Research Article

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Solutions of a coupled system of hybrid boundary value problems with Riesz-Caputo derivative

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Abstract: Riesz-Caputo fractional derivative refers to a fractional derivative that reflects both the past and the future memory effects. This study gives sufficient conditions for the existence of solutions for a coupled system of fractional order hybrid differential equations involving the Riesz-Caputo fractional derivative. For this motive, the results are obtained via classical results due to Dhage.

Keywords: fixed point theorem, coupled system of hybrid boundary value problems, existence of solutions, Riesz-Caputo fractional derivative

MSC 2020: 34A08, 34A12

1 Introduction

The fractional differential equations have become a significant field of investigation due to their frequent use in biophysics, economics, chemistry, mechanics, control theory, image processing and signal, etc. [1–3]. Recently, existence theory of solutions to initial and boundary value problems for fractional system has received considerable attention from many researchers [4–11].

As is known to all, Riesz-Caputo derivative refers to a fractional derivative that reflects both the future and the past memory effects; consequently, the Riesz fractional operator plays an important role in characterizing anomalous diffusion, owing to successful applications to subdiffusive, superdiffusive, and evolution problems. Space fractional quantum mechanics is a natural generalization of standard quantum mechanics, which arises when the Brownian trajectories in Feynman path integrals are replaced by Levy flights. The classical Levy flight is a stochastic processes, which in one dimension, is described by a jump length probability density function. The position space representation of the αth power of the momentum operator is given by:

$$\langle x|\hat{P}^{\alpha}|\psi\rangle = -h^{\alpha}D_{|x|}^{\alpha}\psi(x),$$

where $D_{|x|}^{\alpha}$ is the Riesz fractional derivative operator of order α. The Riesz fractional derivative is regarded as an effective tool for studying nonlocal and memory effects in physics, engineering, and applied sciences. Therefore, many scholars are engaged in the study of the solutions of differential equations with the Riesz fractional derivative. Many works carried out so far discuss the numerical solutions of diffusion equations, which contain the Riesz derivative, and fractional variational problems, which contain the Riesz-Caputo derivative, and only a few works report on the existence results of fractional boundary value problems, which contain the Riesz-Caputo derivative.

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Gu et al. [12] discussed a new class of differential equations that contains the Riesz-Caputo fractional derivative:

$$ {}_{0}^{RC}D_{1}^{\alpha}x(\xi) = h(\xi, x(\xi)), \quad \xi \in [0,1], \quad 0 < \alpha \le 1, $$
$$ x(0) = x_0, \quad x(1) = x_1. $$

On the other hand, hybrid differential equation is a class of dynamical systems with quadratic perturbation. Nowadays, due to the wide range of application of hybrid differential equations in several areas of real-life problems, more researchers began to study the existence of solutions for hybrid differential equations with different perturbations (readers can refer to [13–20]).

Houas [21] studied the results for the coupled hybrid system that contains integral boundary conditions as follows:

$$ D^{\alpha_1}\left(\frac{u_1(t)}{g_1(t,u_1(t),u_2(t))}\right) = f_1(t,u_1(t),u_2(t)), \quad t \in [0,1], \quad \alpha_1 \in (0,1), $$
$$ D^{\alpha_2}\left(\frac{u_2(t)}{g_2(t,u_1(t),u_2(t))}\right) = f_2(t,u_1(t),u_2(t)), \quad t \in [0,1], \quad \alpha_1 \in (0,1), $$
$$ u_1(0) = \int_0^{\theta_1} A_1(s)u_1(s)\mathrm{d}s, \quad \theta_1 \in (0,1), $$
$$ u_2(0) = \int_0^{\theta_2} A_2(s)u_2(s)\mathrm{d}s, \quad \theta_2 \in (0,1), $$

where $D^{\alpha_i}(i = 1, 2)$ stands for the Caputo's derivative.

Derbazi et al. [22] considered the solutions for the fractional boundary value problem as follows:

$$ {}^{C}D_{0^+}^{\alpha}\left[\frac{x(t) - f(t,x(t))}{g(t,x(t))}\right] = h(t,x(t)), \quad 1 < \alpha \le 2, \quad t \in J = [0,T], $$
$$ a_1\left[\frac{x(t) - f(t,x(t))}{g(t,x(t))}\right]_{t=0} + b_1\left[\frac{x(t) - f(t,x(t))}{g(t,x(t))}\right]_{t=T} = \lambda_1, $$
$$ a_2{}^{C}D_{0^+}^{\beta}\left[\frac{x(t) - f(t,x(t))}{g(t,x(t))}\right]_{t=\eta} + b_2{}^{C}D_{0^+}^{\beta}\left[\frac{x(t) - f(t,x(t))}{g(t,x(t))}\right]_{t=T} = \lambda_2, \quad 0 < \eta < T, $$

where ${}^{C}D_{0^+}^{\alpha}$ and ${}^{C}D_{0^+}^{\beta}$ stand for the Caputo fractional derivative.

Recently, Baleanu et al. [23] discussed the hybrid fractional coupled system as follows:

$$ D^{\omega}\left(\frac{x(t)}{H(t,x(t),z(t))}\right) = -K_1(t,x(t),z(t)), \quad \omega \in (2,3], $$
$$ D^{\varepsilon}\left(\frac{z(t)}{g(t,x(t),z(t))}\right) = -K_2(t,x(t),z(t)), \quad \varepsilon \in (2,3], $$
$$ \left.\frac{x(t)}{H(t,x(t),z(t))}\right|_{t=1} = 0, \quad \left.D^{\mu}\left(\frac{x(t)}{H(t,x(t),z(t))}\right)\right|_{t=\delta_1} = 0, \quad x^{(2)}(0) = 0, $$
$$ \left.\frac{z(t)}{g(t,x(t),z(t))}\right|_{t=1} = 0, \quad \left.D^{\nu}\left(\frac{z(t)}{g(t,x(t),z(t))}\right)\right|_{t=\delta_2} = 0, \quad x^{(2)}(0) = 0, $$

where $D^{\omega}, D^{\varepsilon}, D^{\mu}$, and D^{ν} stand for the Caputo's fractional derivative.

