Research Article

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On the monotonic properties and oscillatory behavior of solutions of neutral differential equations

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Abstract: In this work, we study new asymptotic properties of positive solutions of the even-order neutral differential equation with the noncanonical operator. The new properties are iterative, which means they can be used several times. We also use these properties to obtain new criteria for oscillation of the studied equation.

Keywords: neutral differential equations, oscillatory, even-order, non-canonical case

MSC 2020: 34C10, 34K11

1 Introduction

A delay differential equation (DDE) is an equation in which the solution and/or its derivatives at earlier times influence the time derivatives at the present time. Therefore, it is a better way to model engineering and physical problems. For example, we find that the neutral DDEs arise in many phenomena including problems in electrical networks that contain lossless transmission lines (as in high-speed computers where such lines are used to interconnect switching circuits), see [1].

The aim of this research is to discuss and analyze the asymptotic and oscillatory behavior of solutions of the neutral differential equation (NDE) of even-order

$$(\ell(u)(w^{(n-1)}(u))^r)' + q(u)x^r(\varrho(u)) = 0, \quad u \geq u_0, \tag{1}$$

where $w \coloneqq x + p \cdot (x \circ \tau)$ and $(x \circ \tau)(u) \coloneqq x(\tau(u))$. Throughout this work, we assume that

(A1) r is a ratio of odd positive integer;
(A2) $n \geq 4$ is an even natural number;
(A3) $q \in C([u_0, \infty))$ and $q(u) \geq 0$;

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(A4) $p \in C^1([u_0, \infty))$ and $0 \le p(u) < 1$;
(A5) $\ell \in C^1([u_0, \infty)), \ell(u) > 0$, $\ell'(u) \ge 0$, and

$$\int_{u_0}^{\infty} \frac{1}{\ell^{1/r}(b)} \mathrm{d}b < \infty;$$

(A6) $\tau, \varrho \in C^1([u_0, \infty))$, $\varrho(u) \le u$, $\varrho'(u) > 0$, and $\lim_{u \to \infty} \tau(u) = \lim_{u \to \infty} \varrho(u) = \infty$.

By a solution of (1), we mean a real-valued function $x \in C^{(n-1)}([U_x, \infty))$, $U_x \ge u_0$, which has the property $\ell(u)(w^{(n-1)}(u))^r \in C^1([U_x, \infty))$, and satisfies (1) on $[U_x, \infty)$. We consider only those solutions x of (1) that satisfy the condition

$$\sup\{|x(u)| : u \ge U\} > 0, \quad \text{for } U \ge U_x.$$

Definition 1.1. [2] A nontrivial solution x of the differential equation is said to be oscillatory if x has arbitrarily large zeros, that is, there exists an infinite sequence $\{u_n\}_{n=0}^{\infty}$ such that $x(u_n) = 0$ and $\lim_{n \to \infty} u_n = \infty$. Otherwise, it is said to be nonoscillatory. A differential equation is said to be oscillatory if all of its solutions are oscillatory.

Notation 1.1. A solution of (1) is said to be oscillatory if it has arbitrarily large zeros on $[U_x, \infty)$. Otherwise, it is said to be nonoscillatory. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

The highest-order derivative of the unknown function occurs with and without delay in an NDE. The qualitative study of such equations has a lot of practical use in addition to its theoretical value. NDEs are employed in a range of applications in economics, biology, medicine, engineering, and physics, such as lossless transmission lines, bridge vibration, and vibrational motion in flight, as well as the Euler equation in various variational situations [3,4]. Recently, several studies have appeared which investigated the oscillatory behavior of solutions of NDEs of different orders. The neutral equations of the second order have been greatly studied in works [5–7]. Even-order equations have also received great attention and remarkable development, see, for example, [8–10]. While neutral equations of odd order have received less attention compared to equations of even order [11,12].

Baculikova et al. [13] investigated the asymptotic characteristics and oscillation of the equation

$$(\ell(u)(x^{(n-1)}(u))^r)' + q(u) f(x(\tau(u))) = 0, \tag{2}$$

where $f(x)$ is nondecreasing, and

$$-f(-xy) \ge f(xy) \ge f(x) f(y), \quad \text{for } xy > 0.$$

Moreover, they considered the canonical case

$$\int_{u_0}^{\infty} \frac{1}{\ell^{1/r}(b)} \mathrm{d}b = \infty,$$

and noncanonical case

$$\int_{u_0}^{\infty} \frac{1}{\ell^{1/r}(b)} \mathrm{d}b < \infty. \tag{3}$$

Theorem 1.1. ([13], Theorem 4) *Let* (3) *hold. Assume that, for some $\delta \in (0, 1)$ and every $u_1 \ge u_0$, both*

$$y'(u) + q(u) f\left(\frac{\delta}{(n-1)!} \frac{\varrho^{n-1}(u)}{\ell^{1/r}(\varrho(u))}\right) f(y^{1/r}(\varrho(u))) = 0$$

and

$$y'(u) + q(u)\left(\frac{\delta}{(n-1)!}\right)^{\beta}\frac{(\varrho^{n-1}(u))^{\beta}}{\ell^{\beta/r}(\varrho(u))}y^{\beta/r}(\varrho(u)) = 0$$

are oscillatory. Assume further that there exists $\zeta(u) \in C([u_0, \infty))$ *such that*

$$y'(u) + \ell^{-1/r}(u)\left(\int_{u_1}^{u} q(b)\mathrm{d}b\right)^{1/r} f^{1/r}(J_{n-2}(\tau(u)))f^{1/r}(y(\zeta_{n-2}(\tau(u)))) = 0$$

is oscillatory for every $u_1 \geq u_0$ *and*

$$\zeta(u) \text{ is nondecreasing}, \ \zeta(u) > u \quad \text{and} \quad \zeta_{n-2}(\tau(u)) < u,$$

where

$$\zeta_1(u) = \zeta(u), \quad \zeta_{i+1}(u) = \zeta_i(\zeta(u)),$$
$$J_1(u) = \zeta(u) - u, \quad J_{i+1}(u) = \int_{u}^{\zeta(u)} J_i(b)\mathrm{d}b,$$

and $\zeta(u) \in C([u_0, \infty))$. *Then, equation* (2) *is oscillatory.*

Zhang et al. [14] studied the equation

$$(\ell(u)(x^{(n-1)}(u))^r)' + q(u)x^{\beta}(\tau(u)) = 0, \tag{4}$$

such that $\beta \leq r$, where β is a ratio of odd positive integer.

Theorem 1.2. ([14], Theorem 2.1) *Let* (3) *hold. Assume that the differential equation*

$$y'(u) + q(u)\left(\frac{\lambda_0 \tau^{n-1}(u)}{(n-1)!\ell^{1/r}(\tau(u))}\right)^{\beta} y^{\beta/r}(\tau(u)) = 0,$$

is oscillatory for some constant $\lambda_0 \in (0,1)$. *If*

$$\limsup_{u\to\infty}\int_{u_0}^{u}\left(M^{\beta-r}q(b)\left(\frac{\lambda_1\tau^{n-2}(b)}{(n-2)!}\right)^{\beta}\delta^r(b) - \frac{r^{r+1}}{(r+1)^{r+1}}\frac{1}{\delta(b)\ell^{1/r}(b)}\right)\mathrm{d}b = \infty$$

and

$$\limsup_{u\to\infty}\int_{u_0}^{u}\left(M^{\beta-r}q(b)\ell^r(b) - \frac{r^{r+1}}{(r+1)^{r+1}}\frac{(\ell'(b))^{r+1}}{\ell(b)\ell_*^r(b)}\right)\mathrm{d}b = \infty$$

hold for some constant $\lambda_1 \in (0,1)$ *and for every constant* $M > 0$, *where* $\delta(u) = \int_u^{\infty}\frac{1}{\ell^{1/r}(b)}\mathrm{d}b$ *and*

$$\ell(u) \coloneqq \frac{\int_u^{\infty}(b-u)^{n-3}\delta(b)\mathrm{d}b}{(n-3)!}, \quad \ell_*(u) \coloneqq \frac{\int_u^{\infty}(b-u)^{n-4}\delta(b)\mathrm{d}b}{(n-4)!},$$

then, (4) *is oscillatory.*

Zhang et al. [15] used the Riccati technique to establish some new oscillation conditions for all solutions of the equation

$$(\ell(u)(x''(u))^r)' + q(u)x^r(\tau(u)) = 0. \tag{5}$$

Theorem 1.3. ([15], Theorem 2.1) *Let* (3) *hold. Assume that there exists a positive function* $\rho \in C^1[u_0, \infty)$ *such that*

$$\int_{u_0}^{\infty}\left(q(b)\left(\frac{\tau^2(b)}{b^2}\right)^r \rho(b) - \frac{2^r}{(r+1)^{r+1}}\frac{\ell(b)(\rho_+'(b))^{r+1}}{(k_1\rho(b)b^2)^r}\right)\mathrm{d}b = \infty,$$

for some constant $k_1 \in (0, 1)$. *Assume further that there exists a positive function* $\theta \in C^1[u_0, \infty)$ *such that*

$$\int_{u_0}^{\infty}\left(\theta(b)\int_b^{\infty}\left(\frac{1}{\ell(\vartheta)}\int_{\vartheta}^{\infty}q(\varsigma)\left(\frac{\tau^2(\varsigma)}{\varsigma^2}\right)^r\mathrm{d}\varsigma\right)^{1/r}\mathrm{d}\vartheta - \frac{(\theta_+'(b))^2}{4\theta(b)}\right)\mathrm{d}b = \infty.$$

