence theorem for solving SVIP in real Hilbert spaces, using the following introduced algorithm: for a given $x_0 \in H_1$ the sequence $\{x_n\}$ is generated by:

$$x_{n+1} = J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n), \quad \lambda > 0, \tag{6}$$

where A^* is the adjoint of A, the stepsize $\gamma \in (0, \frac{2}{L})$ with $L = \|A^*A\|$. Recently, after Byrne's algorithm, Chuang [7] was motivated to introduce an algorithm that is a modification of (6) for solving SVIP; the iterative steps are outlined as follows:

$$\begin{cases} y_n = J_{\lambda_n}^{B_1}(x_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n), \\ D(x_n, y_n) = x_n - y_n - \gamma_n[A^*(I - J_{\lambda_n}^{B_2})Ax_n - A^*(I - J_{\lambda_n}^{B_2})Ay_n], \\ x_{n+1} = J_{\lambda_n}^{B_1}(x_n - \beta_n D(x, y_n)), \end{cases} \tag{7}$$

where $\beta_n = \frac{\langle x_n - y_n, D(x_n, y_n)\rangle}{\|D(x_n, y_n)\|^2}$. Moreover, inspired by the work of Byrne et al., Kazmi and Rizvi [8] proposed an algorithm for approximating a common solution of SVIP, whereby the two sequences $\{u_n\}$ and $\{x_n\}$ generated by the algorithm were both proved to converge strongly to $z \in F(S) \cap \Gamma$, where Γ is the solution set of SVIP and $F(S)$ is the fixed point of a nonexpansive mapping S. The algorithm is presented as follows: For a given $x_0 \in H_1$, let the sequences $\{u_n\}$ and $\{x_n\}$ be generated by:

$$\begin{cases} u_n = J_{\lambda}^{B_1}(x_n + \gamma A^*(J_{\lambda}^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad n \geq 0. \end{cases} \tag{8}$$

Wen and Chen [9] introduced a modified general iterative method for solving SVIP and nonexpansive semigroups that are defined as follows:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)J_{\lambda}^{B_1}[x_n + \varepsilon A^*(J_{\lambda}^{B_2} - I)Ax_n]\mathrm{d}s, \tag{9}$$

where $\gamma, \alpha_n \in [0, 1]$ and B is a strongly bounded linear operator on H_1 and a strong convergence theorem was proved. Recently, Alofi et al. [10] extended the study of SVIP from real Hilbert space to Banach spaces. The authors proposed the following algorithm for solving SVIP between the two spaces, namely, Hilbert and Banach spaces:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n)J_{\lambda_n}^{B_1}(x_n - \lambda_n A^* J_E(I - J_{\mu}^{B_2})Ax_n)), \tag{10}$$

where J_E is the duality mapping on a Banach space, $\{u_n\}$ is a sequence in Hilbert space such that $u_n \to u$ and the step size λ_n satisfies $0 < \lambda_n L < 2$. Also, Suantai et al. [11] introduced a viscosity modification in Banach spaces presented as follows:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{\lambda_n}^{B_1}(x_n - \lambda_n A^* J_E(I - J_{\mu}^{B_2})Ax_n), \tag{11}$$

where $0 < \lambda_n L < 2$ and f is a contraction. Alvarez [12], and Alvarez and Attouch [13] were motivated by the second-order time dynamical system, the heavy ball method, to introduce an inertial term that significantly updates some previous algorithms for generating the next algorithm. Furthermore, Tang [14] introduced an algorithm with an inertial term for solving SVIP in Banach spaces:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = J_{\lambda_n}^{B_1}(w_n - \lambda_n A^* J_E(I - J_{\mu}^{B_2})Aw_n), \end{cases} \tag{12}$$

where $\{\theta_n\}$ is in $(0, \bar{\theta}_n)$ and $\sum_{n=1}^{\infty}\varepsilon_n < \infty$,

$$\theta_n = \begin{cases} \min\{\theta, \varepsilon_n \max\{\|x_n - x_{n-1}\|, \|x_n - x_{n-1}\|^2\}^{-1}\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{otherwise}. \end{cases} \tag{13}$$

The study of the inertial scheme has shown its importance in continuous optimization due to its good convergence properties in that domain.

Motivated by the above work, we introduce a new inertial self-adaptive projection algorithm for solving problems (1) and (2) in p-uniformly convex and uniformly smooth real Banach spaces. We highlight our contributions in this article as follows:

(i) The SVIP is studied in p-uniformly convex and uniformly smooth real Banach spaces, which is more general than the real Hilbert spaces and 2-uniformly convex natural Banach spaces. This extends the results of [8,11,14] to mention a few.

(ii) The stepsize of our proposed algorithm is determined by a self-adaptive process that is more efficient and applicable than the methods used in [15,16].

(iii) We used an inertial technique to accelerate the proposed algorithm's convergence rate and compare its performance with other methods in the literature.

The rest of the article is organized as follows: Section 2 presents some essential notions and preliminary results needed in the form. In Section 3, we offer our iterative algorithm and its convergence analysis. In Section 4, we present some numerical experiments to illustrate the performance of the proposed algorithm. Finally, in Section 5, we offer the conclusion of the article.

2 Preliminaries

In this section, we present some important and useful definitions together with lemmas used in this article. Let E be a real Banach space with the norm $||\cdot||$ and E^* be the dual with the norm $||\cdot||_*$. We denote the strong and weak convergence of a sequence $\{x_n\} \subset E$ to $x \in E$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x^*, x\rangle = ||x||^2 = ||x^*||_*^2\}, \tag{14}$$

for all $x \in E$. Let $U = \{x \in E : ||x|| = 1\}$, E is said to be smooth if the limit

$$\lim_{\tau\to 0} \frac{||x + \tau y|| - ||x||}{\tau} \tag{15}$$

exists for all $x, y \in U$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{||x + \tau y|| + ||x - \tau y||}{2} - 1 : ||x|| = ||y|| = 1\right\}.$$

E is called *uniformly smooth* if $\lim_{\tau\to 0^+}\frac{\rho_E(\tau)}{\tau} = 0$ and *q-uniformly smooth* if there exists $C_q > 0$ such that $\rho_E(\tau) \le C_q\tau^q$. Every uniformly smooth Banach space is smooth and reflexive.

E is said to be *strictly convex* if for all $x, y \in E$, $x \ne y$, $||x|| = ||y|| = 1$, we have $||\lambda x + (1 - \lambda)y|| < 1 \quad \forall\lambda \in (0, 1)$. Let $1 < q \le 2 \le p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : ||x|| = ||y|| = 1, ||x - y|| \ge \varepsilon\right\}.$$

E is said to be *uniformly convex* if $\delta_E(\varepsilon) > 0$ and *p-uniformly convex* if there exists a constant $C_p > 0$ such that $\delta_E(\varepsilon) \ge C_p\varepsilon$, for any $\varepsilon \in (0, 2]$. The L_p space is 2-uniformly convex for $1 < p \le 2$ and p-uniformly convex for $p \ge 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive; more examples can be found in [17].

The generalized duality mapping $J_p^E : E \to 2^{E^*}$ is defined by

$$J_p^E(x) = \{u^* \in E^* : \langle u^*, x\rangle = ||x||^p, ||u^*||_* = ||x||^{p-1}\}. \tag{16}$$

If $p = 2$, (16) becomes the normalized duality mapping (14). It is known that $J_p^E(x) = ||x||^{p-2}J(x)$ for all $x \in E$, $x \neq 0$. It is well known that E is uniformly smooth if and only if J_p^E is norm-to-norm uniformly continuous on bounded subsets of E and E is smooth if and only if J_p^E is single-valued. Furthermore, E is p-uniformly convex (rep. smooth) if and only if E^* is q-uniformly smooth (rep. convex). Also, if E is p-uniformly convex and uniformly smooth, then the duality mapping J_p^E is norm-to-norm uniformly continuous on bounded subsets of E [18,19]. Examples of generalized duality mapping can be found in, for instance, [20,21]. The following lemma was proved by Xu and Roach [19].

Lemma 2.1. *Let $x, y \in E$. If E is a q-uniformly smooth Banach space, then there exists a $C_q > 0$ such that*

$$||x - y||^q \leq ||x||^q - q\langle J_q^{E^*}(x), y\rangle + C_q||y||^q.$$

Definition 2.2. A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be:
(1) *proper* if its effective domain $D(f) = \{x \in E : f(x) < +\infty\}$ is nonempty,
(2) *convex* if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ for every $\lambda \in (0, 1)$, $x, y \in D(f)$,
(3) *lower semicontinuous* at $x_0 \in D(f)$ if $f(x_0) \leq \liminf_{x\to x_0} f(x)$.