By virtue of the aforementioned documents, in this work, the authors present the solutions for the hybrid fractional coupled system that contains the Riesz-Caputo derivative as follows:

$$ {}^{RC}_{0}D^{\gamma}_{1}\left(\frac{x(t)-f_1(t,x(t),y(t))}{g_1(t,x(t),y(t))}\right)=h_1(t,x(t),y(t)),\quad 0<\gamma\le 1, \tag{1} $$

$$ {}^{RC}_{0}D^{\nu}_{1}\left(\frac{y(t)-f_2(t,x(t),y(t))}{g_2(t,x(t),y(t))}\right)=h_2(t,x(t),y(t)),\quad 0<\nu\le 1, \tag{2} $$

$$ x(0)=a_1,\quad x(1)=b_1, \tag{3} $$

$$ y(0)=a_2,\quad y(1)=b_2, \tag{4} $$

where $0<\gamma\le 1$, $0<\nu\le 1$, $0\le t\le 1$, ${}^{RC}_{0}D^{\gamma}_{1}$, and ${}^{RC}_{0}D^{\nu}_{1}$ are Riesz-Caputo's fractional derivative of orders $\gamma, \nu, f_1, f_2, h_1, h_2 \in C([0,1]\times R^2, R)$, $g_1, g_2 \in C([0,1]\times R^2, R\backslash\{0\})$ and a_1, a_2, b_1, and b_2 are the real constants. We will use "Dhage's fixed point theorem" to find the sufficient conditions for the existence of solutions. This is the first time for us to present the existence of solutions of Systems (1)–(4).

2 Some lemmas

Suppose $n-1<\alpha\le n, n\in N$ for $\alpha>0$, and $n=[\omega]$ and $[\cdot]$ stands for the ceiling of a number.

Definition 2.1. [24] The Riesz-Caputo fractional derivative for a function $x(t), 0\le t\le T$, can be written as follows:

$$ \begin{aligned} {}^{RC}_{0}D^{\alpha}_{T}x(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^T \frac{x^{(n)}(u)}{|t-u|^{\alpha+1-n}}\mathrm{d}u \\ &= \frac{1}{2}({}^{C}_{0}D^{\alpha}_{t} + (-1)^n t^{C}D^{\alpha}_{T})x(t), \end{aligned} $$

where ${}^{C}_{0}D^{\alpha}_{t}$ is the left Caputo derivative and ${}^{C}_{t}D^{\alpha}_{T}$ is the right Caputo derivatives

$$ {}^{C}_{0}D^{\alpha}_{t}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{x^{(n)}(u)}{(t-u)^{\alpha+1-n}}\mathrm{d}u $$

and

$$ t^{C}D^{\alpha}_{T}x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^T \frac{x^{(n)}(u)}{(u-t)^{\alpha+1-n}}\mathrm{d}u. $$

Particularly, for $x(t)\in C(0,T)$, then

$$ {}^{RC}_{0}D^{\alpha}_{T}x(t) = \frac{1}{2}({}^{C}_{0}D^{\alpha}_{t} -^{C}_{t} D^{\alpha}_{T})x(t), $$

if $0<\alpha\le 1$.

Definition 2.2. [24] The right, left, and fractional Riemann-Liouville integrals of order α can be written as follows:

$$ {}_{t}I^{\alpha}_{T}x(t) = \frac{1}{\Gamma(\alpha)}\int_t^T (u-t)^{\alpha-1}x(u)\mathrm{d}u, $$

$$ {}_0I_t^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}x(u)\mathrm{d}u, $$

$$ {}_0I_T^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^T |u-t|^{\alpha-1}x(u)\mathrm{d}u. $$

Lemma 2.1. [24] *Let* $x(t) \in C^n[0, T]$, *we have*

$$ {}_0I_t^{\alpha}\,{}_0^CD_t^{\alpha}x(t) = x(t) - \sum_{l=0}^{n-1}\frac{x^{(l)}(0)}{l!}(t-0)^l $$

and

$$ {}_tI_T^{\alpha}\,{}_t^CD_T^{\alpha}x(t) = (-1)^n\left[x(t) - \sum_{l=0}^{n-1}\frac{(-1)^l x^{(l)}(T)}{l!}(T-t)^l\right]. $$

From the aforementioned equations, thus we have

$$ \begin{aligned} {}_0I_T^{\alpha}\,{}_0^{RC}D_T^{\alpha}x(t) &= \frac{1}{2}\left({}_0I_t^{\alpha}\,{}_0^CD_t^{\alpha} + {}_tI_{T0}^{\alpha C}D_t^{\alpha}\right)x(t) + (-1)^n\frac{1}{2}\left({}_0I_t^{\alpha}\,{}_t^CD_T^{\alpha} + {}_tI_T^{\alpha}\,{}_t^CD_T^{\alpha}\right)x(t) \\ &= \frac{1}{2}\left({}_0I_t^{\alpha}\,{}_0^CD_t^{\alpha} + (-1)^n\,{}_tI_T^{\alpha}\,{}_t^CD_T^{\alpha}\right)x(t). \end{aligned} $$

Particularly, for $x(t) \in C(0, T)$, then

$$ {}_0I_{T0}^{\alpha RC}D_T^{\alpha}x(t) = x(t) - \frac{1}{2}(x(0) + x(T)), \tag{5} $$

if $0 < \alpha \le 1$.

Lemma 2.2. [25] *Let* S *be a closed convex, bounded, and nonempty subset of a Banach algebra* E, *and let* $A, C : E \to E$ *and* $B : S \to E$ *be three operators such that*
(*a*) A *and* C *are Lipschitzian with Lipschitz constants* δ *and* ρ, *respectively,*
(*b*) B *is compact and continuous,*
(*c*) $x = AxBy + Cx \Rightarrow x \in S$ *for all* $y \in S$,
(*d*) $\delta M + \rho < 1$, *where* $M = \|B(S)\|$.

Then, there is a solution for the operator equation $AxBx + Cx = x$ *in* S.

3 Main results

The Banach space $C[0, 1]$ is recorded as X. The norm is defined as follows: $\|x\| = \max_{t\in[0,1]}|x(t)|$. Under the following norm definition $\|(x, y)\| = \|x\| + \|y\|$, we have $E = X \times X$, which is a Banach space.

Definition 3.1. [26] If there is a constant $\mu > 0$ that satisfies

$$ \|F(w, z) - F(\bar{w}, \bar{z})\| \le \mu(\|w - \bar{w}\| + \|z - \bar{z}\|) $$

for all $(w, z), (\bar{w}, \bar{z}) \in E$, we call $F : E \to E$ as μ-Lipschitz. Moreover, if $\mu < 1$, we call F as a strict contraction.