If

$$\int_{u_0}^{\infty}\left(q(b)\left(\int_b^{\infty}\int_b^{\infty}\ell(v)\mathrm{d}v\mathrm{d}b\right)^u - \frac{r^{r+1}}{(r+1)^{r+1}}\frac{\int_b^{\infty}\ell(v)\mathrm{d}v}{\int_b^{\infty}\int_b^{\infty}\ell(v)\mathrm{d}v\mathrm{d}b}\right)\mathrm{d}b = \infty$$

and

$$\int_{u_0}^{\infty}\left(q(b)\left(\frac{k_2}{2}\tau^2(b)\right)^{\gamma}\ell^r(b) - \frac{r^{r+1}}{(r+1)^{r+1}\ell(b)\ell^{1/r}(b)}\right)\mathrm{d}b = \infty,$$

for some constant $k_2 \in (0, 1)$, $\ell(u) := \int_l^{\infty}\frac{1}{\ell^{1/\gamma}(b)}\mathrm{d}b, \theta_+'(l) := \max\{0, \theta'(l)\}$, *and* $\rho_+'(l) := \max\{0, \rho'(l)\}$, *then every solution of* (5) *is oscillatory.*

For even-order NDEs, Zafer [16], Karpuz et al. [17], Zhang et al. [18], and Li et al. [19] studied the oscillation of the NDE

$$w^{(n)}(u) + q(u)x(\varrho(u)) = 0.$$

The oscillation properties of the even-order quasi-linear NDE

$$(\ell(u)|(w(u))^{(n-1)}|^{r-1}(w(u))^{(n-1)})' + q(u)|x(\varrho(u))|^{r-1}x(\varrho(u)) = 0$$

were studied by Meng and Xu [20]. Li and Rogovchenko [21] studied the asymptotic behavior of solutions of the NDE

$$(\ell(u)(w^{(n-1)}(u))^r)' + q(u)x^{\beta}(\varrho(u)) = 0, \tag{6}$$

where $\beta \le r$.

Theorem 1.4. ([21], Theorem 8) *Let* $0 < r = \beta \le 1$ *and* (3) *hold, and there exist three functions* η_1, η_2, $\eta_3 \in C([u_0, \infty), \mathbb{R})$ *such that*

$$\eta_3(u) \ge \varrho(u), \quad \eta_3(u) > \tau(u).$$

Suppose also that conditions

$$\limsup_{u\to\infty}\int_{\eta_1(u)}^{u}\widetilde{Q}_{\theta}(b)\mathrm{d}b > \frac{1}{\tau_*}\frac{((n-1)!)^{\beta}(\tau_* + p_0^{\beta})}{\mathrm{e}}$$

and

$$\limsup_{u\to\infty}\int_u^{\tau^{-1}(\eta_2(u))}Q_{\theta}(b)A^{\beta}(\eta_2(b))\mathrm{d}b > \frac{1}{\tau_*}\frac{((n-2)!)^{\beta}(\tau_* + p_0^{\beta})}{\mathrm{e}}$$

hold. If

$$\limsup_{u\to\infty}\int_{u}^{\tau^{-1}(\eta_3(u))}\overline{Q_\theta}(b)\mathrm{d}b > \frac{1}{\tau_*}\frac{((n-3)!)^{\beta}(\tau_*+p_0^{\beta})}{\mathrm{e}},$$

then (6) *is oscillatory.*

Note that

$$0 \le p(u) \le p_0 < \infty \quad and \quad \tau(u) \ge \tau_*,$$

$$Q_\theta(u) \coloneqq Q(u)(\varrho^{n-2}(u))^{\beta}, \quad \widetilde{Q}_\theta(u) \coloneqq Q(u)\left(\frac{(\eta_1(u))^{n-1}}{\ell^{1/\beta}(\eta_1(u))}\right)^{\beta},$$

$$\overline{Q_\theta}(b) \coloneqq Q(u)\left(\int_{\eta_3(u)}^{\infty}(b-\eta_3(u))^{n-3}A(b)\mathrm{d}b\right), \quad and \quad A(u) \coloneqq \int_{u}^{\infty}\frac{1}{\ell^{1/r}(b)}\mathrm{d}b.$$

In this article, we derive new monotonic properties of a class of the positive solutions of (1). Then, we improve these properties by giving them an iterative nature. Moreover, using these new properties enables us to create oscillation criteria for all solutions of the studied equation. Finally, we give some examples that support our results.

2 Auxiliary results

In this section, we will establish some important lemmas that we will use to prove the main results.

Lemma 2.1. ([22], Lemma 2.2.3) *Suppose that $f \in C^m([u_0,\infty),\ell^+)$, $f^{(m)}(u)$ is of fixed sign and not identically zero on $[u_0,\infty)$ and that there exists $u_1 \ge u_0$ such that $f^{(m-1)}(u)f^{(m)}(u) \le 0$ for all $u_1 \ge u_0$. If $\lim_{u\to\infty} f(u) \ne 0$, then, for every $\delta \in (0,1)$, there exists $u_\delta \in [u_1,\infty)$ such that*

$$f(u) \ge \frac{\delta}{(m-1)!}u^{m-1}|f^{(m-1)}(u)|,$$

for $u \in [u_\delta,\infty)$.

Lemma 2.2. [23] *Let A and B be real numbers and $A > 0$. Then,*

$$Ab^{(r+1)/r} - Bb \ge -\frac{r^r}{(r+1)^{r+1}}\frac{B^{r+1}}{A^r}. \tag{7}$$

The qualitative study of solutions of NDEs begins with the classification of the signs of the associated function’s derivatives $w^{(i)}$, $i = 1, 2, \ldots, n$. By using (Kneser’s theorem) Lemma 2.2.1 in [22], we can obtain the following classification of derivatives of w.

Lemma 2.3. *Assume that x is a positive solution to equation* (1). *Then, $\ell(u)(w^{(n-1)}(u))^r$ is nonincreasing and w satisfies one of the following cases*:

($\mathbf{N}_1$) $w^{(\ell)}(u) > 0$, *for* $\ell = 0, 1, n-1$ *and* $w^{(n)}(u) < 0$;
($\mathbf{N}_2$) $w^{(\ell)}(u) > 0$, *for* $\ell = 0, 1, n-2$ *and* $w^{(n-1)}(u) < 0$;
($\mathbf{N}_3$) $(-1)^{\ell}w^{(\ell)}(u) > 0$ *for* $\ell = 0, 1, 2, \ldots, n-1$,

eventually.

Here, we define the class Ω as the category of all positive solutions of (1) with w satisfying $\mathbf{N}_2$. Further, we define

$$L_0(u) := \int_u^\infty \frac{1}{\ell^{1/r}(b)}\mathrm{d}b,$$
$$L_m(u) := \int_u^\infty L_{m-1}(b)\mathrm{d}b,\ m = 1, 2, \ldots, n-2,$$
$$Q(u) := q(u)(1 - p(\varrho(u)))^r,$$

and

$$Q^*(u) := q(u)\left(1 - p(\varrho(u))\frac{L_{n-2}(\tau(\varrho(u)))}{L_{n-2}(\varrho(u))}\right)^r.$$

Lemma 2.4. *Assuming x belongs to Ω, we obtain the following, eventually*

B1,1 $x(u) \geq (1 - p(u))w(u)$;
B1,2 $w(u) \geq \frac{\mu_0}{(n-2)!}u^{n-2}w^{(n-2)}(u)$, for all $\mu_0 \in (0, 1)$;
B1,3 $(\ell(u)(w^{(n-1)}(u))^r)' \leq -Q(u)w^r(\varrho(u))$;
B1,4 $w^{(n-2)}(u) \geq -L_0(u)\ell^{1/r}(u)w^{(n-1)}(u)$;
B1,5 $w^{(n-2)}(u)/L_0(u)$ *is increasing.*

Proof. ($\mathbf{B}_{1,1}$): As a result of the facts $x \in \Omega$ and $\tau(u) \leq u$, we find $w'(u) > 0$ and $x(\tau(u)) \leq w(\tau(u)) \leq w(u)$. Consequently, we obtain

$$\begin{aligned} x(u) &= w(u) - p(u)x(\tau(u)) \\ &\geq w(u) - p(u)w(\tau(u)) \\ &\geq (1 - p(u))w(u). \end{aligned}$$

($\mathbf{B}_{1,2}$): Using Lemma 2.1 with $m = n - 1$ and $g = w$, we have

$$w(u) \geq \frac{\mu_0}{(n-2)!}u^{n-2}w^{(n-2)}(u),$$

for all $\mu_0 \in (0, 1)$.