Let $x \in$ *intdomf*. For any $y \in E$, the right-hand derivative of f at x denoted by $f^0(x, y)$ is defined by:

$$f^0(x, y) = \lim_{\tau\to 0^+} \frac{f(x + \tau y) - f(x)}{\tau}. \tag{17}$$

If the limit as $\tau \to 0$ in (17) exists for any y, then the function f is said to be *Gâteaux differentiable* at x. Thus, the gradient of f at x is the function $\nabla f(x)$, which is defined by $\langle \nabla f(x), y\rangle = f^0(x, y)$ for any $y \in E$. Therefore, the function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in$ intdomf [22].

Given a Gâteaux differentiable function f, the bifunction $\Delta_f : E \times E \to [0, +\infty)$ given as:

$$\Delta_f(x, y) = f(x) - f(y) - \langle \nabla f(y), x - y\rangle, \quad \forall x, y \in E \tag{18}$$

is called the *Bregman distance* with respect to f [23]. Moreover, let $f(x) = \frac{1}{p}||x||^p$, then, the duality mapping J_p^E is the derivative of f. The Bregman distance with respect to p that is $\Delta_p : E \times E \to [0, +\infty)$ is defined by:

$$\begin{aligned}\Delta_p(x, y) &= \frac{||x||^p}{p} - \frac{||y||^p}{p} - \langle J_p^E(y), x - y\rangle \\ &= \frac{||x||^p}{p} + \frac{||y||^p}{q} - \langle J_p^E(y), x\rangle.\end{aligned} \tag{19}$$

Note that $\Delta_p(x, y) \geq 0$ and $\Delta_p(x, y) = 0$ if and only if $x = y$. Using (19), the three-point identity equality is given by:

$$\begin{aligned}\Delta_p(x, y) + \Delta_p(y, z) - \Delta_p(x, z) &= \frac{||x||^p}{p} - \frac{||y||^p}{p} - \langle J_p^E(y), x - y\rangle + \frac{||y||^p}{p} - \frac{||z||^p}{p} - \langle J_p^E(z), y - z\rangle \\ &\quad - \frac{||x||^p}{p} + \frac{||z||^p}{p} + \langle J_p^E(z), x - z\rangle \\ &= \langle J_p^E(z) - J_p^E(y), x - y\rangle \quad \forall x, y, z \in E.\end{aligned} \tag{20}$$

Furthermore,

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle J_p^E(x) - J_p^E(y), x - y\rangle \quad \forall x, y \in E.$$

For p-uniformly convex space, the metric and Bregman distance satisfy the following [24]:

$$\tau||x - y||^p \leq \Delta_p(x, y) \leq \langle J_p^E(x) - J_p^E(y), x - y\rangle, \tag{21}$$

where $\tau > 0$ is some fixed number. Lets say, if $f(x) = ||x||^2$, the Bregman distance is the *Lyapunov functional* $\phi : E \times E \to [0, +\infty)$ defined by:

$$\phi(x,y) = \|x\|^2 - 2\langle Jy, x\rangle + \|y\|^2. \tag{22}$$

Let E be a uniformly convex Banach space with the Gâteaux differentiable norm and $A : E \to 2^{E^*}$ be a maximal monotone operator, for more information see [25]. The role of the resolvent $\mathrm{Res}_p^{\lambda A}: E \to 2^E$ defined in (3) is of importance in the approximation theory of zero points of maximal monotone operators in Banach spaces. It is well known that the resolvent operator satisfies the following properties, see, e.g., [26–28]:

$$\langle \mathrm{Res}_p^{\lambda A}x - y, J_p^E(x - \mathrm{Res}_p^{\lambda A}x)\rangle \geq 0, \quad \forall y \in A^{-1}(0). \tag{23}$$

Thus, if E is a real Hilbert space, then

$$\langle J_\lambda^A x - y, x - J_\lambda^A x\rangle \geq 0, \quad \forall y \in A^{-1}(0), \tag{24}$$

where $J_\lambda^A = (I - \lambda A)^{-1}$ is the general resolvent and $A^{-1}(0) = \{z \in E : 0 \in Az\}$.

Let C be a nonempty, closed, and convex subset of E. The metric projection is defined as:

$$P_C x = \operatorname*{argmin}_{y\in C} \|x - y\|, \quad x \in E.$$

The metric projection is the unique minimizer of the distance ([29]) and it is also characterize by the following variational inequality:

$$\langle J_E^p(x - P_C x), z - P_C x\rangle \leq 0, \quad \forall z \in C. \tag{25}$$

Similarly, the Bregman projection:

$$\Pi_C(x) = \operatorname*{argmin}_{y\in C} \Delta_p(y, x), \quad x \in E, \tag{26}$$

is the unique minimizer of the Bregman distance (see [29]). The variational inequality can also characterize it:

$$\langle J_p^E(x) - J_p^E(\Pi_C x), z - \Pi_C x\rangle \geq 0 \quad \forall z \in C,$$

from which one can derive that

$$\Delta_p(y, \Pi_C x) + \Delta_p(\Pi_C x, x) \leq \Delta_p(y, x), \quad \forall y \in C. \tag{27}$$

The metric projection generally differs from the Bregman projection, but in Hilbert spaces, both projections coincide.

Associated with the Bregman distance f_p is the functional $V_p : E \times E^* \to [0, +\infty)$ defined by:

$$V_p(x, \bar{x}) = \frac{1}{p}\|x\|^p - \langle \bar{x}, x\rangle + \frac{1}{q}\|\bar{x}\|^q, \quad x \in E, \quad \bar{x} \in E^*.$$

We can see that $V_p(x, \bar{x}) \geq 0$ and the following properties are satisfied:

$$V_p(x, \bar{x}) = \Delta_p(x, J_q^{E^*}(\bar{x})), \quad \forall x \in E, \quad \bar{x} \in E^*,$$

and

$$V_p(x, \bar{x}) + \langle \bar{y}, J_q^{E^*}(\bar{x}) - x\rangle \leq V_p(x, \bar{x} + \bar{y}), \quad \forall x \in E, \quad \bar{x}, \bar{y} \in E. \tag{28}$$

Also, V_p is convex in the second variable. Then for all $z \in E$,

$$\Delta_p\left(z, J_q^{E^*}\left(\sum_{i=1}^N t_i J_p^E x_i\right)\right) \leq \sum_i^N t_i \Delta_p(z, x_i),$$

where $\{x_i\} \subset E$ and $\{t_i\} \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Let C be a convex subset of $\mathrm{intdom} f_p$, where $f_p = (\frac{1}{p})\|x\|^p, \quad 2 \leq p < \infty$, thus an *asymptotic fixed point* of T is a point $p \in C$ such that if C contains a sequence $\{x_n\}$ which converges weakly to p and

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

$\hat{F}(T)$ denotes the set of asymptotic fixed points of T. Then a point $p \in C$ is called a *strong asymptotic fixed point* of T if C contains a sequence $\{x_n\}$ which converges strongly to p and

$$\lim_{n\to\infty} ||x_n - Tx_n|| = 0.$$

$\tilde{F}(T)$ denotes the set of strong asymptotic fixed points of T. Thus, from the above definitions, we deduce that $F(T) \subset \tilde{F}(T) \subset \hat{F}(T)$, the introduction of these asymptotic fixed points was studied in the previous study [30].

Definition 2.3. A mapping $T : C \to C$ is said to be:
(1) *Bregman quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\Delta_p(x^*, Ty) \le \Delta_p(x^*, y), \quad \forall y \in C, \quad x^* \in F(T),$$

(2) *Bregman weak relatively nonexpansive* if $\tilde{F}(T) \neq \emptyset$, $\tilde{F}(T) = F(T)$, and

$$\Delta_p(x^*, Ty) \le \Delta_p(x^*, y), \quad \forall y \in C, \quad x^* \in F(T),$$

(3) *Bregman relatively nonexpansive* if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$, and

$$\Delta_p(x^*, Ty) \le \Delta_p(x^*, y), \quad \forall y \in C, \quad x^* \in F(T).$$

From Definition (2.3), we have noted that the class of Bregman quasi-nonexpansive contains the class of Bregman weak relatively nonexpansive, and finally the class of Bregman weak relatively nonexpansive contains the class of Bregman relatively nonexpansive, more on Bregman relatively nonexpansive mappings can be found in [31].

The following results are necessary for establishing our main results.