Lemma 3.1. *Let* $h \in C[0,1]$. *Then, the equation*

$$x(t) = g(t, x(t)y(t))\left[\frac{1}{2}\frac{a_1 - f(0, a_1, a_2)}{g(0, a_1, a_2)} + \frac{1}{2}\frac{b_1 - f(1, b_1, b_2)}{g(1, b_1, b_2)} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}h(s)\mathrm{d}s + \frac{1}{\Gamma(\gamma)}\int_t^1 (s-t)^{\gamma-1}h(s)\mathrm{d}s\right] + f(t, x(t), y(t)) \tag{6}$$

is the solution of the boundary value problem:

$${}^{RC}_{0}D^{\gamma}_{1}\left(\frac{x(t) - f(t, x(t), y(t))}{g(t, x(t), y(t))}\right) = h(t), \qquad 0 < \gamma \le 1, \tag{7}$$

$$x(0) = a_1, \quad x(1) = b_1, \tag{8}$$

$$y(0) = a_2, \quad y(1) = b_2. \tag{9}$$

Proof. Integrating on both sides of equation (7) and considering (5), (8), and (9), we obtain

$$\frac{x(t) - f(t, x(t), y(t))}{g(t, x(t), y(t))} ={}_0 I_1^{\gamma} h(t) + \frac{1}{2}\frac{a_1 - f(0, a_1, a_2)}{g(0, a_1, a_2)} + \frac{1}{2}\frac{b_1 - f(1, b_1, b_2)}{g(1, b_1, b_2)}.$$

Consequently, we find that

$$\frac{x(t) - f(t, x(t), y(t))}{g(t, x(t), y(t))} = \frac{1}{2}\frac{a_1 - f(0, a_1, a_2)}{g(0, a_1, a_2)} + \frac{1}{2}\frac{b_1 - f(1, b_1, b_2)}{g(1, b_1, b_2)} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}h(s)\mathrm{d}s + \frac{1}{\Gamma(\gamma)}\int_t^1 (s-t)^{\gamma-1}h(s)\mathrm{d}s.$$

Therefore, (6) holds. □

Define the operator T mapping E to E

$$T(x, y)(t) = (T_1(x, y)(t), \quad T_2(x, y)(t)), \tag{10}$$

where

$$T_1(x, y)(t) = g_1(t, x(t), y(t))\left[\frac{1}{2}\frac{a_1 - f_1(0, a_1, a_2)}{g_1(0, a_1, a_2)} + \frac{1}{2}\frac{b_1 - f_1(1, b_1, b_2)}{g_1(1, b_1, b_2)} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}h_1(s, x(s), y(s))\mathrm{d}s + \frac{1}{\Gamma(\gamma)}\int_t^1 (s-t)^{\gamma-1}h_1(s, x(s), y(s))\mathrm{d}s\right] + f_1(t, x(t), y(t)), \tag{11}$$

$$T_2(x, y)(t) = g_2(t, x(t), y(t))\left[\frac{1}{2}\frac{a_2 - f_2(0, a_1, a_2)}{g_2(0, a_1, a_2)} + \frac{1}{2}\frac{b_2 - f_2(1, b_1, b_2)}{g_2(1, b_1, b_2)} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}h_2(s, x(s), y(s))\mathrm{d}s + \frac{1}{\Gamma(\nu)}\int_t^1 (s-t)^{\nu-1}h_2(s, x(s), y(s))\mathrm{d}s\right] + f_2(t, x(t), y(t)). \tag{12}$$

We give the hypotheses of this study.

(H_1) $f_1, f_2, h_1, h_2 : \quad [0,1] \times R^2 \to R$ and $g_1, g_2 : \quad [0,1] \times R^2 \to R\backslash\{0\}$ are continuous functions;

(H_2) $\phi_0, \phi_1, \bar{\phi}_0, \bar{\phi}_1$ are four positive functions such that

$$|f_1(t, x, y) - f_1(t, \bar{x}, \bar{y})| \le \phi_0(t)[|x - \bar{x}| + |y - \bar{y}|], \tag{13}$$

$$|f_2(t,x,y) - f_2(t,\bar{x},\bar{y})| \le \overline{\phi_0}(t)[|x-\bar{x}| + |y-\bar{y}|], \tag{14}$$

$$|g_1(t,x,y) - g_1(t,\bar{x},\bar{y})| \le \phi_1(t)[|x-\bar{x}| + |y-\bar{y}|], \tag{15}$$

$$|g_2(t,x,y) - g_2(t,\bar{x},\bar{y})| \le \overline{\phi_1}(t)[|x-\bar{x}| + |y-\bar{y}|], \tag{16}$$

for all $t \in [0,1]$, $x, \bar{x}, y$, and $\bar{y}$ are the elements in R;

(H_3) Functions $p_1, p_2 \in L^\infty(J, R^+)$ and four continuous nondecreasing functions $\psi_1, \psi_2, \xi_1, \xi_2 : [0,\infty) \to (0,\infty)$ satisfy

$$|h_1(t,x,y)| \le p_1(t)\psi_1(|x|)\xi_1(|y|), \tag{17}$$

$$|h_2(t,x,y)| \le p_2(t)\psi_2(|x|)\xi_2(|y|), \tag{18}$$

for all $t \in [0,1]$, x and y are the elements in R;

(H_4) The positive number r satisfies

$$\frac{1}{1-\rho}\left(\frac{g_1\Lambda + f_1}{1 - \|\phi_1\|\Lambda - \|\phi_0\|} + \frac{g_2\Lambda + f_2}{1 - \|\overline{\phi_1}\|\Lambda - \|\overline{\phi_0}\|}\right) < r, \tag{19}$$

where

$$\rho = \max\left\{\frac{\|\phi_1\|\Lambda + \|\phi_0\|}{1 - \|\phi_1\|\Lambda - \|\phi_0\|}, \frac{\|\overline{\phi_1}\|\Lambda + \|\overline{\phi_0}\|}{1 - \|\overline{\phi_1}\|\Lambda - \|\overline{\phi_0}\|}\right\}, \tag{20}$$

$$\begin{aligned}\Lambda = {} & \frac{1}{2}\left|\frac{a_1 - f_1(0,a_1,a_2)}{g_1(0,a_1,a_2)}\right| + \frac{1}{2}\left|\frac{b_1 - f_1(1,b_1,b_2)}{g_1(1,b_1,b_2)}\right| + \frac{2\psi_1(r)\xi_1(r)\|p_1\|}{\Gamma(\gamma+1)} \\ & + \frac{1}{2}\left|\frac{a_2 - f_2(0,a_1,a_2)}{g_2(0,a_1,a_2)}\right| + \frac{1}{2}\left|\frac{b_2 - f_2(1,b_1,b_2)}{g_2(1,b_1,b_2)}\right| + \frac{2\psi_2(r)\xi_2(r)\|p_2\|}{\Gamma(\nu+1)},\end{aligned} \tag{21}$$

and $\|\phi_1\|\Lambda + \|\phi_0\| < \frac{1}{2}$, $\|\overline{\phi_1}\|\Lambda + \|\overline{\phi_0}\| < \frac{1}{2}$, $f_1 = \sup_{t\in[0,1]}|f_1(t,0,0)|$, $f_2 = \sup_{t\in[0,1]}|f_2(t,0,0)|$, $g_1 = \sup_{t\in[0,1]}|g_1(t,0,0)|$, $g_2 = \sup_{t\in[0,1]}|g_2(t,0,0)|$.