($\mathbf{B}_{1,3}$): Equation (1) with ($\mathbf{B}_{1,1}$) becomes

$$\begin{aligned} (\ell(u)(w^{(n-1)}(u))^r)' &= -q(u)x^r(\varrho(u)) \\ &\leq -q(u)(1 - p(\varrho(u)))^r w^r(\varrho(u)) \\ &\leq -Q(u)w^r(\varrho(u)). \end{aligned}$$

($\mathbf{B}_{1,4}$): Since $\ell(u)(w^{(n-1)}(u))^r$ is decreasing, we obtain

$$\begin{aligned} w^{(n-2)}(u) &= -\int_u^\infty w^{(n-1)}(b)\mathrm{d}b \\ &= -\int_u^\infty \frac{\ell^{1/r}(b)}{\ell^{1/r}(b)}w^{(n-1)}(b)\mathrm{d}b \\ &\geq -\ell^{1/r}(u)w^{(n-1)}(u)\int_u^\infty \frac{1}{\ell^{1/r}(b)}w\mathrm{d}b \\ &\geq -L_0(u)\ell^{1/r}(u)w^{(n-1)}(u). \end{aligned} \tag{8}$$

($\mathbf{B}_{1,5}$): From ($\mathbf{B}_{1,4}$), we obtain

$$\left(\frac{w^{(n-2)}(u)}{L_0(u)}\right)' = \frac{1}{\ell^{1/r}(u)L_0^2(u)}(L_0(u)\ell^{1/r}(u)w^{(n-1)}(u) + w^{(n-2)}(u)) \geq 0.$$

The proof of this lemma is now complete. □

Lemma 2.5. *Assuming x belongs to Ω and there are $\delta > 0$ and $u_1 \geq u_0$ such that*

$$\frac{1}{r}\ell^{1/r}(u)L_0^{1+r}(u)(\varrho^{n-2}(u))^r Q(u) \geq ((n-2)!)^r\delta, \tag{9}$$

we obtain, for $u \geq u_1$,

B2,1 $\lim_{u\to\infty} w^{(n-2)}(u) = 0$;

B2,2 $w^{(n-2)}(u)/L_0^{\beta_0}(u)$ *is decreasing*;

B2,3 $\lim_{u\to\infty} w^{(n-2)}(u)/L_0^{\beta_0}(u) = 0$;

B2,4 $w^{(n-2)}(u)/L_0^{1-\beta_0}(u)$ *is increasing*,

for $u \geq u_0$, where $\beta_0 = \mu_0\delta^{1/r}$, $\mu_0 \in (0,1)$, and $r \leq 1$.

Proof. ($\mathbf{B}_{2,1}$): Since $x \in \Omega$, we can conclude that ($\mathbf{B}_{1,1}$) − ($\mathbf{B}_{1,5}$) in Lemma 2.4 are satisfied for all $u \geq u_1$, u_1 large enough. Now, since $w^{(n-2)}(u)$ is a positive decreasing function, we conclude that $\lim_{u\to\infty} w^{(n-2)}(u) = c_1 \geq 0$. We claim that $c_1 = 0$. If not, then $w^{(n-2)}(u) \geq c_1 > 0$ eventually, which with ($\mathbf{B}_{1,2}$) gives

$$\begin{aligned} w(u) &\geq \frac{\mu_0}{(n-2)!}u^{n-2}w^{(n-2)}(u) \\ &\geq \frac{\mu_0 c_1}{(n-2)!}u^{n-2}, \end{aligned}$$

for all $\mu_0 \in (0,1)$. Thus, from ($\mathbf{B}_{1,3}$), we obtain

$$\begin{aligned} (\ell(u)(w^{(n-1)}(u))^r)' &\leq -Q(u)w^r(\varrho(u)) \\ &\leq -\left(\frac{\mu_0 c_1}{(n-2)!}\varrho^{n-2}(u)\right)^r Q(u) \\ &\leq -\mu_0^r c_1^r \frac{(\varrho^{n-2}(u))^r}{((n-2)!)^r}Q(u), \end{aligned}$$

which with (9) gives

$$\begin{aligned} (\ell(u)(w^{(n-1)}(u))^r)' &\leq -rc_1^r\mu_0^r\delta\frac{1}{\ell^{1/r}(u)L_0^{1+r}(u)} \\ &\leq -rc_1^r\beta_0^r\frac{1}{\ell^{1/r}(u)L_0^{1+r}(u)}. \end{aligned}$$

Integrating the previous inequality from u_2 to u, we have

$$\begin{aligned} \ell(u)(w^{(n-1)}(u))^r &\leq \ell(u_2)(w^{(n-1)}(u_2))^r - rc_1^r\beta_0^r\int_{u_2}^{u}\frac{1}{\ell^{1/r}(b)L_0^{1+r}(b)}\mathrm{d}b \\ &\leq \beta_0^r c_1^r\left(\frac{1}{L_0^r(u_2)} - \frac{1}{L_0^r(u)}\right). \end{aligned} \tag{10}$$

Since $L_0^{-r}(u) \to \infty$ as $u \to \infty$, there is a $u_3 \geq u_2$ such that $L_0^{-r}(u) - L_0^{-r}(u_2) \geq \varepsilon L_0^{-r}(u)$ for all $\varepsilon \in (0,1)$. Hence, (10) becomes

$$w^{(n-1)}(u) \leq -c_1 \varepsilon^{1/r} \beta_0 \frac{1}{\ell^{1/r}(u) L_0(u)},$$

for all $u \geq u_3$. Integrating the last inequality from u_3 to u, we find

$$\begin{aligned} w^{(n-2)}(u) &\leq w^{(n-2)}(u_3) - c_1 \varepsilon^{1/r} \beta_0 \int_{u_3}^{u} \frac{1}{\ell^{1/r}(b) L_0(b)} \mathrm{d}b \\ &\leq w^{(n-2)}(u_3) - c_1 \varepsilon^{1/r} \beta_0 \ln \frac{L_0(u_3)}{L_0(u)} \to -\infty \text{ as } u \to \infty, \end{aligned}$$

which is a contradiction. Then, $c_1 = 0$.

($\mathbf{B}_{2,2}$) From (9), ($\mathbf{B}_{1,2}$) and ($\mathbf{B}_{1,3}$), we obtain

$$(\ell(u)(w^{(n-1)}(u))^r)' \leq -\frac{r\beta_0^r}{\ell^{1/r}(u) L_0^{1+r}(u)} (w^{(n-2)}(\varrho(u)))^r.$$

By integrating the last inequality from u_1 to u and using the fact $w^{(n-1)}(u) < 0$, we obtain

$$\begin{aligned} \ell(u)(w^{(n-1)}(u))^r &\leq \ell(u_1)(w^{(n-1)}(u_1))^r - r\beta_0^r \int_{u_1}^{u} \frac{1}{\ell^{1/r}(b) L_0^{1+r}(b)} (w^{(n-2)}(\varrho(b)))^r \mathrm{d}b \\ &\leq \ell(u_1)(w^{(n-1)}(u_1))^r - r\beta_0^r (w^{(n-2)}(u))^r \int_{u_1}^{u} \frac{1}{\ell^{1/r}(b) L_0^{1+r}(b)} \mathrm{d}b \\ &\leq \ell(u_1)(w^{(n-1)}(u_1))^r + \frac{\beta_0^r}{L_0^r(u_1)} (w^{(n-2)}(u))^r - \frac{\beta_0^r}{L_0^r(u)} (w^{(n-2)}(u))^r. \end{aligned}$$

Because $w^{(n-2)}(u) \to 0$ as $u \to \infty$ there is a $u_2 \geq u_1$ such that

$$\ell(u_1)(w^{(n-1)}(u_1))^r + \frac{\beta_0^r}{L_0^r(u_1)} (w^{(n-2)}(u))^r \leq 0,$$

for $u \geq u_2$. Therefore, we have

$$\ell(u)(w^{(n-1)}(u))^r \leq -\frac{\beta_0^r}{L_0^r(u)} (w^{(n-2)}(u))^r,$$

or equivalent

$$\ell^{1/r}(u) w^{(n-1)}(u) L_0(u) + \beta_0 w^{(n-2)}(u) \leq 0, \tag{11}$$

and then

$$\left(\frac{w^{(n-2)}(u)}{L_0^{\beta_0}(u)} \right)' = \frac{L_0(u) \ell^{1/r}(u) w^{(n-1)}(u) + \beta_0 w^{(n-2)}(u)}{\ell^{1/r}(u) L_0^{1+\beta_0}(u)} \leq 0.$$

($\mathbf{B}_{2,3}$) Since $w^{(n-2)}(u)/L_0^{\beta_0}(u)$ is a positive decreasing function,

$$\lim_{u \to \infty} w^{(n-2)}(u)/L_0^{\beta_0}(u) = c_2 \geq 0.$$

We claim that $c_2 = 0$. If not, then $w^{(n-2)}(u)/L_0^{\beta_0}(u) \geq c_2 > 0$ eventually. Now, we introduce the function

$$\Psi(u) = \frac{w^{(n-2)}(u) + L_0(u) \ell^{1/r}(u) w^{(n-1)}(u)}{L_0^{\beta_0}(u)}.$$

In view of ($\mathbf{B}_{1,4}$), we note that $\Psi(u) > 0$ and

$$\begin{aligned}\Psi'(u) &= \frac{w^{(n-1)}(u) + L_0(u)(\ell^{1/r}(u)w^{(n-1)}(u))' - w^{(n-1)}(u)}{L_0^{\beta_0}(u)} \\ &\quad + \beta_0\frac{w^{(n-2)}(u) + L_0(u)\ell^{1/r}(u)w^{(n-1)}(u)}{\ell^{1/r}(u)L_0^{1+\beta_0}(u)} \\ &= \frac{(\ell^{1/r}(u)w^{(n-1)}(u))'}{L_0^{\beta_0-1}(u)} + \beta_0\frac{w^{(n-2)}(u)}{\ell^{1/r}(u)L_0^{1+\beta_0}(u)} + \beta_0\frac{w^{(n-1)}(u)}{L_0^{\beta_0}(u)} \\ &= \frac{1}{r}\frac{(\ell(u)(w^{(n-1)}(u))^r)'(\ell^{1/r}(u)w^{(n-1)}(u))^{1-r}}{L_0^{\beta_0-1}(u)} \\ &\quad + \beta_0\frac{w^{(n-2)}(u)}{\ell^{1/r}(u)L_0^{1+\beta_0}(u)} + \beta_0\frac{w^{(n-1)}(u)}{L_0^{\beta_0}(u)}.\end{aligned}$$