Lemma 2.4. [32] *Let E be a smooth and uniformly convex real Banach space. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. Then* $\lim_{n\to\infty}\Delta_p(x_n, y_n) = 0$ *if and only if* $\lim_{n\to\infty}||x_n - y_n|| = 0$.

Lemma 2.5. [19] *Let $q \ge 1$ and $r > 0$ be two fixed real numbers. Then, a Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$, $g(0) = 0$ such that for all $x, y \in B_r$ and $0 \le \lambda \le 1$,*

$$||\lambda x + (1-\lambda)y||^q \le \lambda||x||^q + (1-\lambda)||y||^q - W_q(\lambda)g(||x-y||),$$

where $W_q(\lambda) = \lambda^q(1-\lambda) + \lambda(1-\lambda)^q$ and $B_r = \{x \in E : ||x|| \le r\}$.

3 Main results

In this section, we present our algorithm and the convergence analysis.

Algorithm 3.1. Let E_1, E_2 be p-uniformly convex smooth Banach spaces with their duals E_1^*, E_2^*, respectively. Let $C = C_1$ be a nonempty closed convex subset of E_1. Let $T : E_1 \to E_1$ be a Bregman weak relatively nonexpansive mapping and $A : E_1 \to E_2$ be a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$. Let $B_1 : E_1 \to 2^{E_1^*}$ and $B_2 : E_2 \to 2^{E_2^*}$ be two maximal monotone mappings with their resolvent operators $\mathrm{Res}_p^{\lambda B_1}$ and $\mathrm{Res}_p^{\lambda B_2}$, respectively. Let the solution set $\mathrm{Sol} := \Gamma \cap F(T)$ be nonempty. We choose $x_0, x_1 \in E_1$, let $\{\theta_n\}$ be a real sequence such that $-\theta \le \theta_n \le \theta$ for some $\theta > 0$ and $\{\delta_n\}, \{\beta_n\} \subset (0,1)$ be real sequences satisfying $\liminf_{n\to\infty}\delta_n > 0$ and $\liminf_{n\to\infty}\beta_n > 0$. Please assume that the $(n-1)$th and nth-iterates have been constructed, and then we calculate the $(n+1)$th-iterate. $x_{n+1} \in E_1$, we present

$$\begin{cases} w_n = J_q^{E_1^*}(J_p^{E_1}x_n + \theta_n(J_p^{E_1}x_n - J_p^{E_1}x_{n-1})), \\ y_n = \text{Res}_p^{\lambda B_1} J_q^{E_1^*}(J_p^{E_1}w_n - \mu_n A^* J_p^{E_2}(I - \text{Res}_p^{\lambda B_2})Aw_n), \\ z_n = J_q^{E_1^*}(\beta_n J_p^{E_1}(w_n) + (1-\beta_n)(\delta_n J_p^{E_1}(y_n) + (1-\delta_n)J_p^{E_1}Ty_n)), \\ C_{n+1} = \{w \in C_n : \Delta_p(w, z_n) \le \Delta_p(w, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 1. \end{cases} \tag{29}$$

Assume for small $\varepsilon > 0$, the stepsize μ_n is chosen such that

$$\mu_n^{q-1} \in \left(\varepsilon, \frac{q\|(I - \text{Res}_p^{\lambda B_2})Aw_n\|^p}{C_q\|A^*J_p^{E_2}(I - \text{Res}_p^{\lambda B_2})Aw_n\|_*^q}\right), \quad n \in \Omega, \tag{30}$$

where the index set $\Omega = \{n \in \mathbb{N} : (I - \text{Res}_p^{\lambda B_2})Aw_n \neq 0\}$, otherwise $\mu_n = \mu$, where μ is any non-negative real number.

Lemma 3.2. *The sequence* $\{\mu_n\}$ *defined by* (30) *is well defined.*

Proof. $x^* \in \text{Sol}$, then $x^* = Tx^*$ and $Ax^* = \text{Res}_p^{\lambda B_2}Ax^*$. Thus,

$$\begin{aligned} \|(I - \text{Res}_p^{\lambda B_2})Aw_n\|^p &= \langle J_p^{E_2}(I - \text{Res}_p^{\lambda B_2})Aw_n, Aw_n - \text{Res}_p^{\lambda B_2}Aw_n\rangle \\ &= \langle J_p^{E_2}(I - \text{Res}_p^{\lambda B_2})Aw_n, Aw_n - Ax^* + \text{Res}_p^{\lambda B_2}Ax^* - \text{Res}_p^{\lambda B_2}Aw_n\rangle \\ &= \langle J_p^{E_2}(I - \text{Res}_p^{\lambda B_2})Aw_n, Aw_n - Ax^*\rangle \\ &\quad + \langle J_p^{E_2}(I - \text{Res}_p^{\lambda B_2})Aw_n, \text{Res}_p^{\lambda B_2}Ax^* - \text{Res}_p^{\lambda B_2}Aw_n\rangle \\ &= \langle A^*J_p^{E_2}(I - \text{Res}_p^{\lambda B_2})Aw_n, w_n - x^*\rangle \\ &\quad + \langle J_p^{E_2}(I - \text{Res}_p^{\lambda B_2})Aw_n, \text{Res}_p^{\lambda B_2}Ax^* - \text{Res}_p^{\lambda B_2}Aw_n\rangle \\ &\le \|w_n - x^*\|\|A^*J_p^{E_2}(I - \text{Res}_p^{\lambda B_2})Aw_n\|_* \\ &\quad + \|\text{Res}_p^{\lambda B_2}Ax^* - \text{Res}_p^{\lambda B_2}Aw_n\|\|J_p^{E_2}(I - \text{Res}_p^{\lambda B_2})Aw_n\|_* \\ &= \|w_n - x^*\|\|A^*J_p^{E_2}(I - \text{Res}_p^{\lambda B_2})Aw_n\|_* + \|\text{Res}_p^{\lambda B_2}Ax^* - \text{Res}_p^{\lambda B_2}Aw_n\| \\ &\quad \|(I - \text{Res}_p^{\lambda B_2})Aw_n\|^{p-1}. \end{aligned} \tag{31}$$

As a result, for $n \in \Omega$, then $\|(I - \text{Res}_p^{\lambda B_2})Aw_n\| > 0$, we obtain $\|w_n - x^*\|\|A^*J_p^{E_2}(I - \text{Res}_p^{\lambda B_2})Aw_n\|_* > 0$ and $\|\text{Res}_p^{\lambda B_2}Ax^* - \text{Res}_p^{\lambda B_2}Aw_n\|\|(I - \text{Res}_p^{\lambda B_2})Aw_n\|^{p-1} > 0$. Thus, $\|\text{Res}_p^{\lambda B_2}Ax^* - \text{Res}_p^{\lambda B_2}Aw_n\|\|(I - \text{Res}_p^{\lambda B_2})Aw_n\|^{p-1} > 0$, which results in $\|A^*J_p^{E_2}(I - \text{Res}_p^{\lambda B_2})Aw_n\|_* \neq 0$, implying that μ_n is well defined. □

Lemma 3.3. *For every* $n \ge 1$, $\text{Sol} \subset C_n$ *and* x_{n+1} *defined by Algorithm* (3.1) *is well defined.*

Proof. Let $C_1 = C$ be closed and convex. Suppose C_k is closed and convex for some $k \in \mathbb{N}$. Then,

$$\begin{aligned} C_{k+1} &= \{w \in C_k : \Delta_p(w, z_k) \le \Delta_p(w, v_k)\} \\ &= \left\{w \in C_k : \frac{\|w\|^p}{p} + \frac{\|z_k\|^p}{q} - \langle J_p^{E_1}z_k, w\rangle \le \frac{\|w\|^p}{p} + \frac{\|v_k\|^p}{q} - \langle J_p^{E_1}v_k, w\rangle\right\} \\ &= \{w \in C_k : \|z_k\|^p - \|v_k\|^p \le q\langle J_p^{E_1}z_k - J_p^{E_1}v_k, w\rangle\}, \end{aligned} \tag{32}$$

it follows that C_{k+1} is closed. Let $w_1, w_2 \in C_{k+1}$, and $\lambda_1, \lambda_2 \in (0,1)$ such that $\lambda_1 + \lambda_2 = 1$, then we have:

$$\begin{aligned} \|z_k\|^p - \|v_k\|^p &\leq q\langle J_p^{E_1} z_k - J_p^{E_1} v_k, w_1\rangle \\ &\text{and} \\ \|z_k\|^p - \|v_k\|^p &\leq q\langle J_p^{E_1} z_k - J_p^{E_1} v_k, w_2\rangle, \end{aligned} \tag{33}$$

then from (33), we have

$$\|z_k\|^p - \|v_k\|^p \leq q\langle J_p^{E_1} z_k - J_p^{E_1} v_k, \lambda_1 w_1 + \lambda_2 w_2\rangle. \tag{34}$$