Theorem 3.2. *Problems* (1)–(4) *have a solution, if Conditions* (H_1)–(H_4) *are true.*

Proof. Let

$$S = \{(x,y) \in E : \|(x,y)\| \le r\}. \tag{22}$$

Then, S is a bounded convex closed subset of E.

Define operators $A_i, C_i : E \to X$ and $B_i : S \to X (i = 1, 2)$ by

$$A_1(x(t),y(t)) = g_1(t,x(t),y(t)), \quad A_2(x(t),y(t)) = g_2(t,x(t),y(t)), \tag{23}$$

$$\begin{aligned}B_1(x(t),y(t)) = {} & \frac{1}{2}\frac{a_1 - f_1(0,a_1,a_2)}{g_1(0,a_1,a_2)} + \frac{1}{2}\frac{b_1 - f_1(1,b_1,b_2)}{g_1(1,b_1,b_2)} \\ & + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}h_1(s,x(s),y(s))\mathrm{d}s \\ & + \frac{1}{\Gamma(\gamma)}\int_t^1 (s-t)^{\gamma-1}h_1(s,x(s),y(s))\mathrm{d}s,\end{aligned} \tag{24}$$

$$B_2(x(t),y(t)) = \frac{1}{2}\frac{a_2 - f_2(0,a_1,a_2)}{g_2(0,a_1,a_2)} + \frac{1}{2}\frac{b_2 - f_2(1,b_1,b_2)}{g_2(1,b_1,b_2)} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}h_2(s,x(s),y(s))\mathrm{d}s + \frac{1}{\Gamma(\nu)}\int_t^1 (s-t)^{\nu-1}h_2(s,x(s),y(s))\mathrm{d}s, \tag{25}$$

$$C_1(x(t),y(t)) = f_1(t,x(t),y(t)), \quad \text{and} \quad C_2(x(t),y(t)) = f_2(t,x(t),y(t)). \tag{26}$$

Then, operators T_1 and T_2 given by equations (11) and (12) are equivalent to

$$T_1(x(t),y(t)) = A_1(x(t),y(t))B_1(x(t),y(t)) + C_1(x(t),y(t)), \tag{27}$$

$$T_2(x(t),y(t)) = A_2(x(t),y(t))B_2(x(t),y(t)) + C_2(x(t),y(t)). \tag{28}$$

Therefore, the operator T in equation (10) can be written as follows:

$$T(x,y)(t) = (T_1(x,y)(t), T_2(x,y)(t)) = A(x(t),y(t))B(x(t),y(t)) + C(x(t),y(t)),$$

where $A = (A_1, A_2)$, $B = (B_1, B_2)$, and $C = (C_1, C_2)$. In the following, we prove that the conditions in Lemma 2.2 can be satisfied.

First, we prove that A and C are Lipschitz on the space E. Given $(x_1,y_1),(x_2,y_2) \in E$, and $t \in [0,1]$, then by (H_2), we obtain

$$\begin{aligned}|A_1(x_1,y_1)(t) - A_1(x_2,y_2)(t)| &= |g_1(t,x_1(t),y_1(t)) - g_1(t,x_2(t),y_2(t))| \\ &\le \phi_1(t)(|x_1(t)-x_2(t)| + |y_1(t)-y_2(t)|).\end{aligned} \tag{29}$$

Similarly,

$$\begin{aligned}|A_2(x_1,y_1)(t) - A_2(x_2,y_2)(t)| &= |g_2(t,x_1(t),y_1(t)) - g_2(t,x_2(t),y_2(t))| \\ &\le \overline{\phi_1}(t)(|x_1(t)-x_2(t)| + |y_1(t)-y_2(t)|).\end{aligned} \tag{30}$$

From equations (29) and (30), we obtain

$$\begin{aligned}|A(x_1,y_1)(t) - A(x_2,y_2)(t)| &= |(A_1,A_2)(x_1,y_1) - (A_1,A_2)(x_2,y_2)| \\ &\le (\phi_1(t) + \overline{\phi_1}(t))(|x_1(t)-x_2(t)| + |y_1(t)-y_2(t)|)\end{aligned}$$

for all t on the interval $[0,1]$. Taking the supremum on both sides of the aforementioned formula with respect to t, we have

$$\|A(x_1,y_1) - A(x_2,y_2)\| \le (\|\phi_1\| + \|\overline{\phi_1}\|)(\|x_1-x_2\| + \|y_1-y_2\|),$$

which implies that A is Lipschitz on the space E with Lipschitz constant $\|\phi_1\| + \|\overline{\phi_1}\|$.

We consider $C : E \to X$, $(x_1,y_1),(x_2,y_2) \in E$, then

$$\begin{aligned}|C_1(x_1,y_1)(t) - C_1(x_2,y_2)(t)| &= |f_1(t,x_1(t),y_1(t)) - f_1(t,x_2(t),y_2(t))| \\ &\le \phi_0(t)(|x_1(t)-x_2(t)| + |y_1(t)-y_2(t)|).\end{aligned} \tag{31}$$

Similarly,

$$\begin{aligned}|C_2(x_1,y_1)(t) - C_2(x_2,y_2)(t)| &= |f_2(t,x_1(t),y_1(t)) - f_2(t,x_2(t),y_2(t))| \\ &\le \overline{\phi_0}(t)(|x_1(t)-x_2(t)| + |y_1(t)-y_2(t)|).\end{aligned} \tag{32}$$

From equations (31) and (32), we obtain

$$\begin{aligned}|C(x_1,y_1)(t) - C(x_2,y_2)(t)| &= |(C_1,C_2)(x_1,y_1) - (C_1,C_2)(x_2,y_2)| \\ &\le (\phi_0(t) + \overline{\phi_0}(t))(|x_1(t)-x_2(t)| + |y_1(t)-y_2(t)|)\end{aligned}$$

for all t on the interval $[0, 1]$. Taking the supremum on both sides of the aforementioned formula with respect to t, we obtain

$$\|C(x_1, y_1) - C(x_2, y_2)\| \le (\|\phi_0\| + \|\overline{\phi_0}\|)(\|x_1 - x_2\| + \|y_1 - y_2\|),$$

which implies that C is Lipschitz on the space E with Lipschitz constant $\|\phi_0\| + \|\overline{\phi_0}\|$.