From ($\mathbf{B}_{1,3}$), ($\mathbf{B}_{1,4}$), (9) and (11), we obtain

$$\begin{aligned}(\ell(u)(w^{(n-1)}(u))^r)' &\leq -\left(\frac{\mu_0}{(n-2)!}\varrho^{n-2}(u)\right)^r Q(u)(w^{(n-2)}(\varrho(u)))^r \\ &\leq -r\beta_0^r\frac{1}{\ell^{1/r}(u)L_0^{1+r}(u)}(w^{(n-2)}(\varrho(u)))^r,\end{aligned} \tag{12}$$

and

$$\ell^{1/r}(u)w^{(n-1)}(u) \leq -\beta_0\frac{w^{(n-2)}(u)}{L_0(u)}$$

or equivalently

$$(\ell^{1/r}(u)w^{(n-1)}(u))^{1-r} \geq \left(\beta_0\frac{w^{(n-2)}(u)}{L_0(u)}\right)^{1-r}, \tag{13}$$

which with (12) and (13), we obtain

$$\begin{aligned}\Psi'(u) &\leq -\frac{\beta_0^r}{L_0^{\beta_0-1}(u)}\frac{1}{\ell^{1/r}(u)L_0^{1+r}(u)}(w^{(n-2)}(\varrho(u)))^r\left(\beta_0\frac{w^{(n-2)}(u)}{L_0(u)}\right)^{1-r} \\ &\quad + \beta_0\frac{w^{(n-2)}(u)}{\ell^{1/r}(u)\ell_0^{1+\beta_0}(u)} + \beta_0\frac{w^{(n-1)}(u)}{\ell_0^{\beta_0}(u)}.\end{aligned}$$

Since $w^{(n-1)}(u) < 0$, $\varrho(u) \leq u$, we obtain $w^{(n-2)}(\varrho(u)) \geq w^{(n-2)}(u)$, and then

$$\begin{aligned}\Psi'(u) &\leq -\frac{\beta_0^r}{L_0^{\beta_0-1}(u)}\frac{1}{\ell^{1/r}(u)L_0^{1+r}(u)}(w^{(n-2)}(u))^r\left(\beta_0\frac{w^{(n-2)}(u)}{L_0(u)}\right)^{1-r} \\ &\quad + \beta_0\frac{w^{(n-2)}(u)}{\ell^{1/r}(u)L_0^{1+\beta_0}(u)} + \beta_0\frac{w^{(n-1)}(u)}{L_0^{\beta_0}(u)} \\ &\leq -\beta_0\frac{w^{(n-2)}(u)}{\ell^{1/r}(u)L_0^{1+\beta_0}(u)} + \beta_0\frac{w^{(n-2)}(u)}{\ell^{1/r}(u)L_0^{1+\beta_0}(u)} + \beta_0\frac{w^{(n-1)}(u)}{L_0^{\beta_0}(u)} \\ &\leq \beta_0\frac{w^{(n-1)}(u)}{L_0^{\beta_0}(u)}.\end{aligned}$$

Using the fact that $w^{(n-2)}(u)/L_0^{\beta_0}(u) \geq c_2$, and (11), we obtain

$$\begin{aligned}\Psi'(u) &\leq \beta_0 \frac{w^{(n-1)}(u)}{L_0^{\beta_0}(u)}\\ &\leq \beta_0 \frac{1}{L_0^{\beta_0}(u)}\left(\frac{-\beta_0 w^{(n-2)}(u)}{\ell^{1/r}(u)L_0(u)}\right)\\ &\leq -\frac{w^{(n-2)}(u)}{L_0^{\beta_0}(u)}\frac{\beta_0^2}{\ell^{1/r}(u)L_0(u)}\\ &\leq \frac{-c_2\beta_0^2}{\ell^{1/r}(u)L_0(u)} < 0.\end{aligned}$$

The function $\Psi(u)$ converges to a non-negative constant because it is a positive decreasing function. Integrating the last inequality from u_3 to ∞, we obtain

$$-\Psi(u_3) \leq -\beta_0^2 c_2 \lim_{u\to\infty} \ln \frac{L_0(u_3)}{L_0(u)},$$

or equivalently

$$\Psi(u_3) \geq \beta_0^2 c_2 \lim_{u\to\infty} \ln \frac{L_0(u_3)}{L_0(u)} \to \infty,$$

which is a contradiction and we obtain that $c_2 = 0$.

($\mathbf{B}_{2,4}$) Now, we have

$$\begin{aligned}(\ell^{1/r}(u)w^{(n-1)}(u)L_0(u) + w^{(n-2)}(u))' &= (\ell^{1/r}(u)w^{(n-1)}(u))'L_0(u) - w^{(n-1)}(u) + w^{(n-1)}(u)\\ &= (\ell^{1/r}(u)w^{(n-1)}(u))'L_0(u)\\ &= \frac{1}{r}(\ell(u)(w^{(n-1)}(u))^r)'(\ell^{1/r}(u)w^{(n-1)}(u))^{1-r}L_0(u),\end{aligned}$$

which with (12) and (13), we obtain

$$\begin{aligned}(\ell^{1/r}(u)w^{(n-1)}(u)L_0(u) + w^{(n-2)}(u))' &\leq -\beta_0^r \frac{1}{\ell^{1/r}(u)L_0^{1+r}(u)}(w^{(n-2)}(\varrho(u)))^r\left(\beta_0\frac{w^{(n-2)}(u)}{L_0(u)}\right)^{1-r}L_0(u)\\ &\leq -\beta_0^r \frac{1}{\ell^{1/r}(u)L_0^r(u)}(w^{(n-2)}(u))^r\left(\beta_0\frac{w^{(n-2)}(u)}{L_0(u)}\right)^{1-r}\\ &\leq \frac{-\beta_0}{\ell^{1/r}(u)L_0(u)}w^{(n-2)}(u).\end{aligned}$$

Integrating the last inequality from u to ∞, we obtain

$$-\ell^{1/r}(u)w^{(n-1)}(u)L_0(u) - w^{(n-2)}(u) \leq -\beta_0 \int_u^\infty \frac{1}{\ell^{1/r}(b)L_0(b)} w^{(n-2)}(b)\mathrm{d}b,$$

or equivalently

$$\begin{aligned}\ell^{1/r}(u)w^{(n-1)}(u)L_0(u) + w^{(n-2)}(u) &\geq \beta_0 \int_u^\infty \frac{1}{\ell^{1/r}(b)L_0(b)} w^{(n-2)}(b)\mathrm{d}b\\ &\geq \beta_0 \frac{w^{(n-2)}(u)}{L_0(u)} \int_u^\infty \frac{1}{\ell^{1/r}(b)}\mathrm{d}b\\ &\geq \beta_0 w^{(n-2)}(u),\end{aligned}$$

which mean that

$$\ell^{1/r}(u)w^{(n-1)}(u)L_0(u) + (1-\beta_0)w^{(n-2)}(u) \geq 0.$$

Then,

$$\left(\frac{w^{(n-2)}(u)}{L_0^{1-\beta_0}(u)}\right)' = \frac{L_0(u)\ell^{1/r}(u)w^{(n-1)}(u) + (1-\beta_0)w^{(n-2)}(u)}{\ell^{1/r}(u)L_0^{2-\beta_0}(u)} \geq 0. \tag{14}$$

□

If $\beta_0 \leq 1/2$, we can improve the properties in Lemma 2.5, as in the following lemma.

Lemma 2.6. *Assuming x belongs to Ω and* (9) *holds. If*

$$\lim_{u\to\infty} \frac{L_0(\varrho(u))}{L_0(u)} = \lambda < \infty, \tag{15}$$

and there exists an increasing sequence $\{\beta_\ell\}_{\ell=1}^{m}$ defined as

$$\beta_\ell \coloneqq \beta_0 \frac{\lambda^{\beta_{\ell-1}}}{(1-\beta_{\ell-1})^{1/r}},$$

with $r \leq 1$, $\beta_0 = \mu_0\delta^{1/r}$, $\beta_{m-1} \leq 1/2$, and $\beta_m, \mu_0 \in (0,1)$. Then, eventually,

B3,1 $w^{(n-2)}(u)/L_0^{\beta_m}(u)$ *is decreasing*;

B3,2 $\lim_{u\to\infty} w^{(n-2)}(u)/L_0^{\beta m}(u) = 0$.

Proof. ($\mathbf{B}_{3,1}$) Since $x \in \Omega$, we can conclude that ($\mathbf{B}_{1,1}$) − ($\mathbf{B}_{1,5}$) in Lemma 2.4 are satisfied for all $u \geq u_1$, u_1 is large enough. Furthermore, from Lemma 2.5, we have that ($\mathbf{B}_{2,1}$) − ($\mathbf{B}_{2,4}$) hold.