By convexity, $\lambda_1 w_1 + \lambda_2 w_2 \in C_k$. From (34), we can therefore conclude that $\lambda_1 w_1 + \lambda_2 w_2 \in C_{k+1}$ and thus, C_{k+1} is convex. Therefore, C_n is convex for all $n \in \mathbb{N}$. Also, since Sol $\neq \varnothing$, it implies that $C_{n+1} \neq \varnothing$. In order to show that Sol $\subset C_n$, $\forall n \geq 1$, we let $x^* \in$ Sol, then $x^* \in F(T)$ and $Ax^* \in F(\mathrm{Res}_p^{\lambda B_2})$ therefore by our construction from (30), we have

$$\begin{aligned} \Delta_p(x^*, z_n) &= \Delta_p(x^*, \beta_n J_p^{E_1}(w_n) + (1-\beta)(\delta_n J_p^{E_1}(y_n) + (1-\delta_n) J_p^{E_1} T y_n)) \\ &\leq \beta_n \Delta_p(x^*, w_n) + (1-\beta_n)(\delta_n \Delta_p(x^*, y_n) + (1-\delta_n)\Delta_p(x^*, y_n)) \\ &= \beta_n \Delta_p(x^*, w_n) + (1-\beta_n)\Delta_p(x^*, y_n) \\ &\leq \Delta_p(x^*, w_n) + \Delta_p(x^*, y_n). \end{aligned} \tag{35}$$

Using (26), Lemma 2.1, and the definition of Bregman distance (19), we have the following:

$$\begin{aligned} \Delta_p(x^*, y_n) &= \Delta_p(x^*, \mathrm{Res}_p^{\lambda B_1} J_q^{E_1^*}(J_p^{E_1} w_n - \mu_n A^* J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n)) \\ &= \frac{\|x^*\|^p}{p} - \langle J_p^{E_1} w_n - \mu_n A^* J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n, x^*\rangle \\ &\quad + \frac{\|J_p^{E_1} w_n - \mu_n A^* J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q}{q} \\ &\leq \frac{\|x^*\|^p}{p} - \langle J_p^{E_1} w_n - \mu_n A^* J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n, x^*\rangle + \frac{\|J_p^{E_1} w_n\|_*^q}{q} \\ &\quad - \mu_n\langle J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n, Aw_n\rangle + \frac{C_q}{q}\mu_n^q \|A^* J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q \\ &= \frac{\|x^*\|^p}{p} - \langle J_p^{E_1} w_n, x^*\rangle + \frac{\|J_p^{E_1} w_n\|_*^q}{q} - \mu_n\langle J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n, Aw_n - Ax^*\rangle \\ &\quad + \frac{C_q}{q}\mu_n^q \|A^* J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q \\ &= V_p(x^*, J_p^{E_1} w_n) - \mu_n\langle J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n, Aw_n - Ax^*\rangle + \frac{C_q}{q}\mu_n^q \|A^* J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q \\ &= \Delta_p(x^*, w_n) - \mu_n\langle J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n, Aw_n - Ax^*\rangle \\ &\quad + \frac{C_q}{q}\mu_n^q \|A^* J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q. \end{aligned} \tag{36}$$

From property (23), it follows that

$$\langle J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n, \mathrm{Res}_p^{\lambda B_2} Aw_n - Ax^*\rangle \geq 0.$$

Thus, we have

$$\begin{aligned} \langle J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n, Aw_n - Ax^*\rangle &= \langle J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n, Aw_n - \mathrm{Res}_p^{\lambda B_2} Aw_n + \mathrm{Res}_\alpha Aw_n - Ax^*\rangle \\ &= \|Aw_n - \mathrm{Res}_p^{\lambda B_2} Aw_n\|^p + \langle J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n, \mathrm{Res}_p^{\lambda B_2} Aw_n - Ax^*\rangle \\ &\geq \|Aw_n - \mathrm{Res}_p^{\lambda B_2} Aw_n\|^p, \end{aligned} \tag{37}$$

substituting (37) into (36), then

$$\begin{aligned}\Delta_p(x^*, y_n) &\leq \Delta_p(x^*, w_n) - \mu_n\|Aw_n - \mathrm{Res}_p^{\lambda B_2}Aw_n\|^p + \frac{C_q}{q}\mu_n^q\|A^*J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q \\ &\leq \Delta_p(x^*, w_n) - \mu_n\left(\|Aw_n - \mathrm{Res}_p^{\lambda B_2}Aw_n\|^p - \frac{C_q}{q}\mu_n^{q-1}\|A^*J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q\right)\end{aligned} \tag{38}$$

$$\leq \Delta_p(x^*, w_n), \tag{39}$$

the condition on the step-size (30) was used on (38) to obtain (39). Thus, from (35) and (39), we have

$$\Delta_p(x^*, z_n) \leq \Delta_p(x^*, w_n) + \Delta_p(x^*, y_n) \leq \Delta_p(x^*, w_n), \tag{40}$$

which shows that $\mathrm{Sol} \subset C_{n+1}, \ \forall n \in \mathbb{N}$. □

Lemma 3.4. *The sequences $\{x_n\}, \{z_n\}, \{y_n\}$, and $\{w_n\}$ are bounded.*

Proof. We know from Algorithm (3.1) that $x_n = \Pi_{C_n}x_0$ and $C_{n+1} \subseteq C_n, \ \forall n \geq 1$. Therefore, from the Bregman projection (26), we have $\Delta_p(x_n, x_0) \leq \Delta_p(x_{n+1}, x_0)$, which then shows that $\{\Delta_p(x_n, x_0)\}$ is nondecreasing. Now since $\mathrm{Sol} \subset C_{n+1}$, this implies that $\Delta_p(x_n, x_0) \leq \Delta_p(x_{n+1}, x_0) \leq \Delta_p(x^*, x_0), \ \forall x^* \in \mathrm{Sol}$. From (21), we then conclude that $\{x_n\}$ is bounded and thus from our construction $\{z_n\}, \{y_n\}$, and $\{w_n\}$ are also bounded. □

Lemma 3.5. *Let the sequences $\{x_n\}, \{z_n\}, \{y_n\}$, and $\{w_n\}$ be as defined in Algorithm (3.1). Assuming that for small $\varepsilon > 0$,*

$$\mu_n \in \left(\varepsilon, \left(\frac{q\|Aw_n - \mathrm{Res}_p^{\lambda B_2}Aw_n\|^p}{C_q\|A^*J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q} - \varepsilon\right)^{\frac{1}{q-1}}\right), \quad n \in \Omega. \tag{41}$$

Then, we have
(i) $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$;
(ii) $\lim_{n\to\infty}\|w_n - x_n\| = 0$;
(iii) $\lim_{n\to\infty}\|y_n - Ty_n\| = 0$;
(iv) $\lim_{n\to\infty}\|x_n - y_n\| = 0$;
(v) $\lim_{n\to\infty}\|A^*J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_* = 0$ *and* $\lim_{n\to\infty}\|(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\| = 0$.

Proof. Following from Lemma 3.4, it is said that $\{\Delta_p(x_n, x_0)\}$ is a nondecreasing sequence in $\mathbb{R}$. Thus, $\lim_{n\to\infty}\Delta_p(x_n, x_0)$ exists. Now we then use (27) and have the following:

$$\Delta_p(x_{n+1}, \Pi_{C_n}x_0) + \Delta_p(\Pi_{C_n}x_0, x_0) \leq \Delta_p(x_{n+1}, x_0), \tag{42}$$

which yields

$$\Delta_p(x_{n+1}, x_n) \leq \Delta_p(x_{n+1}, x_0) - \Delta_p(x_n, x_0) \to 0. \tag{43}$$

After applying Lemma 2.4, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0, \tag{44}$$

which proves (i). From our construction $w_n = J_q^{E_1^*}[J_p^{E_1}x_n + \theta_n(J_p^{E_1}x_n - J_p^{E_1}x_{n-1})]$. Then, we have the following:

$$J_p^{E_1}w_n - J_p^{E_1}x_n = \theta_n(J_p^{E_1}x_n - J_p^{E_1}x_{n-1}).$$

Using the uniform continuity of $J_p^{E_1}$ on bounded subsets of E_1, we obtain the following:

$$\begin{aligned}\|J_p^{E_1}w_n - J_p^{E_1}x_n\|_* &= \|\theta_n(J_p^{E_1}x_n - J_p^{E_1}x_{n-1})\|_* \\ &\leq \theta\|J_p^{E_1}x_n - J_p^{E_1}x_{n-1}\|_* \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{45}$$