Second, we prove that B is completely continuous mapping S to X. In Step 1, we show that B is a continuous operator on S. To prove this, assume that $\{x_n\}, \{y_n\}$ are convergent sequences and satisfy $x_n \to x, y_n \to y$ as $n \to +\infty$. By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} B_1(x_n, y_n) &= \lim_{n\to\infty}\left(\frac{1}{2}\frac{a_1 - f_1(0, a_1, a_2)}{g_1(0, a_1, a_2)} + \frac{1}{2}\frac{b_1 - f_1(1, b_1, b_2)}{g_1(1, b_1, b_2)}\right. \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^t (t - s)^{\gamma-1} h_1(s, x_n(s), y_n(s))\mathrm{d}s \\
&\quad \left. + \frac{1}{\Gamma(\gamma)}\int_t^1 (s - t)^{\gamma-1} h_1(s, x_n(s), y_n(s))\mathrm{d}s\right) \\
&= \frac{1}{2}\frac{a_1 - f_1(0, a_1, a_2)}{g_1(0, a_1, a_2)} + \frac{1}{2}\frac{b_1 - f_1(1, b_1, b_2)}{g_1(1, b_1, b_2)} \\
&\quad + \frac{1}{\Gamma(\gamma)}\lim_{n\to\infty}\int_0^t (t - s)^{\gamma-1} h_1(s, x_n(s), y_n(s))\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\gamma)}\lim_{n\to\infty}\int_t^1 (s - t)^{\gamma-1} h_1(s, x_n(s), y_n(s))\mathrm{d}s \\
&= \frac{1}{2}\frac{a_1 - f_1(0, a_1, a_2)}{g_1(0, a_1, a_2)} + \frac{1}{2}\frac{b_1 - f_1(1, b_1, b_2)}{g_1(1, b_1, b_2)} \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^t (t - s)^{\gamma-1} \lim_{n\to\infty} h_1(s, x_n(s), y_n(s))\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_t^1 (s - t)^{\gamma-1} \lim_{n\to\infty} h_1(s, x_n(s), y_n(s))\mathrm{d}s \\
&= \frac{1}{2}\frac{a_1 - f_1(0, a_1, a_2)}{g_1(0, a_1, a_2)} + \frac{1}{2}\frac{b_1 - f_1(1, b_1, b_2)}{g_1(1, b_1, b_2)} \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^t (t - s)^{\gamma-1} h_1(s, x(s), y(s))\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_t^1 (s - t)^{\gamma-1} h_1(s, x(s), y(s))\mathrm{d}s = B_1(x, y).
\end{aligned} \tag{33}$$

Similarly, we have

$$
\begin{aligned}
\lim_{n\to\infty} B_2(x_n, y_n) &= \lim_{n\to\infty}\Bigg(\frac{1}{2}\frac{a_2 - f_2(0, a_1, a_2)}{g_2(0, a_1, a_2)} + \frac{1}{2}\frac{b_2 - f_2(1, b_1, b_2)}{g_2(1, b_1, b_2)} \\
&\quad + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} h_2(s, x_n(s), y_n(s))\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\nu)}\int_t^1 (s-t)^{\nu-1} h_2(s, x_n(s), y_n(s))\mathrm{d}s\Bigg) \\
&= \frac{1}{2}\frac{a_2 - f_2(0, a_1, a_2)}{g_2(0, a_1, a_2)} + \frac{1}{2}\frac{b_2 - f_2(1, b_1, b_2)}{g_2(1, b_1, b_2)} \\
&\quad + \frac{1}{\Gamma(\nu)}\lim_{n\to\infty}\int_0^t (t-s)^{\nu-1} h_2(s, x_n(s), y_n(s))\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\nu)}\lim_{n\to\infty}\int_t^1 (s-t)^{\nu-1} h_2(s, x_n(s), y_n(s))\mathrm{d}s \\
&= \frac{1}{2}\frac{a_2 - f_2(0, a_1, a_2)}{g_2(0, a_1, a_2)} + \frac{1}{2}\frac{b_2 - f_2(1, b_1, b_2)}{g_2(1, b_1, b_2)} \\
&\quad + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\lim_{n\to\infty} h_2(s, x_n(s), y_n(s))\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\nu)}\int_t^1 (s-t)^{\nu-1}\lim_{n\to\infty} h_2(s, x_n(s), y_n(s))\mathrm{d}s \\
&= \frac{1}{2}\frac{a_2 - f_2(0, a_1, a_2)}{g_2(0, a_1, a_2)} + \frac{1}{2}\frac{b_2 - f_2(1, b_1, b_2)}{g_2(1, b_1, b_2)} \\
&\quad + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} h_2(s, x(s), y(s))\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\nu)}\int_t^1 (s-t)^{\nu-1} h_2(s, x(s), y(s))\mathrm{d}s = B_2(x, y).
\end{aligned}
\tag{34}
$$

In view of equations (33) and (34), for all $t \in [0, 1]$, the operator $B = (B_1, B_2)$ is a continuous operator.

In Step 2, we prove that in S, the set $B(S)$ is uniformly bounded. For any $(x, y) \in S$, in consideration of (H_3), we obtain

$$\begin{aligned}
|B_1(x(t), y(t))| &\le \frac{1}{2}\left|\frac{a_1 - f_1(0, a_1, a_2)}{g_1(0, a_1, a_2)}\right| + \frac{1}{2}\left|\frac{b_1 - f_1(1, b_1, b_2)}{g_1(1, b_1, b_2)}\right| \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}|h_1(s, x(s), y(s))|\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_t^1 (s-t)^{\gamma-1}|h_1(s, x(s), y(s))|\mathrm{d}s \\
&\le \frac{1}{2}\left|\frac{a_1 - f_1(0, a_1, a_2)}{g_1(0, a_1, a_2)}\right| + \frac{1}{2}\left|\frac{b_1 - f_1(1, b_1, b_2)}{g_1(1, b_1, b_2)}\right| \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}p_1(s)\psi_1(r)\xi_1(r)\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_t^1 (s-t)^{\gamma-1}p_1(s)\psi_1(r)\xi_1(r)\mathrm{d}s \\
&= \frac{1}{2}\left|\frac{a_1 - f_1(0, a_1, a_2)}{g_1(0, a_1, a_2)}\right| + \frac{1}{2}\left|\frac{b_1 - f_1(1, b_1, b_2)}{g_1(1, b_1, b_2)}\right| \\
&\quad + ||p_1||\psi_1(r)\xi_1(r)\frac{1}{\Gamma(\gamma)}\left(\int_0^t (t-s)^{\gamma-1}\mathrm{d}s + \int_t^1 (s-t)^{\gamma-1}\mathrm{d}s\right) \\
&= \frac{1}{2}\left|\frac{a_1 - f_1(0, a_1, a_2)}{g_1(0, a_1, a_2)}\right| + \frac{1}{2}\left|\frac{b_1 - f_1(1, b_1, b_2)}{g_1(1, b_1, b_2)}\right| + \frac{2||p_1||\psi_1(r)\xi_1(r)}{\Gamma(\gamma+1)}.
\end{aligned} \tag{35}$$

