Research Article

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Generalized result on the global existence of positive solutions for a parabolic reaction-diffusion model with an $m \times m$ diffusion matrix

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Abstract: The aim of this work is to study the global existence in time of solutions for the tridiagonal system of reaction-diffusion by order m. Our techniques of proof are based on compact semigroup methods and some L^1-estimates. We show that global solutions exist. Our investigation can be applied for a wide class of nonlinear terms of reaction.

Keywords: global solution, semigroups, local solution, reaction-diffusion systems

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1 Introduction

Reaction-diffusion systems are systems involving constituents locally transformed into each other by chemical reactions and transported in space by diffusion. They arise, in many applications, in chemistry, chemical engineering and biology. They have been the subject of countless studies in the past few decades. One of the most important aspects of this broad field is proving the global existence of solutions under certain assumptions and restrictions.

The aim of this work is to study the global existence in time of solutions of an m-component reaction-diffusion system with tridiagonal symmetric Toeplitz diffusion matrix and homogeneous boundary conditions of the form

$$\begin{cases} \dfrac{\partial U}{\partial t} - A_m \Delta U = F(U) & \text{on }]0, +\infty[\times \Omega, \\ \partial_\eta U = 0 \quad \text{or} \quad U = 0 & \text{on }]0, +\infty[\times \partial\Omega, \\ U(0, x) = U_0(x) & \text{on } \Omega. \end{cases} \tag{1}$$

Our techniques of proof are based on compact semigroup methods and some L^1-estimates, and we show that global solutions exist.

We consider the m-equations of reaction-diffusion system (1), with $m \geq 2$, where Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$, ($n \geq 1$), the vectors U, F, U_0 and the matrix A_m are defined as follows:

$$\begin{cases} U = (u_1, \dots, u_m)^T = ((u_i)_{i=1}^m)^T, \\ F = (F_1, \dots, F_m)^T = ((F_i)_{i=1}^m)^T, \\ U_0 = (u_1^0, \dots, u_m^0)^T = ((u_i^0)_{i=1}^m)^T. \end{cases}$$

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$$A_m = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{m-1} \\ 0 & \cdots & 0 & b_{m-1} & a_m \end{pmatrix}. \tag{2}$$

The constants $(a_i)_{i=1}^m$, $(b_i)_{i=1}^{m-1}$ are supposed to be strictly positive and satisfy the condition

$$4b_i^2 \cos^2\left(\frac{\pi}{m+1}\right) < a_i a_{i+1}, \tag{3}$$

which reflects the parabolicity of the system and implies at the same time that the diffusion matrix A_m is positive definite. That means the eigenvalues $(\lambda_s)_{s=1}^m$, $(\lambda_1 > \lambda_2 > \ldots > \lambda_m)$ of A_m are positive.

In 2×2 diagonal case, Alikakos [1] established global existence and L^∞-bounds of solutions for positive initial data for

$$f(u, v) = -g(u, v) = -uv^\sigma, \quad \text{and} \quad 1 < \sigma < \frac{n+2}{n}.$$

Masuda [2] showed that solutions to this system exist globally for every $\sigma > 1$ and converge to a constant vector as $t \to +\infty$.

Haraux and Youkana [3] have generalized the method of Masuda to nonlinearities $f(u, v) = g(u, v) = -u\Psi(v)$ satisfying

$$\lim_{v \to +\infty} \frac{[\log(1 + \Psi(v))]}{v} = 0.$$

In the same direction, Kouachi [4] has proved the global existence of solutions for two-component reaction-diffusion systems with a general full matrix of diffusion coefficients, nonhomogeneous boundary conditions and polynomial growth conditions on the nonlinear terms and he obtained in [5] the global existence of solutions for the same system with homogeneous Neumann boundary conditions.

In [6,7], we find new developed methods based on truncation functions, fixed point theorems and compactness, etc. to prove the existence of global solutions.

In [8–12], m-equations of reaction-diffusion system (1) with the homogeneous and nonhomogeneous boundary conditions were treated with nonlinearities of exponential growth via a Lyapunov functional.

In this present study, we shall generalize the results obtained in [13–16]. In [13–16], the authors obtained a global existence of solutions for the coupled reaction-diffusion semilinear system with diagonal by order 2, and m, triangular, and full matrix of diffusion coefficients. By combining the compact semigroup methods and some L^1 estimates, we show that global solutions exist.

We prove the existence of global solutions of $m \times m$ tridiagonal symmetric Toeplitz diffusion matrix and homogeneous Neumann or Dirichlet conditions. The reaction terms are assumed to be of polynomial growth.

The initial data are assumed to be in the regions:

$$\sum\nolimits_{\mathfrak{S},\mathfrak{Z}} = \left\{ U_0 \in \mathbb{R}^m \ : \ \begin{cases} w_z^0 = \langle V_z, U_0 \rangle \leq 0 & \text{if } z \in \mathfrak{Z} \\ w_s^0 = \langle V_s, U_0 \rangle \geq 0 & \text{if } s \in \mathfrak{S} \end{cases} \right\}, \tag{4}$$

where

$$\mathfrak{S} \cap \mathfrak{Z} = \phi, \quad \mathfrak{S} \cup \mathfrak{Z} = \{1, 2, \ldots, m\}.$$

The notation $\langle ., . \rangle$ denotes the inner product in $\mathbb{R}^m$ and $V_s = (v_{s1}, \ldots, v_{sm})^T$ the eigenvector of the diffusion matrix A_m associated with the eigenvalue $(\lambda_s)_{s=1}^m$. Hence, we can see that there are 2^m regions.