Following from the uniform continuity of $J_q^{E_1^*}$ on bounded subsets of E_1^* and (44), we result with

$$\lim_{n\to\infty} \|w_n - x_n\| = 0, \tag{46}$$

which proves (ii). Now we combine (i) and (ii), which yields $\lim_{n\to\infty}\|x_{n+1} - w_n\| = 0$. Moreover, now that we have $x_{n+1} \in C_{n+1}$, we obtain the following:

$$\Delta_p(x_{n+1}, z_n) \le \Delta_p(x_{n+1}, w_n) \to 0 \quad \text{as } n \to \infty. \tag{47}$$

Thus with Lemma 2.4 we have that $\lim_{n\to\infty}\|x_{n+1} - z_n\| = 0$, which then together with (44) give us:

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{48}$$

Furthermore, from (46) and (48), the following results:

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{49}$$

Let $v_n = J_p^{E_1^*}(\delta_n J_p^{E_1}(y_n) + (1-\delta_n)J_p^{E_1}Ty_n)$, then

$$\begin{aligned}
\Delta_p(x^*, z_n) &= \Delta_p(x^*, J_p^{E_1^*}(\delta_n J_p^{E_1}(y_n) + (1-\delta_n)J_p^{E_1}Ty_n)) \\
&= V_p(x^*, \delta_n J_p^{E_1}(y_n) + (1-\delta_n)J_p^{E_1}Ty_n) \\
&= \frac{\|x^*\|^p}{p} - \langle x^*, \delta_n J_p^{E_1}(y_n) + (1-\delta_n)J_p^{E_1}Ty_n\rangle + \frac{1}{q}\|\delta_n J_p^{E_1}(y_n) + (1-\delta_n)J_p^{E_1}Ty_n\|_*^q \\
&\le \frac{\|x^*\|^p}{p} - \delta_n\langle x^*, J_p^{E_1}(y_n)\rangle - (1-\delta_n)\langle x^*, J_p^{E_1}Ty_n\rangle \\
&\quad + \frac{1}{q}\delta_n\|y_n\|^q + \frac{1}{q}(1-\delta_n)\|Ty_n\|^q - \frac{W_q(\delta_n)}{q}g(\|J_p^{E_1}(y_n) - J_p^{E_1}Ty_n\|_*) \\
&= \delta_n\Delta_p(x^*, y_n) + (1-\delta_n)\Delta_p(x^*, Ty_n) - \frac{W_q(\delta_n)}{q}g(\|J_p^{E_1}(y_n) - J_p^{E_1}Ty_n\|_*) \\
&\le \Delta_p(x^*, y_n) - \frac{W_q(\delta_n)}{q}g(\|J_p^{E_1}(y_n) - J_p^{E_1}Ty_n\|_*) \\
&\le \Delta_p(x^*, w_n) - \frac{W_q(\delta_n)}{q}g(\|J_p^{E_1}(y_n) - J_p^{E_1}Ty_n\|_*),
\end{aligned} \tag{50}$$

where (50) was obtained using Lemma 2.5 and (39). Hence,

$$\Delta_p(x^*, z_n) \le \Delta_p(x^*, w_n). \tag{51}$$

Following from (50), we obtain the following results:

$$\begin{aligned}
\frac{W_q(\delta_n)}{q}g(\|J_p^{E_1}(y_n) - J_p^{E_1}Ty_n\|_*) &\le \Delta_p(x^*, w_n) - \Delta_p(x^*, z_n) \\
&= \langle J_p^{E_1}z_n - J_p^{E_1}w_n, x^* - w_n\rangle - \Delta_p(z_n, w_n) \\
&\le \langle J_p^{E_1}z_n - J_p^{E_1}w_n, x^* - w_n\rangle \\
&= \|x^* - w_n\|\|J_p^{E_1}z_n - J_p^{E_1}w_n\|_*.
\end{aligned} \tag{52}$$

Now that $J_q^{E_1}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, when we take the limit of (52) as $n \to \infty$ we have $\frac{W_q(\beta_n)}{q}g(\|J_p^{E_1}(y_n) - J_p^{E_1}Ty_n\|_*) \to 0$. Therefore, we obtain that:

$$g(\|J_p^{E_1}(y_n) - J_p^{E_1}Ty_n\|_*) \to 0 \quad \text{as } n \to \infty.$$

Now, according to the continuity of g, it then implies that

$$\|J_p^{E_1}(y_n) - J_p^{E_1}Ty_n\|_* \to 0 \quad \text{as } n \to \infty. \tag{53}$$

Now that $J_q^{E_1}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, (53) implies that

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0, \tag{54}$$

which proves (iii).

We then use Lemma 28 and (54), which implies that $\lim_{n\to\infty}\Delta_p(y_n, Ty_n) = 0$. Thus,

$$\begin{aligned}\Delta_p(y_n, z_n) &= \Delta_p(y_n, J_q^{E_1^*}[\beta_n J_p^{E_1}(w_n) + (1-\beta_n)(\delta_n J_p^{E_1}(y_n) + (1-\delta_n)J_p^{E_1}Ty_n)]) \\ &\le \beta_n\Delta_p(y_n, w_n) + (1-\beta_n)\delta_n\Delta_p(y_n, y_n) + (1-\beta_n)(1-\delta_n)\Delta_p(y_n, Ty_n) \\ &= \beta_n\Delta_p(y_n, w_n) + (1-\beta_n)(1-\delta_n)\Delta_p(y_n, Ty_n) \to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{55}$$

Moreover, applying Lemma 2.4, we obtain the following results:

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{56}$$

Following from (46) and (54), we have the following:

$$\lim_{n\to\infty} \|x_n - y_n\| = 0, \tag{57}$$

which proves (iv). Thus, from (49) and (56), we obtain that:

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{58}$$

Furthermore, from (38), we obtain the following:

$$\begin{aligned}&\mu_n\left(\|Aw_n - \mathrm{Res}_p^{\lambda B_2}Aw_n\|^p - \frac{C_q}{q}\mu_n^{q-1}\|A^*J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q\right) \\ &\le \Delta_p(x^*, w_n) - \Delta_p(x^*, y_n) \\ &= \langle J_p^{E_1}y_n - J_p^{E_1}w_n, x^* - w_n\rangle - \Delta_p(y_n, w_n) \\ &\le \langle J_p^{E_1}y_n - J_p^{E_1}w_n, x^* - w_n\rangle \\ &= \|x^* - w_n\|\|J_p^{E_1}y_n - J_p^{E_1}w_n\|_*.\end{aligned} \tag{59}$$

We now then apply the limit as $n\to\infty$ in (59) and also use (58), we have the following:

$$\lim_{n\to\infty}\left(\|Aw_n - \mathrm{Res}_p^{\lambda B_2}Aw_n\|^p - \frac{C_q}{q}\mu_n^{q-1}\|A^*J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q\right) = 0. \tag{60}$$

Now, we put in mind the choice of our step size for the following to hold:

$$\mu_n^{q-1} \le \frac{q\|Aw_n - \mathrm{Res}_p^{\lambda B_2}Aw_n\|^p}{C_q\|A^*J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q} - \varepsilon. \tag{61}$$

Simplification of (61) leads to

$$\frac{\varepsilon C_q}{q}\|A^*J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q < \left(\|Aw_n - \mathrm{Res}_p^{\lambda B_2}Aw_n\|^p - \frac{C_q}{q}\mu_n^{q-1}\|A^*J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q\right). \tag{62}$$

Applying the limit as $n\to\infty$ in (62) and also applying (60), we have that

$$\lim_{n\to\infty} \|A^*J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q = 0; \tag{63}$$

furthermore,

$$\lim_{n\to\infty} \|A^* J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_* = 0$$
$$\text{and} \tag{64}$$
$$\lim_{n\to\infty} \|(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\| = 0,$$

which proves (v). □

4 Strong convergence of Algorithm (3.1)

Theorem 4.1. *The sequence $\{x_n\}$ generated by Algorithm* (3.1) *converges strongly to $v \in \mathrm{Sol}$, where $v = \Pi_{\mathrm{Sol}} x_0$.*

Proof. We know that $\{\Delta_p(x_n, x_0)\}$ is nondecreasing and bounded in $\mathbb{R}$, it then implies that there exists $L \in \mathbb{R}$ such that $\Delta_p(x_n, x_0) \to L$ as $n \to \infty$. We now use (27) to obtain that for every $m, n \in \mathbb{N}$,

$$\Delta_p(x_m, x_n) = \Delta_p(x_m, \Pi_{C_n} x_0) \le \Delta_p(x_m, x_0) - \Delta_p(x_n, x_0) \to 0. \tag{65}$$