$$\begin{aligned}
|B_2(x(t), y(t))| &\le \frac{1}{2}\left|\frac{a_2 - f_2(0, a_1, a_2)}{g_2(0, a_1, a_2)}\right| + \frac{1}{2}\left|\frac{b_2 - f_2(1, b_1, b_2)}{g_2(1, b_1, b_2)}\right| \\
&\quad + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}|h_2(s, x(s), y(s))|\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\nu)}\int_t^1 (s-t)^{\nu-1}|h_2(s, x(s), y(s))|\mathrm{d}s \\
&\le \frac{1}{2}\left|\frac{a_2 - f_2(0, a_1, a_2)}{g_2(0, a_1, a_2)}\right| + \frac{1}{2}\left|\frac{b_2 - f_2(1, b_1, b_2)}{g_2(1, b_1, b_2)}\right| \\
&\quad + ||p_2||\psi_2(r)\xi_2(r)\frac{1}{\Gamma(\nu)}\left(\int_0^t (t-s)^{\nu-1}\mathrm{d}s + \int_t^1 (s-t)^{\nu-1}\mathrm{d}s\right) \\
&= \frac{1}{2}\left|\frac{a_2 - f_2(0, a_1, a_2)}{g_2(0, a_1, a_2)}\right| + \frac{1}{2}\left|\frac{b_2 - f_2(1, b_1, b_2)}{g_2(1, b_1, b_2)}\right| + \frac{2||p_2||\psi_2(r)\xi_2(r)}{\Gamma(\nu+1)}.
\end{aligned} \tag{36}$$

Thus, $||B(x, y)|| \le \Lambda$ for all $(x, y) \in S$ with Λ given in equation (21), which implies that the set $B(S)$ is a uniformly bounded set in S.

In the following, we show that the operator B is equicontinuous. To prove this, given $0 \le t_1 \le t_2 \le 1$, $(x, y) \in S$, we can find

$$\begin{aligned}
|B_1(x,y)(t_2) - B_1(x,y)(t_1)| &= \Bigg| \frac{1}{\Gamma(\gamma)} \int_0^{t_2} (t_2 - s)^{\gamma-1} h_1(s, x(s), y(s)) \mathrm{d}s + \frac{1}{\Gamma(\gamma)} \int_{t_2}^{1} (s - t_2)^{\gamma-1} h_1(s, x(s), y(s)) \mathrm{d}s \\
&\quad - \frac{1}{\Gamma(\gamma)} \int_0^{t_1} (t_1 - s)^{\gamma-1} h_1(s, x(s), y(s)) \mathrm{d}s - \frac{1}{\Gamma(\gamma)} \int_{t_1}^{1} (s - t_1)^{\gamma-1} h_1(s, x(s), y(s)) \mathrm{d}s \Bigg| \\
&\le \frac{1}{\Gamma(\gamma)} \int_0^{t_1} ((t_2 - s)^{\gamma-1} - (t_1 - s)^{\gamma-1}) |h_1(s, x(s), y(s))| \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_{t_1}^{t_2} ((t_2 - s)^{\gamma-1} - (s - t_1)^{\gamma-1}) |h_1(s, x(s), y(s))| \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_{t_2}^{1} ((s - t_2)^{\gamma-1} - (s - t_1)^{\gamma-1}) |h_1(s, x(s), y(s))| \mathrm{d}s \\
&\le \frac{\|p_1\| \psi_1(r) \xi_1(r)}{\Gamma(\gamma)} \left(\int_0^{t_1} ((t_2 - s)^{\gamma-1} - (t_1 - s)^{\gamma-1}) \mathrm{d}s + \int_{t_1}^{t_2} ((t_2 - s)^{\gamma-1} - (s - t_1)^{\gamma-1}) \mathrm{d}s \right. \\
&\quad \left. + \int_{t_2}^{1} ((s - t_2)^{\gamma-1} - (s - t_1)^{\gamma-1}) \mathrm{d}s \right) \\
&= \frac{\|p_1\| \psi_1(r) \xi_1(r)}{\Gamma(\gamma + 1)} [(t_2^{\gamma} - (t_2 - t_1)^{\gamma} - t_1^{\gamma}) + (1 - t_2)^{\gamma} - (1 - t_1)^{\gamma} + (t_2 - t_1)^{\gamma}] \\
&= \frac{\|p_1\| \psi_1(r) \xi_1(r)}{\Gamma(\gamma + 1)} [t_2^{\gamma} - t_1^{\gamma} + (1 - t_2)^{\gamma} - (1 - t_1)^{\gamma}].
\end{aligned} \tag{37}$$

Similarly, we obtain

$$\begin{aligned}
|B_2(x,y)(t_2) - B_2(x,y)(t_1)| &= \Bigg| \frac{1}{\Gamma(\nu)} \int_0^{t_2} (t_2 - s)^{\nu-1} h_2(s, x(s), y(s)) \mathrm{d}s + \frac{1}{\Gamma(\nu)} \int_{t_2}^{1} (s - t_2)^{\nu-1} h_2(s, x(s), y(s)) \mathrm{d}s \\
&\quad - \frac{1}{\Gamma(\nu)} \int_0^{t_1} (t_1 - s)^{\nu-1} h_2(s, x(s), y(s)) \mathrm{d}s - \frac{1}{\Gamma(\nu)} \int_{t_1}^{1} (s - t_1)^{\nu-1} h_2(s, x(s), y(s)) \mathrm{d}s \Bigg| \\
&\le \frac{\|p_2\| \psi_2(r) \xi_2(r)}{\Gamma(\nu + 1)} \left(\int_0^{t_1} ((t_2 - s)^{\nu-1} - (t_1 - s)^{\nu-1}) \mathrm{d}s + \int_{t_1}^{t_2} ((t_2 - s)^{\nu-1} - (s - t_1)^{\nu-1}) \mathrm{d}s \right. \\
&\quad \left. + \int_{t_2}^{1} ((s - t_2)^{\nu-1} - (s - t_1)^{\nu-1}) \mathrm{d}s \right) \\
&= \frac{\|p_2\| \psi_2(r) \xi_2(r)}{\Gamma(\nu + 1)} [t_2^{\nu} - t_1^{\nu} + (1 - t_2)^{\nu} - (1 - t_1)^{\nu}].
\end{aligned} \tag{38}$$

Using equations (37) and (38), we find that

$$\begin{aligned}
|B(x,y)(t_2) - B(x,y)(t_1)| &\le \frac{\|p_1\| \psi_1(r) \xi_1(r)}{\Gamma(\gamma + 1)} [t_2^{\gamma} - t_1^{\gamma} + (1 - t_2)^{\gamma} - (1 - t_1)^{\gamma}] \\
&\quad + \frac{\|p_2\| \psi_2(r) \xi_2(r)}{\Gamma(\nu + 1)} [t_2^{\nu} - t_1^{\nu} + (1 - t_2)^{\nu} - (1 - t_1)^{\nu}].
\end{aligned}$$

Thus, when $t_2 \to t_1$, we have $|B(x,y)(t_2) - B(x,y)(t_1)| \to 0$, so that we can find that B is equicontinuous; therefore, we have the operator B that is compact according to the Arzela-Ascoli theorem.