Now, assume that $\beta_0 \leq 1/2$, and

$$\beta_1 \coloneqq \beta_0 \frac{\lambda^{\beta_0}}{(1-\beta_0)^{1/r}}.$$

Next, we will prove ($\mathbf{B}_{3,1}$) and ($\mathbf{B}_{3,2}$) at $m = 1$. As in the proof of Lemma 2.5, we obtain

$$(\ell(u)(w^{(n-1)}(u))^r)' \leq -r\beta_0^r \frac{1}{\ell^{1/r}(u)L_0^{1+r}(u)}(w^{(n-2)}(\varrho(u)))^r.$$

Integrating the last inequality from u_1 to u, and using ($\mathbf{B}_{2,2}$) and (15), we obtain

$$\begin{aligned}
\ell(u)(w^{(n-1)}(u))^r &\leq \ell(u_1)(w^{(n-1)}(u_1))^r - r\beta_0^r \int_{u_1}^{u} \frac{1}{\ell^{1/r}(b)L_0^{1+r}(b)}(w^{(n-2)}(\varrho(b)))^r \mathrm{d}b \\
&\leq \ell(u_1)(w^{(n-1)}(u_1))^r - r\beta_0^r \int_{u_1}^{u} \frac{1}{\ell^{1/r}(b)L_0^{1+r}(b)} \ell_0^{r\beta_0}(\varrho(b)) \left(\frac{w^{(n-2)}(b)}{L_0^{\beta_0}(b)}\right)^r \mathrm{d}b \\
&\leq \ell(u_1)(w^{(n-1)}(u_1))^r - r\beta_0^r \left(\frac{w^{(n-2)}(u)}{L_0^{\beta_0}(u)}\right)^r \int_{u_1}^{u} \frac{\ell_0^{-1-r+r\beta_0}(b)}{\ell^{1/r}(b)} \frac{\ell_0^{r\beta_0}(\varrho(b))}{L_0^{r\beta_0}(b)} \mathrm{d}b \\
&\leq \ell(u_1)(w^{(n-1)}(u_1))^r - r\beta_0^r \lambda^{r\beta_0} \left(\frac{w^{(n-2)}(u)}{L_0^{\beta_0}(u)}\right)^r \int_{u_1}^{u} \frac{\ell_0^{-1-r+r\beta_0}(b)}{\ell^{1/r}(b)} \mathrm{d}b \\
&\leq \ell(u_1)(w^{(n-1)}(u_1))^r - \frac{\beta_0^r \lambda^{r\beta_0}}{1-\beta_0} \left(\frac{w^{(n-2)}(u)}{L_0^{\beta_0}(u)}\right)^r \left(\frac{1}{L_0^{r(1-\beta_0)}(u)} - \frac{1}{L_0^{r(1-\beta_0)}(u_1)}\right) \\
&\leq \ell(u_1)(w^{(n-1)}(u_1))^r + \beta_1^r \frac{1}{L_0^{r(1-\beta_0)}(u_1)} \left(\frac{w^{(n-2)}(u)}{L_0^{\beta_0}(u)}\right)^r - \beta_1^r \left(\frac{w^{(n-2)}(u)}{L_0(u)}\right)^r.
\end{aligned}$$

Using the fact that $w^{(n-2)}(u)/L_0^{\beta_0}(u) \to 0$ as $u \to \infty$, we have that

$$\ell(u_1)(w^{(n-1)}(u_1))^r + \beta_1^r \frac{1}{L_0^{r(1-\beta_0)}(u_1)} \left(\frac{w^{(n-2)}(u)}{L_0^{\beta_0}(u)} \right)^r \le 0.$$

Therefore, we have

$$\ell(u)(w^{(n-1)}(u))^r \le -\beta_1^r \left(\frac{w^{(n-2)}(u)}{L_0(u)} \right)^r,$$

or equivalently

$$\ell^{1/r}(u)w^{(n-1)}(u)L_0(u) + \beta_1 w^{(n-2)}(u) \le 0,$$

and then

$$\left(\frac{w^{(n-2)}(u)}{L_0^{\beta_1}(u)} \right)' = \frac{L_0(u)\ell^{1/r}(u)w^{(n-1)}(u) + \beta_1 w^{(n-2)}(u)}{\ell^{1/r}(u)L_0^{1+\beta_1}(u)} \le 0.$$

By repeating the same approach used previously, we can prove that

$$\lim_{u\to\infty} \frac{w^{(n-2)}(u)}{L_0^{\beta_1}(u)} = 0$$

and

$$\left(\frac{w^{(n-2)}(u)}{L_0^{1-\beta_1}(u)} \right)' \ge 0.$$

Similarly, if $\beta_{k-1} < \beta_k \le 1/2$, then we can prove

$$\ell^{1/r}(u)w^{(n-1)}(u)L_0(u) + \beta_k w^{(n-2)}(u) \le 0 \tag{16}$$

and

$$\lim_{u\to\infty} \frac{w^{(n-2)}(u)}{L_0^{\beta_k}(u)} = 0,$$

for $k = 2, 3, \ldots, m$. The proof of lemma is complete. □

Lemma 2.7. *Assume that x is a positive solution of* (1) *and w satisfies case* ($\mathbf{N}_3$), *then*

$$\left(\frac{w(u)}{L_{n-2}(u)} \right)' \ge 0. \tag{17}$$

Proof. Assume that x is a positive solution of (1) and w satisfies case ($\mathbf{N}_3$). From (1), we have $\ell(u)(w^{(n-1)}(u))^r$ is decreasing, and hence

$$\begin{aligned} \ell^{1/r}(u)w^{(n-1)}(u)\int_u^\infty \frac{1}{\ell^{1/r}(b)}\mathrm{d}b &\ge \int_u^\infty \frac{1}{\ell^{1/r}(b)}\ell^{1/r}(b)w^{(n-1)}(b)\mathrm{d}b \\ &= \lim_{u\to\infty} w^{(n-2)}(u) - w^{(n-2)}(u). \end{aligned} \tag{18}$$

Since $w^{(n-2)}(u)$ is a positive decreasing function, we have that $w^{(n-2)}(u)$ converges to a nonnegative constant when $u \to \infty$. Thus, (18) becomes

$$-w^{(n-2)}(u) \le \ell^{1/r}(u)w^{(n-1)}(u)L_0(u), \tag{19}$$

from (19), we find

$$\left(\frac{w^{(n-2)}(u)}{L_0(u)}\right)' = \frac{(\ell^{1/r}(u)L_0(u)w^{(n-1)}(u) + w^{(n-2)}(u))}{\ell^{1/r}(u)L_0^2(u)} \geq 0,$$

which leads to

$$\begin{aligned}-w^{(n-3)}(u) &\geq \int_u^\infty \frac{w^{(n-2)}(b)}{L_0(b)}L_0(b)\mathrm{d}b\\ &\geq \frac{w^{(n-2)}(u)}{L_0(u)}\int_u^\infty L_0(b)\mathrm{d}b\\ &= \frac{w^{(n-2)}(u)}{L_0(u)}L_1(b).\end{aligned}$$

This implies

$$\left(\frac{w^{(n-3)}(u)}{L_1(u)}\right)' = \frac{L_1(u)w^{(n-2)}(u) + w^{(n-3)}(u)L_0(u)}{L_1^2(u)} \leq 0.$$

Similarly, we repeat the same previous process $(n-4)$ times, we obtain

$$\left(\frac{w'(u)}{L_{n-3}(u)}\right)' \leq 0.$$

Now

$$\begin{aligned}-w(u) &\leq \int_u^\infty \frac{w'(b)}{L_{n-3}(b)}L_{n-3}(b)\mathrm{d}b\\ &\leq \frac{w'(u)}{L_{n-3}(u)}\int_u^\infty L_{n-3}(b)\mathrm{d}b\\ &= \frac{w'(u)}{L_{n-3}(u)}L_{n-2}(u).\end{aligned}$$

This implies

$$\left(\frac{w(u)}{L_{n-2}(u)}\right)' = \frac{L_{n-2}(u)w'(u) + w(u)L_{n-3}(u)}{L_{n-2}^2(u)} \geq 0. \qquad \square$$

3 Conditions for emptying class Ω

In the following, we present some theorems that prove that there are no positive solutions which satisfy case $\mathbf{N}_2$.

Theorem 3.1. *Assume that* (9) *holds. If*

$$\beta_0 > 1/2, \tag{20}$$

for some $\mu_0 \in (0,1)$, *then the class* Ω *is empty, where* β_0 *is defined as in Lemma* 2.5.

Proof. Assume the contrary that x belongs to Ω. From Lemma 2.5, we have that the functions $w^{(n-2)}(u)/\ell_0^{\beta_0}(u)$ and $w^{(n-2)}(u)/L_0^{1-\beta_0}(u)$ are decreasing and increasing for $u \geq u_1$, respectively. Then, we obtain

$$\ell^{1/r}(u)w^{(n-1)}(u)L_0(u) + \beta_0 w^{(n-2)}(u) \leq 0, \tag{21}$$

and

$$\ell^{1/r}(u)w^{(n-1)}(u)L_0(u) + (1-\beta_0)w^{(n-2)}(u) \geq 0. \tag{22}$$

Combining (21) and (22), we obtain

$$\begin{aligned} 0 &\leq \ell^{1/r}(u)w^{(n-1)}(u)L_0(u) + (1-\beta_0)w^{(n-2)}(u) \\ &= \ell^{1/r}(u)w^{(n-1)}(u)L_0(u) + \beta_0 w^{(n-2)}(u) + w^{(n-2)}(u) - 2\beta_0 w^{(n-2)}(u) \\ &\leq (1-2\beta_0)w^{(n-2)}(u). \end{aligned}$$

Since $w^{(n-2)}(u) > 0$, we obtain $1 - 2\beta_0 \geq 0$, which means that $\beta_0 \leq 1/2$, which is a contradiction. The proof is complete. □

Theorem 3.2. *Assume that* (9) *and* (15) *hold. If there exists a positive integer number m such that*

$$\Psi'(u) + \frac{1}{r}\frac{\mu_0^r\beta_m^{1-r}}{((n-2)!)^r(1-\beta_m)}\frac{L_0(u)}{L_0^{1-r}(\varrho(u))}(\varrho^{n-2}(u))^r Q(u)\Psi(\varrho(u)) = 0, \tag{23}$$

is oscillatory, then the class Ω is empty, where $r \leq 1$ and β_m is defined as in Lemma 2.6.

Proof. Assume the contrary that $x \in \Omega$. From Lemma 2.6, we have that $(\mathbf{B}_{3,1})$ and $(\mathbf{B}_{3,2})$ hold.