Thus, following from Lemma 2.4, we obtain that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Therefore, it shows that $\{x_n\}$ is a Cauchy sequence in C. Now, because C is a closed convex subset of a Banach space, it then implies that there exists $v \in C$ such that $x_n \to v$ as $n \to \infty$. We follow from Lemma 3.5 that $w_n \to v$ and $y_n \to v$ as $n \to \infty$. Using the linearity of A, we obtain that $Aw_n \to Av$ as $n \to \infty$. Now that in Lemma 3.5, we have proven that $\|y_n - Ty_n\| \to 0$ as $n \to \infty$, with T being the Bregman weak relatively nonexpansive simply means that $v \in F(T)$. Moreover, we have also proven that $\|(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\| \to 0$ as $n \to \infty$, implying that $Av \in \tilde{F}(v)$; thus, $Av \in F(v)$. Therefore, it implies that $v \in \mathrm{Sol}$. Now, we prove that $v = \Pi_{\mathrm{Sol}} x_0$. To prove that suppose that there exists $y \in \mathrm{Sol}$ such that $y = \Pi_{\mathrm{Sol}} x_0$. Then, we have

$$\Delta_p(y, x_0) \le \Delta_p(v, x_0). \tag{66}$$

Thus, $\Delta_p(x_n, x_0) \le \Delta_p(y, x_0)$, reason being that $\mathrm{Sol} \in C_n$ for all $n \ge 1$. We then use the lower semicontinuity of the norm to obtain

$$\begin{aligned} \Delta_p(v, x_0) &= \frac{\|v\|^p}{p} + \frac{\|x_0\|^p}{q} - \langle J_p^{E_1} x_0, v\rangle \\ &\le \liminf_{n\to\infty} \left(\frac{\|x_n\|^p}{p} + \frac{\|x_0\|^p}{q} - \langle J_p^{E_1} x_0, x_n\rangle \right) \\ &= \liminf_{n\to\infty} \Delta_p(x_n, x_0) \\ &\le \limsup_{n\to\infty} \Delta_p(x_n, x_0) \\ &\le \Delta_p(y, x_0). \end{aligned} \tag{67}$$

Now, following from (66) and (67), we obtain the following:

$$\Delta_p(y, x_0) \le \Delta_p(v, x_0) \le \Delta_p(y, x_0), \tag{68}$$

which implies that $v = y$. Therefore, $v = \Pi_{\mathrm{Sol}} x_0$. □

We now present the following arguments of our main results below. If $\theta_n = 0$, we have the non-inertial shrinking projection algorithm.

Corollary 4.2. *Let E_1, E_2 be p-uniformly convex and uniformly smooth Banach spaces with their duals E_1^*, E_2^*, respectively. Let $C = C_1$ be a nonempty closed and convex subset of E_1 and also $T : E_1 \to E_1$ be a Bregman weak relatively nonexpansive mapping with $A : E_1 \to E_2$ be a bounded linear operator and its adjoint $A^* : E_2^* \to E_1^*$. Let $B_1 : E_1 \to 2^{E_1^*}$ and $B_2 : E_2 \to 2^{E_2^*}$ be two maximal monotone mappings with their resolvent operators* $\mathrm{Res}_p^{\lambda B_1}$ *and* $\mathrm{Res}_p^{\lambda B_2}$, *respectively. Choose $x_0 \in E_1$, let $\{\gamma_n\}, \{\delta_n\}, \{\beta_n\} \subset (0, 1)$ be real sequences satisfying* $\liminf_{n\to\infty} \delta_n > 0$

and $\liminf_{n\to\infty}\beta_n > 0$. *Assume that the nth-iterate,* $x_n \in E_1$ *has been constructed already, we then calculate the* $(n+1)$*th-iterate,* $x_{n+1} \in E_1$, *we present:*

$$\begin{cases} w_n = J_q^{E_1^*} x_n, \\ y_n = \mathrm{Res}_p^{\lambda B_1} J_q^{E_1^*}(J_p^{E_1} w_n - \mu_n A^* J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n), \\ z_n = J_q^{E_1^*}(\beta_n J_p^{E_1}(w_n) + (1-\beta_n)(\delta_n J_p^{E_1}(y_n) + (1-\delta_n) J_p^{E_1} T y_n)), \\ C_{n+1} = \{w \in C_n : \Delta_p(w, z_n) \le \Delta_p(w, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 1. \end{cases} \tag{69}$$

Assume for small $\varepsilon > 0$, *we choose the stepsize* μ_n *such that:*

$$\mu_n \in \left(\varepsilon, \left(\frac{q\|Aw_n - \mathrm{Res}_p^{\lambda B_2} Aw_n\|^p}{C_q \|A^* J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n\|_*^q} - \varepsilon\right)^{\frac{1}{q-1}}\right), \quad n \in \Omega, \tag{70}$$

whereby the index set is $\Omega = \{n \in \mathbb{N} : Aw_n - \mathrm{Res}_p^{\lambda B_2} Aw_n \neq 0\}$, *otherwise* $\mu_n = \mu$, *with* μ *being any non-negative real number. Therefore,* $\{x_n\}$ *converges strongly to* $v \in \mathrm{Sol}$, *where* $v = \Pi_{\mathrm{Sol}} x_0$.

Next, we check the performance of the convergence for the sequence $\{x_n\}$ generated by Algorithm 3.1 when we let $\mathrm{Res}_p^{\lambda B_1}, \mathrm{Res}_p^{\lambda B_2}$ be metric projection mappings onto a closed convex subset Q_1 of E_1 and Q_2 of E_2, respectively, in Algorithm 3.1. The following results were obtained: the solution to the split feasibility and fixed point problems.

Corollary 4.3. *We follow from our construction of Algorithm* 3.1, *and let* Q_1 *and* Q_2 *be nonempty closed convex subsets of* E_1 *and* E_2, *respectively, with* $\mathrm{Res}_p^{\lambda B_1} = P_{Q_1}$ *and* $\mathrm{Res}_p^{\lambda B_2} = P_{Q_2}$. *Therefore, assume that* $\Theta = \{x \in C : x \in F(T), Ax \in Q\} \neq \emptyset$. *Thus, we can conclude that the sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges strongly to* $v \in \Theta$, *where* $v = \Pi_\Theta x_0$.

Corollary 4.4. *Let* H_1, H_2 *be real Hilbert spaces and* $C = C_1$ *be a nonempty closed convex subset of* H_1. $A : H_1 \to H_2$ *be a bounded linear operator and the adjoint operator of* A *is* $A^* : H_2 \to H_1$. *Let* $B_1 : H_1 \to 2^{H_1}$ *and* $B_2 : H_2 \to 2^{H_2}$ *be two multi-valued maximal monotone mappings, with their resolvent operators* $J_\alpha^{B_1}, J_\alpha^{B_2}$, *respectively. We choose* $x_0, x_1 \in H_1$, *let* $\{\theta_n\}$ *be a real sequence such that* $-\theta \le \theta_n \le \theta$ *for some* $\theta > 0$ *and* $\{\delta_n\}, \{\beta_n\} \subset (0,1)$ *be real sequences satisfying* $\liminf_{n\to\infty}\delta_n > 0$ *and* $\liminf_{n\to\infty}\beta_n > 0$. *Assume that the* $(n-1)$ *th and* n *th-iterates have been constructed, then we calculate the* $(n+1)$ *th-iterate.* $x_{n+1} \in H_1$, *we present*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = J_\lambda^{B_1}(w_n - \mu_n A^*(I - J_\lambda^{B_2})Aw_n), \\ z_n = \beta_n(w_n) + (1-\beta_n)(\delta_n y_n + (1-\delta_n)Ty_n), \\ C_{n+1} = \{w \in C_n : \|w - z_n\| \le \|w - w_n\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \ge 1. \end{cases} \tag{71}$$

Assume for small $\varepsilon > 0$, *the stepsize* μ_n *is chosen such that*

$$\mu_n \in \left(0, \frac{2\|(I - J_\lambda^{B_2})Aw_n\|^2}{\|A^*(I - J_\lambda^{B_2})Aw_n\|^2}\right), \quad n \in \Omega, \tag{72}$$

where the index set $\Omega = \{n \in \mathbb{N} : (I - J_\alpha^{B_2})Aw_n \neq 0\}$, *otherwise* $\mu_n = \mu$, *with* μ *being any non-negative real number.*

5 Numerical illustrations

In this section, we present some numerical examples to illustrate the convergence and efficiency of the proposed algorithms.