Third, we prove that conditions (c) of Lemma 2.2 holds.

Take any two elements x, y from E and any two elements $\bar{x}, \bar{y}$ from S, which satisfy $(x,y) = A(x,y)B(x,y) + C(x,y)$; using (H_2), (H_3), and (H_4), we find

$$\begin{aligned}
|x(t)| &\le |A_1(x(t),y(t))||B_1(\bar{x}(t),\bar{y}(t))| + |C_1(x(t),y(t))| \\
&\le |g_1(t,x(t)y(t))| \left(\frac{1}{2} \left| \frac{a_1 - f_1(0,a_1,a_2)}{g_1(0,a_1,a_2)} \right| + \frac{1}{2} \left| \frac{b_1 - f_1(1,b_1,b_2)}{g_1(1,b_1,b_2)} \right| \right. \\
&\quad \left. + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} |h_1(s,\bar{x}(s),\bar{y}(s))| ds + \frac{1}{\Gamma(\gamma)} \int_t^1 (s-t)^{\gamma-1} |h_1(s,\bar{x}(s),\bar{y}(s))| ds \right) + |f_1(t,x(t)y(t))| \\
&\le (|g_1(t,x(t)y(t)) - g_1(t,0,0)| + |g_1(t,0,0)|) \left(\frac{1}{2} \left| \frac{a_1 - f_1(0,a_1,a_2)}{g_1(0,a_1,a_2)} \right| + \frac{1}{2} \left| \frac{b_1 - f_1(1,b_1,b_2)}{g_1(1,b_1,b_2)} \right| \right. \\
&\quad \left. + \frac{\|p_1\| \psi_1(r) \xi_1(r)}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} ds + \int_t^1 (s-t)^{\gamma-1} ds \right) + (|f_1(t,x(t)y(t)) - f_1(t,0,0)| + |f_1(t,0,0)|) \\
&\le (\|\phi_1\|(|x(t)| + |y(t)|) + g_1)\Lambda + \|\phi_0\|(|x(t)| + |y(t)|) + f_1.
\end{aligned} \tag{39}$$

Taking the supremum on both sides of the aforementioned formula with respect to t,

$$\|x\| \le (\|\phi_1\|(\|x\| + \|y\|) + g_1)\Lambda + \|\phi_0\|(\|x\| + \|y\|) + f_1,$$

this equals to

$$\|x\| \le \frac{(\|\phi_1\|\Lambda + \|\phi_0\|)\|y\| + g_1\Lambda + f_1}{1 - \|\phi_1\|\Lambda - \|\phi_0\|}.$$

Similarly, we obtain

$$\begin{aligned}
|y(t)| &\le |A_2(x(t),y(t))||B_2(\bar{x}(t),\bar{y}(t))| + |C_2(x(t),y(t))| \\
&\le |g_2(t,x(t)y(t))| \left(\frac{1}{2} \left| \frac{a_2 - f_2(0,a_1,a_2)}{g_2(0,a_1,a_2)} \right| + \frac{1}{2} \left| \frac{b_2 - f_2(1,b_1,b_2)}{g_2(1,b_1,b_2)} \right| \right. \\
&\quad \left. + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |h_2(s,\bar{x}(s),\bar{y}(s))| ds + \frac{1}{\Gamma(\nu)} \int_t^1 (s-t)^{\nu-1} |h_2(s,\bar{x}(s),\bar{y}(s))| ds \right) + |f_2(t,x(t)y(t))| \\
&\le (|g_2(t,x(t)y(t)) - g_2(t,0,0)| + |g_2(t,0,0)|) \left(\frac{1}{2} \left| \frac{a_2 - f_2(0,a_1,a_2)}{g_2(0,a_1,a_2)} \right| \right. \\
&\quad \left. + \frac{1}{2} \left| \frac{b_2 - f_2(1,b_1,b_2)}{g_2(1,b_1,b_2)} \right| + \frac{\|p_2\| \psi_2(r) \xi_2(r)}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} ds + \int_t^1 (s-t)^{\nu-1} ds \right) \\
&\quad + (|f_2(t,x(t)y(t)) - f_2(t,0,0)| + |f_2(t,0,0)|) \\
&\le (\|\overline{\phi_1}\|(|x(t)| + |y(t)|) + g_2)\Lambda + \|\overline{\phi_0}\|(|x(t)| + |y(t)|) + f_2.
\end{aligned} \tag{40}$$

Thus,

$$\|y\| \le (\|\overline{\phi_1}\|(\|x\| + \|y\|) + g_2)\Lambda + \|\overline{\phi_0}\|(\|x\| + \|y\|) + f_2,$$

which equals to

$$\|y\| \le \frac{(\|\overline{\phi_1}\|\Lambda + \|\overline{\phi_0}\|)\|x\| + g_2\Lambda + f_2}{1 - \|\overline{\phi_1}\|\Lambda - \|\overline{\phi_0}\|},$$

so,

$$\begin{aligned}\|(x,y)\| &= \|x\| + \|y\| \\ &\le \frac{(\|\phi_1\|\Lambda + \|\phi_0\|)\|y\| + g_1\Lambda + f_1}{1 - \|\phi_1\|\Lambda - \|\phi_0\|} + \frac{(\|\overline{\phi_1}\|\Lambda + \|\overline{\phi_0}\|)\|x\| + g_2\Lambda + f_2}{1 - \|\overline{\phi_1}\|\Lambda - \|\overline{\phi_0}\|} \\ &= \frac{(\|\phi_1\|\Lambda + \|\phi_0\|)\|y\|}{1 - \|\phi_1\|\Lambda - \|\phi_0\|} + \frac{g_1\Lambda + f_1}{1 - \|\phi_1\|\Lambda - \|\phi_0\|} \\ &\quad + \frac{(\|\overline{\phi_1}\|\Lambda + \|\overline{\phi_0}\|)\|x\|}{1 - \|\overline{\phi_1}\|\Lambda - \|\overline{\phi_0}\|} + \frac{g_2\Lambda + f_2}{1 - \|\overline{\phi_1}\|\Lambda - \|\overline{\phi_0}\|}.\end{aligned} \tag{41}$$

Therefore, we have

$$\|(x,y)\| \le \frac{1}{1-\rho}\left(\frac{g_1\Lambda + f_1}{1 - \|\phi_1\|\Lambda - \|\phi_0\|} + \frac{g_2\Lambda + f_2}{1 - \|\overline{\phi_1}\|\Lambda - \|\overline{\phi_0}\|}\right) < r.$$

Fourth, we show that $\delta M + \rho < 1$.