Now, we define the function

$$\Psi(u) = \ell^{1/r}(u)w^{(n-1)}(u)L_0(u) + w^{(n-2)}(u).$$

It follows from $(\mathbf{B}_{1,4})$ that $\Psi(u) > 0$ for $u \geq u_1$. From $(\mathbf{B}_{3,1})$, we obtain

$$\ell^{1/r}(u)w^{(n-1)}(u)L_0(u) \leq -\beta_m w^{(n-2)}(u).$$

Then, from the definition of $\Psi(u)$, we have

$$\begin{aligned} \Psi(u) &= \ell^{1/r}(u)w^{(n-1)}(u)L_0(u) + \beta_m w^{(n-2)}(u) - \beta_m w^{(n-2)}(u) + w^{(n-2)}(u) \\ &\leq (1-\beta_m)w^{(n-2)}(u). \end{aligned} \tag{24}$$

Using Lemma 2.4, we find that $(\mathbf{B}_{1,1}) - (\mathbf{B}_{1,5})$ hold. From $(\mathbf{B}_{1,2})$ and $(\mathbf{B}_{1,3})$, we obtain

$$\begin{aligned} \Psi'(u) &= (\ell^{1/r}(u)w^{(n-1)}(u))'L_0(u) \\ &\leq \frac{1}{r}(\ell(u)(w^{(n-1)}(u))^r)'(\ell^{1/r}(u)w^{(n-1)}(u))^{1-r}L_0(u) \\ &\leq -\frac{1}{r}Q(u)w^r(\varrho(u))(\ell^{1/r}(u)w^{(n-1)}(u))^{1-r}L_0(u) \\ &\leq -\frac{1}{r}Q(u)w^r(\varrho(u))\left(\beta_m\frac{w^{(n-2)}(u)}{L_0(u)}\right)^{1-r}L_0(u) \\ &\leq -\frac{1}{r}\beta_m^{1-r}Q(u)L_0(u)w^r(\varrho(u))\left(\frac{w^{(n-2)}(u)}{L_0(u)}\right)^{1-r} \\ &\leq -\frac{1}{r}\beta_m^{1-r}Q(u)L_0(u)\left(\frac{\mu_0}{(n-2)!}\varrho^{n-2}(u)\right)^r(w^{(n-2)}(\varrho(u)))^r\left(\frac{w^{(n-2)}(u)}{L_0(u)}\right)^{1-r}, \end{aligned}$$

from $(\mathbf{B}_{1,5})$ in Lemma 2.4, we note that $w^{(n-2)}(u)/L_0(u)$ is increasing, then

$$\frac{w^{(n-2)}(\varrho(u))}{L_0(\varrho(u))} \le \frac{w^{(n-2)}(u)}{L_0(u)}$$

and

$$\left(\frac{w^{(n-2)}(\varrho(u))}{L_0(\varrho(u))}\right)^{1-r} \le \left(\frac{w^{(n-2)}(u)}{L_0(u)}\right)^{1-r},$$

then

$$\begin{aligned}\Psi'(u) &\le -\frac{1}{r}\beta_m^{1-r}Q(u)L_0(u)\left(\frac{\mu_0}{(n-2)!}\varrho^{n-2}(u)\right)^r (w^{(n-2)}(\varrho(u)))^r\left(\frac{w^{(n-2)}(\varrho(u))}{L_0(\varrho(u))}\right)^{1-r}\\ &\le -\frac{1}{r}\frac{\beta_m^{1-r}\mu_0^r}{r((n-2)!)^r}Q(u)\frac{L_0(u)}{L_0^{1-r}(\varrho(u))}(\varrho^{n-2}(u))^r w^{(n-2)}(\varrho(u)),\end{aligned}$$

which, from (24), gives

$$\Psi'(u) + \frac{1}{r}\frac{\mu_0^r\beta_m^{1-r}}{((n-2)!)^r(1-\beta_m)}\frac{L_0(u)}{L_0^{1-r}(\varrho(u))}(\varrho^{n-2}(u))^r Q(u)\Psi(\varrho(u)) \le 0. \tag{25}$$

Hence, $\Psi(u)$ is a positive solution of the differential inequality (25). Using ([24], Corollary 1), we see that equation (23) also has a positive solution, which is a contradiction. This contradiction completes the proof of the theorem. □

Corollary 3.1. *If*

$$\liminf_{u\to\infty}\int_{\varrho(u)}^{u}\frac{L_0(b)(\varrho^{n-2}(b))^r Q(b)}{L_0^{1-r}(\varrho(b))}\mathrm{d}b > \frac{r\beta_m^{r-1}(1-\beta_m)((n-2)!)^r}{\mathrm{e}} \tag{26}$$

holds, then the class Ω *is empty.*

Proof. From Theorem 2.1.1 in [25], condition (26) guarantees that (23) is oscillatory. This completes the proof. □

Theorem 3.3. *If*

$$\limsup_{u\to\infty}\int_{u_0}^{u}\left[\left(\frac{\lambda\varrho^{n-2}(b)}{(n-2)!}\right)^r\frac{L_0^{r\beta_m}(\varrho(b))}{L_0^{-r(1-\beta_m)}(b)}Q(b) - \frac{r^{r+1}}{(1+r)^{1+r}}\frac{1}{L_0(b)\ell^{1/r}(b)}\right]\mathrm{d}b = \infty \tag{27}$$

holds for some constant $\lambda \in (0,1)$, *then the class* Ω *is empty.*

Proof. Assume the contrary that $x \in \Omega$ and assume that the case $\mathbf{N}_2$ holds. Define the function Ψ by

$$\Psi(u) = \frac{\ell(u)(w^{(n-1)}(u))^r}{(w^{(n-2)}(u))^r},\ u \ge u_1. \tag{28}$$

Then, $\Psi(u) < 0$ for $u \ge u_1$. Since $\ell(u)(w^{(n-1)}(u))^r$ is decreasing, we obtain

$$\ell^{1/r}(b)w^{(n-1)}(b) \le \ell^{1/r}(u)w^{(n-1)}(u),$$

for $b \ge u \ge u_1$. By dividing the last inequality by $\ell^{1/r}(b)$ and integrating it from u to l, we obtain

$$w^{(n-2)}(l) \le w^{(n-2)}(u) + \ell^{1/r}(u)w^{(n-1)}(u)\int_u^l \frac{1}{\ell^{1/r}(b)}\mathrm{d}b.$$

Putting $l \to \infty$, we have

$$0 \le w^{(n-2)}(u) + \ell^{1/r}(u)w^{(n-1)}(u)L_0(u),$$

which produces

$$-\frac{\ell^{1/r}(u)w^{(n-1)}(u)}{w^{(n-2)}(u)}L_0(u) \le 1.$$

Therefore, from (28), we see that

$$-\Psi(u)L_0^r(u) \le 1. \tag{29}$$

From (28), we obtain

$$\begin{aligned}\Psi'(u) &= \frac{(\ell(u)(w^{(n-1)}(u))^r)'}{(w^{(n-2)}(u))^r} - r\frac{\ell(u)(w^{(n-1)}(u))^{r+1}}{(w^{(n-2)}(u))^{r+1}}\\ &= \frac{-q(u)x^r(\varrho(u))}{(w^{(n-2)}(u))^r} - r\frac{\ell(u)(w^{(n-1)}(u))^{r+1}}{(w^{(n-2)}(u))^{r+1}}\\ &\le \frac{-Q(u)w^r(\varrho(u))}{(w^{(n-2)}(u))^r} - r\frac{\Psi^{(r+1)/r}}{\ell^{1/r}(u)}.\end{aligned}$$

From Lemma 2.1, we obtain

$$w(u) \ge \frac{\lambda}{(n-2)!}u^{n-2}w^{(n-2)}(u)$$

and

$$w(\varrho(u)) \ge \frac{\lambda}{(n-2)!}\varrho^{n-2}(u)w^{(n-2)}(\varrho(u)),$$

for every $\lambda \in (0,1)$ and for all sufficiently large u. Then,

$$\Psi'(u) \le -Q(u)\left(\frac{\lambda}{(n-2)!}\varrho^{n-2}(u)\right)^r\frac{(w^{(n-2)}(\varrho(u)))^r}{(w^{(n-2)}(u))^r} - r\frac{\Psi^{(r+1)/r}(u)}{\ell^{1/r}(u)}.$$

Since $w^{(n-2)}(u)/\ell_0^{\beta_m}(u)$ is decreasing, then

$$w^{(n-2)}(u) \le \frac{w^{(n-2)}(\varrho(u))}{L_0^{\beta_m}(\varrho(u))}L_0^{\beta_m}(u), \tag{30}$$

for $\varrho(u) \le u$, thus

$$\Psi'(u) \le -Q(u)\frac{L_0^{r\beta_m}(\varrho(u))}{L_0^{r\beta_m}(u)}\left(\frac{\lambda}{(n-2)!}\varrho^{n-2}(u)\right)^r - r\frac{\Psi^{(r+1)/r}(u)}{\ell^{1/r}(u)}.$$

Multiplying the last inequality by $L_0^r(u)$ and integrating it from u_1 to u, we obtain

$$\begin{aligned}&L_0^r(u)\Psi(u) - L_0^r(u_1)\Psi(u_1) + r\int_{u_1}^{u}\frac{L_0^{r-1}(b)}{\ell^{1/r}(b)}\Psi(b)\mathrm{d}b\\ &+ \int_{u_1}^{u}Q(b)\frac{L_0^{r\beta_m}(\varrho(b))}{L_0^{-r(1-\beta_m)}(b)}\left(\frac{\lambda\varrho^{n-2}(b)}{(n-2)!}\right)^r\mathrm{d}b + r\int_{u_1}^{u}\frac{\Psi^{(r+1)/r}(b)}{\ell^{1/r}(b)}L_0^r(b)\mathrm{d}b \le 0.\end{aligned}$$