Example 5.1. Let $E_1 = E_2 = \mathbb{R}$ and $C = C_1 = [0, 3]$, with $T : E_1 \to E_1$ defined by:

$$Tx = \begin{cases} 0 & \text{if } x \neq 3, \\ 2 & \text{if } x = 3, \end{cases} \tag{73}$$

$\forall x \in E_1$. Then, T is weak relatively nonexpansive. Define the operator $B_1 : E_1 \to 2^{E_2^*}, B_2 : E_2 \to 2^{E_2^*}$ by $B_i x = \frac{1}{2i}x$, $\forall x \in E_i$, for $i = 1, 2$. Thus, the resolvent operator is given by $\mathrm{Res}_p^{\lambda B_i} x = \left(\frac{2i}{2i+\lambda}\right)x$, for $i = 1, 2$. Let $A : E_1 \to E_2$ be a mapping defined by $Ax = \frac{2}{3}x$, $\forall x \in E_1$. Choose $\theta_n = \frac{(-1)^n + 3}{10n}$, $\delta_n = \frac{2n}{3n+5}$, and $\beta_n = \frac{1}{4n}$. Then, Algorithm 3.1 results in

$$\begin{cases} w_n = x_n + \dfrac{(-1)^n + 3}{10n}(x_n - x_{n-1}), \\ y_n = \mathrm{Res}_p^{\lambda B_1}[w_n - \mu_n A^*(I - \mathrm{Res}_p^{\lambda B_2})Aw_n], \\ z_n = \dfrac{1}{4n}w_n + \dfrac{4n-1}{4n}\left(\dfrac{2n}{3n+5} + \dfrac{n+5}{3n+5}\right), \\ C_{n+1} = \{w \in C_n : |w - z_n| \leq |w - w_n|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \ \forall n \geq 1, \end{cases} \tag{74}$$

where the step size μ_n is chosen such that

$$\mu_n \in \left(0, \frac{2|Aw_n - J_\alpha^{B_2}Aw_n|^2}{|A^*(I - J_\alpha^{B_2})Aw_n|^2}\right).$$

We compare the performance of the proposed method with (10) of Alofi et al. [10], (11) of [11], and (12) of [14]. For (10), we take $\beta_n = \frac{3n}{7n+2}$, $\alpha_n = \frac{1}{4n}$, $\lambda_n = \frac{1}{4}$; for (11), we take $\alpha_n = \frac{1}{4n}$, $\beta_n = \frac{2n}{3n+5}$, $\gamma_n = 1 - \alpha_n - \beta_n$ and for (12), we take $\theta_n = \frac{(-1)^n + 3}{10n}$. We test the algorithms using the following initial points:

Case I: $x_0 = 3$ and $x_1 = 0.5$,
Case II: $x_0 = 1$ and $x_1 = \frac{1}{3}$,
Case III: $x_0 = 0.5$ and $x_1 = 2$,
Case IV: $x_0 = \sqrt{2}$ and $x_1 = \sqrt{5}$.

Table 1: Numerical results for Example 5.1

		Case I	**Case II**	**Case III**	**Case IV**
Proposed alg.	Iter.	18	19	21	20
	CPU (s)	0.0039	0.0079	0.0126	0.0056
Alofi et al. alg.	Iter.	19	85	128	111
	CPU (s)	0.0058	0.0107	0.0232	0.0165
Suantai et al. alg.	Iter.	21	94	146	116
	CPU (s)	0.0073	0.0127	0.0230	0.0178
Tang alg.	Iter.	20	75	105	84
	CPU (s)	0.0050	0.0106	0.0175	0.0126

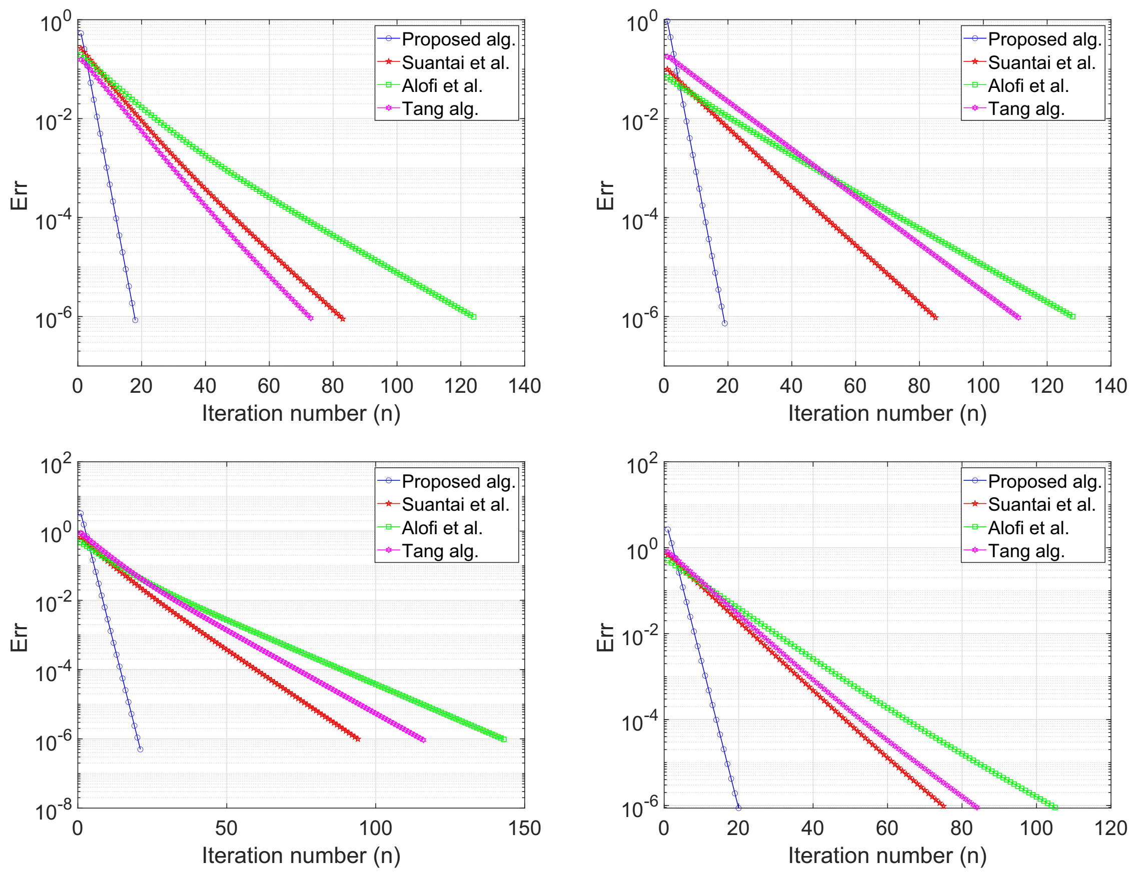

Figure 1: Example 5.1, top left: Case I; top right: Case II, bottom left: Case III; bottom right: Case IV.

We used Err $\coloneqq |x_{n+1} - x_n| < 10^{-6}$ as a stopping criterion for the algorithms. The numerical results are shown in Table 1 and Figure 1.

Example 5.2. Let $E_1 = E_2 = \ell_2(\mathbb{R})$, where $\ell_2(\mathbb{R}) = \{\sigma = (\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_i, ..)\sigma_i \in \mathbb{R} : \sum_{i=1}^{\infty} |\sigma_i|^2 < \infty\}, \|\sigma\|_{\ell_2} = (\sum_{i=1}^{\infty} |\sigma_i|^2)^{\frac{1}{2}}$, $\forall \sigma \in E_1$. Then, E_1 and E_2 are 2-uniformly convex and uniformly smooth, and the duality mapping J_p^E and its dual become the identity mapping on E. Let $C = C_1 = \{x \in E_1 : \|x\|_{\ell_2} \leq 1\}$ and $Q = \{x \in E_2 : \langle x, b \rangle \neq r\}$. It is known that the indicator function on C and Q, i.e., i_C and i_Q are proper, convex, and lower semicontinous. Moreover, the subdifferential ∂i_C and ∂i_Q are maximally monotone. The resolvent operators ∂i_C and ∂i_Q are the metric projections that are defined by

$$P_C(x) = \begin{cases} \dfrac{x}{\|x\|_{\ell_2}^2} & \text{if} \quad \|x\|_{\ell_2}^2 > 1, \\ x & \text{if} \quad \|x\|_{\ell_2}^2 \leq 1, \end{cases}$$

and

$$P_Q(x) = \begin{cases} x - \dfrac{\langle x, b\rangle b}{\|b\|_{\ell_2}^2} & \text{if} \quad \langle x, b \rangle \neq r \\ x & \text{if} \quad \langle x, b \rangle = r. \end{cases}$$