Since $M = \|B(S)\| = \sup_{(x,y)\in S}(\sup_{t\in[0,1]}|B(x,y)(t)|) < \Lambda$, using (H_4), we find

$$(\|\phi_1\| + \|\overline{\phi_1}\|)M + \|\phi_0\| + \|\overline{\phi_0}\| \le (\|\phi_1\| + \|\overline{\phi_1}\|)\Lambda + \|\phi_0\| + \|\overline{\phi_0}\| < 1.$$

Therefore, all the conditions of Lemma 2.2 are satisfied, so that $T(x,y)(t) = A(x,y)B(x,y) + C(x,y)$ has a coupled fixed point in S; consequently, Problems (1)–(4) have a solution defined on [0,1]. □

4 Example

We consider the hybrid boundary value systems with the Riesz–Caputo derivative:

$$ {}^{RC}_{0}D^{\frac{1}{2}}_{1}\left(\frac{x(t) - \frac{t^2}{100}(|x(t)| + |y(t)| + 10)}{\frac{|x(t)|}{1+|x(t)|} + \frac{|y(t)|}{1+|y(t)|} + 10}\right) = \frac{t}{10{,}000}\sin x(t)\frac{|y(t)|}{1 + |y(t)|}, \tag{42}$$

$$ {}^{RC}_{0}D^{\frac{1}{2}}_{1}\left(\frac{y(t) - \frac{t^2}{50}(|x(t)| + |y(t)| + 10)}{\frac{|x(t)|}{1+|x(t)|} + \frac{|y(t)|}{1+|y(t)|} + 11}\right) = \frac{t}{10{,}000}\cos\left(\frac{t}{4}\right)\sin x(t)\frac{|y(t)|}{1 + |y(t)|}, \tag{43}$$

$$x(0) = 0, \quad x(1) = 1, \tag{44}$$

$$y(0) = 0, \quad y(1) = 1. \tag{45}$$

We note that $\gamma = \nu = \frac{1}{2}$, $a_1 = 0$, $a_2 = 0$, $b_1 = 1$, $b_2 = 1$,

$$f_1(t,x,y) = \frac{t^2}{100}(|x| + |y| + 10), \quad f_2(t,x,y) = \frac{t^2}{50}(|x| + |y| + 10),$$

$$g_1(t,x,y) = \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|} + 10, \quad g_2(t,x,y) = \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|} + 11,$$

$$h_1(t,x,y) = \frac{t}{10{,}000}\sin x(t)\frac{|y|}{1+|y|}, \quad \text{and} \quad h_2(t,x,y) = \frac{t}{10{,}000}\cos\left(\frac{t}{4}\right)\sin x(t)\frac{|y|}{1+|y|}.$$

We can show that

$$\begin{aligned} f_1(t,x,y) - f_1(t,\bar{x},\bar{y}) &\le \frac{t^2}{100}(|x - \bar{x}| + |y - \bar{y}|), \\ f_2(t,x,y) - f_2(t,\bar{x},\bar{y}) &\le \frac{t^2}{50}(|x - \bar{x}| + |y - \bar{y}|), \\ g_1(t,x,y) - g_1(t,\bar{x},\bar{y}) &\le |x - \bar{x}| + |y - \bar{y}|, \\ g_2(t,x,y) - g_2(t,\bar{x},\bar{y}) &\le |x - \bar{x}| + |y - \bar{y}|. \end{aligned}$$

Hence, we obtain $\varphi_0(t) = \frac{t^2}{100}$, $\overline{\varphi_0}(t) = \frac{t^2}{50}$, $\varphi_1(t) = 1$, $\overline{\varphi_1}(t) = 1$, $\Gamma(\gamma + 1) = \Gamma(\nu + 1) = \Gamma(\frac{3}{2}) = 0.886$, then $\|\varphi_0\| = \frac{1}{100}$, $\|\overline{\varphi_0}\| = \frac{1}{50}$, $\|\varphi_1\| = 1$, and $\|\overline{\varphi_1}\| = 1$,

$$|h_1(t,x,y)| \le p_1(t)\psi_1(|x|)\xi_1(|y|), \quad |h_2(t,x,y)| \le p_2(t)\psi_2(|x|)\xi_2(|y|),$$

where

$$p_1(t) = \frac{t}{10{,}000}, \quad \psi_1(|x|) = |x|, \quad \xi_1(|y|) = |y|, \quad p_2(t) = \frac{t}{10{,}000}\cos\left(\frac{t}{4}\right), \quad \psi_2(|x|) = |x|, \quad \xi_2(|y|) = |y|,$$

then $\|p_1\| = \frac{1}{10{,}000}$, $\|p_2\| = \frac{1}{10{,}000}$, and

$$f_1 = \sup_{t\in[0,1]} |f_1(t,0,0)| = \sup_{t\in[0,1]} \frac{t^2}{10} = \frac{1}{10}, \quad f_2 = \sup_{t\in[0,1]} |f_2(t,0,0)| = \sup_{t\in[0,1]} \frac{t^2}{5} = \frac{1}{5},$$

$$g_1 = \sup_{t\in[0,1]} |g_1(t,0,0)| = \sup_{t\in[0,1]} 10 = 10, \quad g_2 = \sup_{t\in[0,1]} |g_2(t,0,0)| = \sup_{t\in[0,1]} 11 = 11,$$

$$f_1(0,0,0) = 0, \quad f_2(0,0,0) = 0, \quad f_1(1,1,1) = \frac{12}{100}, \quad f_2(1,1,1) = \frac{12}{50}, \quad g_1(1,1,1) = 11, \quad \text{and} \quad g_2(1,1,1) = 12.$$

We choose $r = 10$; by calculation $\Lambda = 0.1168$, $\rho = 0.158$; thus,

$$\frac{1}{1-\rho}\left(\frac{g_1\Lambda + f_1}{1 - \|\phi_1\|\Lambda - \|\phi_0\|} + \frac{g_2\Lambda + f_2}{1 - \|\overline{\phi_1}\|\Lambda - \|\overline{\phi_0}\|}\right) = 3.768 < 10 = r.$$

So far, we have proved that all conditions in Theorem 3.2 are satisfied, which implies that Problems (42)–(45) have a solution.

5 Conclusion

We have presented the existence results of a coupled system of hybrid boundary value problems with the Riesz-Caputo derivative with the aid of the Dhage fixed point theorem. The novelty of this kind of fractional derivative is Riesz-Caputo fractional derivative that can reflect both the past and the future memory effects.

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