Using inequality (7) with

$$A \coloneqq \frac{L_0^r(b)}{\ell^{1/r}(b)}, \quad B \coloneqq \frac{L_0^{r-1}(b)}{\ell^{1/r}(b)}, \quad \text{and} \quad b \coloneqq -\Psi(b),$$

we have

$$\int_{u_1}^{u}\left[\left(\frac{\lambda\varrho^{n-2}(b)}{(n-2)!}\right)^{r}\frac{L_0^{r\beta_m}(\varrho(b))}{L_0^{-r(1-\beta_m)}(b)}Q(b)-\frac{r^{r+1}}{(1+r)^{1+r}}\frac{1}{L_0(b)\ell^{1/r}(b)}\right]\mathrm{d}b \le L_0^r(u_1)\Psi(u_1)+1,$$

due to (29), which contradicts (27). This completes the proof of the theorem. □

Example 3.1. Consider the NDE

$$(u^{4r}((x(u)+p_0x(\tau_0u))''')^{r})'+q_0u^{r-1}x^{r}(\varrho_0u)=0,\quad u\ge 1,\tag{31}$$

where $0 \le p_0 < 1$, τ_0, $\varrho_0 \in (0,1)$, and $q_0 > 0$. By comparing (1) and (31), we see that $n = 4$, $\ell(u) = u^{4r}$, $q(u) = q_0u^{r-1}$, $p(u) = p_0$, $\varrho(u) = \varrho_0u$, and $\tau(u) = \tau_0u$. It is easy to find that

$$L_0(u)=\frac{1}{3u^3},\quad L_1(u)=\frac{1}{6u^2},\quad L_2(u)=\frac{1}{6u},$$

and

$$Q(u)=q_0u^{r-1}(1-p_0)^{r}.$$

For (9), we set

$$\delta=\frac{1}{r}\frac{\varrho_0^{2r}q_0}{2^r3^{r+1}}(1-p_0)^{r}.$$

Form (15), we have $\lambda = 1/\varrho_0^3$. Now, we define the sequence $\{\beta_r\}_{r=1}^{m}$ as

$$\beta_\ell=\beta_0\frac{1}{(1-\beta_{\ell-1})^{1/r}}\left(\frac{1}{\varrho_0}\right)^{3\beta_{\ell-1}},$$

with

$$\beta_0=\frac{\mu_0}{6r^{1/r}3^{1/r}}\varrho_0^2q_0^{1/r}(1-p_0).$$

Then, condition (20) reduces to

$$q_0>\frac{r3^{r+1}}{\varrho_0^{2r}(1-p_0)^{r}},\tag{32}$$

and condition (26) becomes

$$\begin{aligned}\liminf_{u\to\infty}&\int_{\varrho(u)}^{u}\frac{L_0(b)(\varrho^{n-2}(b))^{r}Q(b)}{L_0^{1-r}(\varrho(b))}\mathrm{d}b\\&=\liminf_{u\to\infty}\int_{\varrho_0u}^{u}\frac{1}{3b^3}\varrho_0^{2r}b^{2r}3^{1-r}b^{3-3r}\varrho_0^{3-3r}q_0b^{r-1}(1-p_0)^{r}\mathrm{d}b\\&=\frac{\varrho_0^{3-r}}{3^r}q_0(1-p_0)^{r}\liminf_{u\to\infty}\int_{\varrho_0u}^{u}\frac{1}{b}\mathrm{d}b\\&=\frac{\varrho_0^{3-r}}{3^r}q_0(1-p_0)^{r}\ln\frac{1}{\varrho_0},\end{aligned}$$

which leads to

$$6^{-r}\varrho_0^{3-r}q_0(1-p_0)^{r}\ln\frac{1}{\varrho_0}>\frac{r\beta_m^{r-1}(1-\beta_m)}{\mathrm{e}}.\tag{33}$$

While condition (27) becomes

$$\begin{aligned}
&\limsup_{u\to\infty}\int_{u_0}^{u}\left[\left(\frac{\lambda\varrho^{n-2}(b)}{(n-2)!}\right)^r\frac{L_0^{r\beta_m}(\varrho(b))}{L_0^{-r(1-\beta_m)}(b)}Q(b)-\frac{r^{r+1}}{(1+r)^{1+r}}\frac{1}{L_0(b)\ell^{1/r}(b)}\right]\mathrm{d}b\\
&=\limsup_{u\to\infty}\int_{u_0}^{u}\left[\frac{\lambda^r}{2^r}\varrho_0^{2r}b^{2r}\frac{1}{3^rb^{3r}}\frac{1}{\varrho_0^{3r\beta_m}}q_0b^{r-1}(1-p_0)^r-\frac{r^{r+1}}{(1+r)^{1+r}}3b^3\frac{1}{b^4}\right]\mathrm{d}b\\
&=\limsup_{u\to\infty}\int_{u_0}^{u}\left[\frac{\lambda^r}{6^r}\frac{1}{\varrho_0^{3r\beta_m-3r}}q_0(1-p_0)^r-\frac{3r^{r+1}}{(1+r)^{1+r}}\right]\frac{1}{b}\mathrm{d}b\\
&=\left[\frac{\lambda^r}{6^r}\frac{1}{\varrho_0^{3r\beta_m-2r}}q_0(1-p_0)^r-\frac{3r^{r+1}}{(1+r)^{1+r}}\right]\limsup_{u\to\infty}\ln\frac{u}{u_0}=\infty,
\end{aligned}$$

which is achieved if

$$\frac{\lambda^r}{6^r}\frac{1}{\varrho_0^{3r\beta_m-2r}}q_0(1-p_0)^r>\frac{3r^{r+1}}{(1+r)^{1+r}}. \tag{34}$$

Using Theorem 3.1, Corollary 3.1, and Theorem 3.3, we note that the class Ω is empty if either (32) or (33) or (34) holds, respectively.

4 Oscillation theorem

In the following theorems, we use the results from the previous sections to obtain new oscillation criteria for (1).

Theorem 4.1. *Let* (20) *holds. Assume that*

$$\liminf_{u\to\infty}\int_{\varrho(u)}^{u}Q(b)\frac{(\varrho^{n-1}(b))^r}{\ell(\varrho(b))}\mathrm{d}b>\frac{((n-1)!)^r}{\mathrm{e}} \tag{35}$$

and

$$\limsup_{u\to\infty}\int_{u_1}^{u}\left[Q^*(b)L_{n-2}^r(b)-\frac{r^{r+1}}{(r+1)^{r+1}}\frac{L_{n-3}(b)}{L_{n-2}(b)}\right]\mathrm{d}b=\infty \tag{36}$$

hold for some constant $\lambda\in(0,1)$. Then, every solution of (1)*is oscillatory.*

Proof. Assume that equation (1) has a non-oscillatory solution x. Without loss of generality, we may assume that x is eventually positive. It follows from equation (1) that there exist three possible cases as in Lemma 2.3.

Assume that case ($\mathbf{N}_1$) holds. From Lemma 2.1, we obtain

$$w(u)\geq\frac{\lambda}{(n-1)!}u^{n-1}w^{(n-1)}(u), \tag{37}$$

for every $\lambda\in(0,1)$ and for all sufficiently large u. From (1) and (37), we obtain

$$\begin{aligned}
(\ell(u)(w^{(n-1)}(u))^r)'&=-q(u)x^r(\varrho(u))\\
&\leq-Q(u)w^r(\varrho(u))\\
&\leq-Q(u)\frac{\lambda^r(\varrho^{n-1}(u))^r}{((n-1)!)^r\ell(\varrho(u))}(\ell(\varrho(u))(w^{(n-1)}(\varrho(u)))^r).
\end{aligned}$$

Letting $\Psi(u)\coloneqq\ell(u)(w^{(n-1)}(u))^r$, we see that

$$\Psi'(u) + Q(u)\frac{\lambda^r(\varrho^{n-1}(u))^r}{((n-1)!)^r\ell(\varrho(u))}\Psi(\varrho(u)) \le 0, \tag{38}$$

This is a contradiction because condition (35) guarantees that (38) has no positive solution according to Theorem 2.1.1 in [25].

Assume that case ($\mathbf{N}_2$) holds. The proof of the case ($\mathbf{N}_2$) is the same as that of Theorem 2.5.Assume that case ($\mathbf{N}_3$) holds. Since $\ell(u)(w^{(n-1)}(u))^r$ is decreasing, we obtain

$$\ell^{1/r}(b)w^{(n-1)}(b) \le \ell^{1/r}(u)w^{(n-1)}(u),$$

for $b \ge u \ge u_1$. By dividing the last inequality by $\ell^{1/r}(b)$ and integrating it from u to l, we obtain

$$w^{(n-2)}(l) \le w^{(n-2)}(u) + \ell^{1/r}(u)w^{(n-1)}(u)\int_u^l \frac{1}{\ell^{1/r}(b)}\mathrm{d}b.$$

Putting $l \to \infty$, we have

$$0 \le w^{(n-2)}(u) + \ell^{1/r}(u)w^{(n-1)}(u)L_0(u),$$

which leads to

$$w^{(n-2)}(u) \ge -\ell^{1/r}(u)w^{(n-1)}(u)L_0(u). \tag{39}$$

Integrating (39) from u to ∞ yields

$$\begin{aligned} -w^{(n-3)}(u) &\ge -\int_u^\infty \ell^{1/r}(b)w^{(n-1)}(b)L_0(b)\mathrm{d}b \\ &\ge -\ell^{1/r}(u)w^{(n-1)}(u)\int_u^\infty L_0(b)\mathrm{d}b \\ &\ge -\ell^{1/r}(u)w^{(n-1)}(u)L_1(u). \end{aligned}$$