Let $T : E_1 \to E_1$ be defined by

$$Tx = \begin{cases} \dfrac{n}{n+1}x & \text{if } x = x_n, \\ -x & \text{if } x \neq x_n. \end{cases} \tag{75}$$

Then T is Bregman weak relative nonexpansive mapping. Define the operator $B_1 : E_1 \to 2^{E_2^*}$, $B_2 : E_2 \to 2^{E_2^*}$ by $B_i x = \frac{1}{2i}x$, $\forall x \in E_i$, for $i = 1, 2$. Let $A : E_1 \to E_2$ be a mapping defined by $Ax = \frac{1}{4}x$. Choose $\theta_n = \frac{2n+1}{10n}$, $\delta_n = \frac{2n+1}{7n+4}$, and $\beta_n = \frac{1}{n+1}$. Therefore, Algorithm (3.1) results in

$$\begin{cases} w_n = J_q^{E_1^*}[J_p^{E_1}x_n + \dfrac{2n+1}{10n}(J_p^{E_1}x_n - J_p^{E_1}x_{n-1})], \\ y_n = \mathrm{Res}_p^{\lambda B_1} J_q^{E_1^*}[J_p^{E_1}w_n - \mu_n A^* J_p^{E_2}(I - \mathrm{Res}_p^{\lambda B_2})Aw_n], \\ z_n = J_q^{E_1^*}\left[\dfrac{1}{10n+1}J_p^{E_1}w_n + \dfrac{10n}{10n+1}(J_p^{E_1}\dfrac{2n+1}{7n+4}y_n + J_p^{E_1}\dfrac{5n+3}{7n+4}Ty_n)\right], \\ C_{n+1} = \{w \in C_n : \Delta_p(w, z_n) \leq \Delta_p(w, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \ \forall n \geq 1, \end{cases} \tag{76}$$

where the stepsize μ_n is chosen as defined in (30). We also compare the performance of the proposed method with (10) of Alofi et al. [10], (11) of [11], and (12) of [14]. For (10), we take $\beta_n = \frac{2n}{5n+1}$, $\alpha_n = \frac{1}{4n+1}$, $\lambda_n = \frac{1}{2}$; for (11), we take $\alpha_n = \frac{1}{4n+1}$, $\beta_n = \frac{1}{5n+1}$, $\gamma_n = 1 - \alpha_n - \beta_n$, and for (12), we take $\theta_n = \frac{2n+1}{10}$. We test the algorithms using the following initial points:

Case I: $x_0 = (1, 2, 3, \ldots)$ and $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$,

Case II: $x_0 = (\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \ldots)$ and $x_1 = (5, 5, 5, \ldots)$,

Case III: $x_0 = (2, 2, 2, \ldots)$ and $x_1 = (1, 0, 1, \ldots)$,

Case IV: $x_0 = (1, 1, 1, \ldots)$ and $x_1 = (\frac{1}{5}, \frac{1}{10}, \frac{1}{20})$.

We used $\mathrm{Err} := \|x_{n+1} - x_n\|_{\ell_2} < 10^{-6}$ as a stopping criterion for the algorithms. The numerical results are shown in Table 2 and Figure 2.

6 Discussion

(i) Our proposed algorithm is designed in such a way that it contains an inertial term, which helps our algorithm to converge at a faster rate than those studied in the literature. This proposed algorithm is also known to be self-adaptive; that is to say, the stepsize designed in our algorithm does not depend on the prior knowledge of the norm of the bounded linear operator, which is difficult to compute, thus making our algorithm very easy to compute. Therefore, our proposed algorithm is easy to compute and converges faster than other algorithms from the literature in solving the SVIP.

Table 2: Numerical results for Example 5.2

		Case I	**Case II**	**Case III**	**Case IV**
Proposed alg.	Iter.	12	11	14	13
	CPU (s)	0.1918	0.1550	0.6452	0.7741
Alofi et al. alg.	Iter.	22	21	24	20
	CPU (s)	0.2335	0.2170	0.9537	1.9741
Suantai et al. alg	Iter.	25	24	27	22
	CPU (s)	0.2202	0.2481	1.0483	1.9990
Tang alg.	Iter.	17	15	21	18
	CPU (s)	0.2762	0.2703	0.7656	1.9978

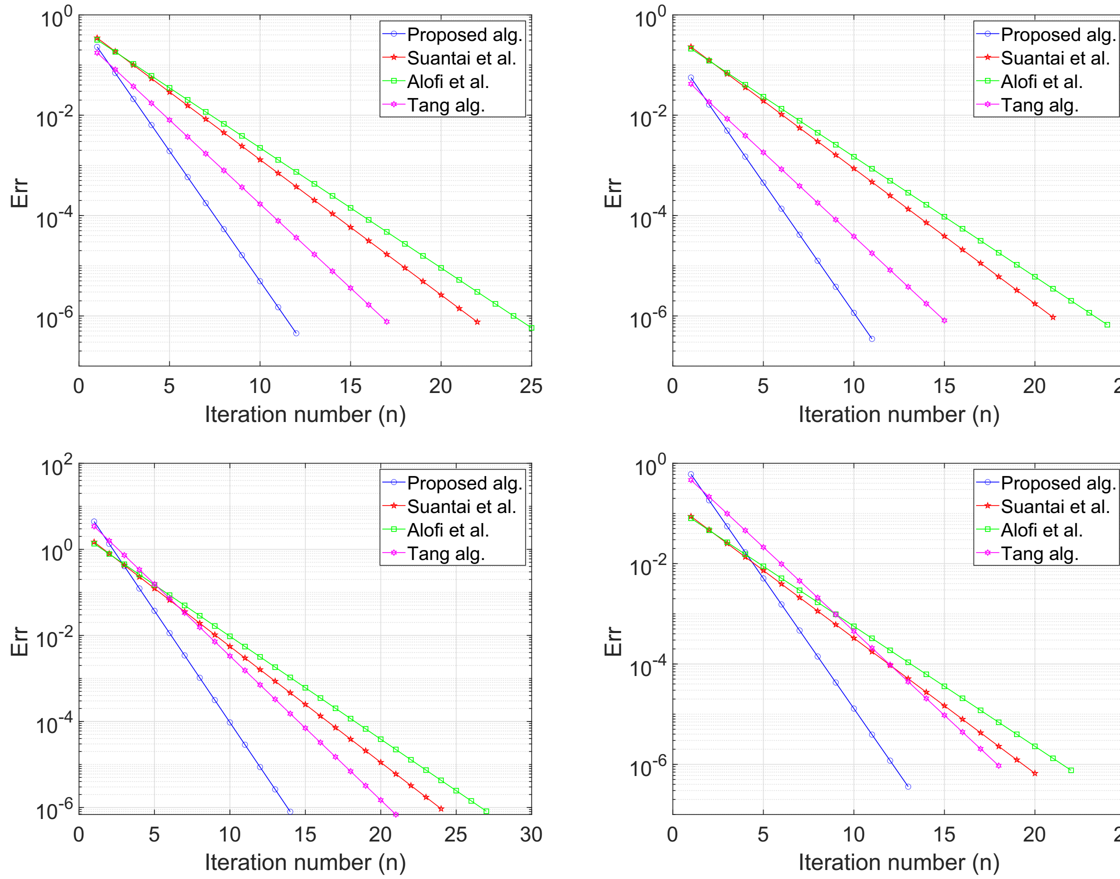

Figure 2: Example 5.2, top left: Case I; top right: Case II, bottom left: Case III; bottom right: Case IV.

(ii) Our results in this article are quite general compared to the result of Shehu [33], which studied the (multiple sets) split feasibility problems in Banach spaces. More so, the proposed algorithm in the current article does not depend on the prior estimate of the norm of the bounded linear operator, and its performance is improved with the aid of the inertial extrapolation process. These are stated as future motivation in [34] even in the case of split feasibility problems.

7 Conclusion

We proposed an inertial projection-type algorithm for solving split variational inclusion problems in p-uniformly convex and uniformly smooth Banach spaces. The algorithm is designed such that the stepsize is chosen self-adaptively at each iteration, and a strong convergence result is proved under some mild conditions. We provide some numerical experiments to illustrate the efficiency of the proposed algorithm and compare its performance with other recent methods in the literature. This result improves and extends the corresponding results in [8,11,14] and other similar results in the literature. In our future work, we would like to extend the results in this article from p-uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces.

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