Similarly, integrating the previous inequality from u to ∞, a total of $(n-4)$ times, we obtain

$$-w'(u) \ge -\ell^{1/r}(u)w^{(n-1)}(u)L_{n-3}(u). \tag{40}$$

Integrating (40) from u to ∞ provides

$$w(u) \ge -\ell^{1/r}(u)w^{(n-1)}(u)L_{n-2}(u). \tag{41}$$

Define the function Ψ by

$$\Psi(u) = \frac{\ell(u)(w^{(n-1)}(u))^r}{w^r(u)}, u \ge u_1. \tag{42}$$

Then, $\Psi(u) < 0$ for $u \le u_1$. Differentiating (42), we obtain

$$\Psi'(u) = \frac{(\ell(u)(w^{(n-1)}(u))^r)'}{w^r(u)} - r\frac{\ell(u)(w^{(n-1)}(u))^r w'(u)}{w^{r+1}(u)}.$$

It follows from (1) and (42) that

$$\Psi'(u) \le -q(u)\frac{x^r(\varrho(u))}{w^r(u)} - r\frac{\ell(u)(w^{(n-1)}(u))^r}{w^r(u)}\frac{\ell^{1/r}(u)w^{(n-1)}(u)}{w(u)}L_{n-3}(u). \tag{43}$$

Since

$$w(u) = x(u) + p(u)x(\tau(u)),$$

then

$$x(u) = w(u) - p(u)x(\tau(u)) \ge w(u) - p(u)w(\tau(u)), \tag{44}$$

from (17) in Lemma 2.7 we see that $w(u)/L_{n-2}(u)$ is increasing, consequently

$$\frac{w(u)}{L_{n-2}(u)} \geq \frac{w(\tau(u))}{L_{n-2}(\tau(u))},$$

for $\tau(u) \leq u$. From (44), we find

$$x(u) \geq \left(1 - p(u)\frac{\ell_{n-2}(\tau(u))}{\ell_{n-2}(u)}\right)w(u)$$

and

$$x(\varrho(u)) \geq \left(1 - p(\varrho(u))\frac{\ell_{n-2}(\tau(\varrho(u)))}{\ell_{n-2}(\varrho(u))}\right)w(\varrho(u)),$$

also

$$\begin{aligned} q(u)x^r(\varrho(u)) &\geq q(u)\left(1 - p(\varrho(u))\frac{\ell_{n-2}(\tau(\varrho(u)))}{\ell_{n-2}(\varrho(u))}\right)^r w^r(\varrho(u)) \\ &= Q^*(u)w^r(\varrho(u)). \end{aligned}$$

Now, we see that (43) becomes

$$\Psi'(u) \leq -Q^*(u)\frac{w^r(\varrho(u))}{w^r(u)} - r\frac{\ell(u)(w^{(n-1)}(u))^r}{w^r(u)}\frac{\ell^{1/r}(u)w^{(n-1)}(u)}{w(u)}L_{n-3}(u).$$

Multiplying the last inequality by $L^r_{n-2}(u)$ and integrating it from u_1 to u, we obtain

$$\begin{aligned} &L^r_{n-2}(u)\Psi(u) - L^r_{n-2}(u_1)\Psi(u_1) + r\int_{u_1}^{u} L^{r-1}_{n-2}(b)L_{n-3}(b)\Psi(b)\mathrm{d}b \\ &+ \int_{u_1}^{u} Q^*(b)L^r_{n-2}(b)\mathrm{d}b + r\int_{u_1}^{u} L_{n-3}(b)L^r_{n-2}(b)\Psi^{(r+1)/r}(b)\mathrm{d}b \leq 0. \end{aligned}$$

Using the inequality (7) with

$$A \coloneqq L_{n-3}(b)L^r_{n-2}(b), \quad B \coloneqq L^{r-1}_{n-2}(b)L_{n-3}(b) \quad \text{and} \quad b \coloneqq -\Psi(b),$$

we have

$$\int_{u_1}^{u}\left[Q^*(b)L^r_{n-2}(b) - \frac{r^{r+1}}{(r+1)^{r+1}}\frac{L_{n-3}(b)}{L_{n-2}(b)}\right]\mathrm{d}b \leq L^r_{n-2}(u_1)\Psi(u_1) + 1,$$

due to (41), which contradicts (36). Therefore, every solution of (1) is oscillatory. □

Theorem 4.2. *Let* (9) *and* (15) *hold. Assume that* (26), (35), *and* (36) *hold for some constant* $\lambda \in (0, 1)$. *Then, every solution of* (1) *is oscillatory.*

Theorem 4.3. *Let* (9) *and* (15) *hold. Assume that* (27), (35), *and* (36) *hold for some constant* $\lambda \in (0, 1)$, *then, every solution of* (1) *is oscillatory.*

Example 4.1. Consider the NDE

$$\left(u^{4/3}\left(\left(x(u) + \frac{1}{4}x\left(\frac{1}{2}u\right)\right)'''\right)^{1/3}\right)' + \frac{5}{u^{2/3}}x\left(\frac{1}{3}u\right) = 0, \tag{45}$$

where $p(u) = \frac{1}{4}$, $q(u) = \frac{5}{u^{2/3}}$, and $\tau(u) = \frac{1}{2}u$. It is easy to see that

$$Q(u)=\left(\frac{3}{4}\right)^{1/3}\frac{5}{u^{2/3}}.$$

For (9), we set $\delta = 1.20187$, and $\beta_0 = 1.73609\mu_0$. Therefore, $\beta_0 > 1/2$ for all $\mu_0 \in (0.3, 1)$ and then condition (20) holds. Conditions (35) and (36) reduce to

$$\liminf_{u\to\infty}\frac{1}{((n-1)!)^r}\int_{\varrho(u)}^{u}Q(b)\frac{(\varrho^{n-1}(b))^r}{\ell(\varrho(b))}\mathrm{d}b=\frac{1}{6^{1/3}}\liminf_{u\to\infty}\int_{\frac{u}{3}}^{u}\frac{3^{1/3}}{2^{2/3}}\frac{5}{b^{2/3}}\frac{3^{4/3}}{b^{4/3}}\frac{b}{3}\mathrm{d}b\simeq 4.1>\frac{1}{\mathrm{e}}$$

and

$$\begin{aligned}&\limsup_{u\to\infty}\int_{u_1}^{u}\left[Q^*(b)L_{n-2}^r(b)-\frac{r^{r+1}}{(r+1)^{r+1}}\frac{L_{n-3}(b)}{L_{n-2}(u)}\right]\mathrm{d}b\\&=\limsup_{u\to\infty}\int_{u_1}^{u}\left(\frac{5}{2^{1/3}}\frac{1}{b^{2/3}}\frac{1}{6^{1/3}b^{1/3}}-\frac{1}{2^{8/3}}\frac{1}{6b^2}6b\right)\mathrm{d}b\\&=2.03\limsup_{u\to\infty}\left(\ln\frac{u}{u_1}\right)=\infty,\end{aligned}$$

respectively. Thus, from Theorem 4.1, we conclude that every solution of (45) is oscillatory.

Example 4.2. Consider the NDE (31). Condition (35) becomes

$$\begin{aligned}\liminf_{u\to\infty}\frac{1}{((n-1)!)^r}\int_{\varrho(u)}^{u}Q(b)\frac{(\varrho^{n-1}(b))^r}{\ell(\varrho(b))}\mathrm{d}b&=\frac{1}{6^r}\liminf_{u\to\infty}\int_{\varrho_0 u}^{u}q_0b^{r-1}(1-p_0)^r\frac{\varrho^{3r}b^{3r}}{b^{4r}\varrho_0^{4r}}\mathrm{d}b\\&=\frac{1}{6^r}\frac{q_0(1-p_0)^r}{\varrho_0^r}\ln\frac{1}{\varrho_0},\end{aligned}$$

which leads to

$$\frac{1}{6^r}\frac{q_0(1-p_0)^r}{\varrho_0^r}\ln\frac{1}{\varrho_0}>\frac{1}{\mathrm{e}}.\tag{46}$$

While condition (36) is abbreviated to

$$\begin{aligned}&\limsup_{u\to\infty}\int_{u_1}^{u}\left[Q^*(b)L_{n-2}^r(b)-\frac{r^{r+1}}{(r+1)^{r+1}}\frac{L_{n-3}(b)}{L_{n-2}(b)}\right]\mathrm{d}b\\&=\limsup_{u\to\infty}\int_{u_1}^{u}\left(q_0\left(1-\frac{p_0}{\tau_0}\right)^r\frac{1}{6^r}-\frac{r^{r+1}}{(r+1)^{r+1}}\right)\frac{1}{b}\mathrm{d}b\\&=\left(q_0\left(1-\frac{p_0}{\tau_0}\right)^r\frac{1}{6^r}-\frac{r^{r+1}}{(r+1)^{r+1}}\right)\limsup_{u\to\infty}\left(\ln\frac{u}{u_1}\right)=\infty,\end{aligned}$$

which is achieved when

$$q_0\left(1-\frac{p_0}{\tau_0}\right)^r\frac{1}{6^r}>\frac{r^{r+1}}{(r+1)^{r+1}}.\tag{47}$$

From Theorem 4.1, we see that every solution of (31) is oscillatory if (32), (46), and (34) hold.

5 Conclusion

The higher-order NDEs have not received much attention compared to the delay equations or the second-order equations in general. In particular, the equations of higher-order NDEs in the noncanonical case receive almost no attention. In this study, we create new oscillation conditions for solutions of (1). The new criteria were created based on finding new monotonic properties of a class of positive solutions to (1). Moreover, we provided some examples that support our research and illustrate the significance of the results.

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