2 Statement of the main result

2.1 Assumptions

The following assumptions are also made on the function Ψ defined by

$$\Psi = ((\Psi_s)_{s=1}^m)^T, \quad \Psi_s = \langle (-1)^{i_s} V_s, F\rangle, \quad i_s = 1, 2.$$

(A1) Ψ_s are continuously differentiable on $\mathbb{R}_+^m$ and Ψ_s, $s = \overline{1, m}$, are quasi-positive functions, which means that, for $s = \overline{1, m}$

$$[w_1 \geq 0, \ldots, w_{s-1} \geq 0, w_{s+1} \geq 0, \ldots, w_m \geq 0],$$

which implies

$$\Psi_s(w_1, \ldots, w_{s-1}, 0, w_{s+1}, \ldots, w_m) \geq 0. \tag{5}$$

These conditions on Ψ guarantee local existence of unique, nonnegative classical solutions on a maximal time interval $[0, T_{\max})$, see Hollis and Morgan [17].

(A2) The inequality

$$\langle S, \Psi(W)\rangle \leq C_1(1 + \langle W, 1\rangle), \tag{6}$$

such that

$$W = (w_1, \ldots, w_m), S = (d_1, d_2, \ldots, d_{m-1}, 1),$$

for all $w_s \geq 0$, $s = 1, \ldots, m$ and all constants d_s satisfy $d_s \geq \overline{d}_s$, $s = 1, \ldots, m-1$, where $C_1 \geq 0$ and $\overline{d}_s$ are positive constants sufficiently large.

(A3) W_s^0, $i = \overline{1, m}$ are nonnegative functions in $L^1(\Omega)$.

(A4) There exists a positive constant C_2 independent of w_s, $s = \overline{1, m}$ such that

$$\sum_{s=1}^m \Psi_s(w_1, \ldots, w_m) \leq C_2 \sum_{s=1}^m w_s, \quad \forall w_s \geq 0, s = \overline{1, m}. \tag{7}$$

3 Eigenvalues and eigenvectors of the diffusion matrix

The usual norms in the spaces $L^1(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ are, respectively, by

$$\|u\|_1 = \int_\Omega |u(x)| \mathrm{d}x,$$

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \quad \text{and} \quad \|u\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |u(x)|.$$

For any initial data in $C(\overline{\Omega})$ or $L^\infty(\Omega)$, local existence and uniqueness of solutions to the initial value problem (1) follow from the basic existence theory for abstract semilinear differential equations (Friedman [18], Henry [19], and Pazy [20]).

The aim in this section is to derive a three-term recurrence relation for the characteristic polynomial of the matrix A_m (of dimension $m \times m$) in terms of matrices of dimensions $(m-1) \times (m-1)$ and $(m-2) \times (m-2)$. Additionally, this recurrence relation will help in determining the eigenvectors of the matrix A_m. The eigenvalues of the matrix A_m are the values of λ that satisfy the equation: $\det(A_m - \lambda I_m) = 0$. We denote the characteristic polynomial of A_m, A_{m-1}, A_{m-2} by $\phi_m(\lambda), \phi_{m-1}(\lambda), \phi_{m-2}(\lambda)$, respectively.

Lemma 1. [21] *Let A_m be the tridiagonal matrix defined in* (2). *The eigenvalues of A_m are distinct and interlace strictly with the eigenvalues of A_{m-1} and A_{m-2}, for $m \geq 2$.*

$$\phi_0(\lambda) = 1, \ \phi_1(\lambda) = a_1 - \lambda, \ \phi_m(\lambda) = a_m\phi_{m-1}(\lambda) - b_{m-1}^2\phi_{m-2}(\lambda). \tag{8}$$

Lemma 2. [22] *Let A_m be the real symmetric tridiagonal matrix defined in* (2), *with diagonal entries positive. If*

$$4b_i^2\cos^2\left(\frac{\pi}{m+1}\right) < a_i a_{i+1}, \quad i = 1,\dots,m-1,$$

then A_m is positive definite.

Lemma 3. *The real symmetric tridiagonal matrix A_m, defined in* (2), *is positive definite if and only if its principal minors* $\det A_s$, *for $s = 1,\dots,m$, are positive.*

Proof. See Andelic and da Fonseca [22]. □

Lemma 4. *The eigenvector of the matrix A_m given in* (2) *associated with eigenvalue λ_s, for $s = 1,\dots,m$ is given by $V_s = (v_{s1},\dots,v_{sm})^T$, where $v_{s\ell}$ ($\ell = 1,\dots,m$) are given by the following expressions*:

$$\begin{cases} v_{sm} = 1, \\ v_{s(m-1)} = \dfrac{\lambda_s - a_m}{b_{m-1}}, \\ v_{s(\ell-1)} = -\dfrac{b_\ell v_{s(\ell+1)} + (a_\ell - \lambda_s)v_{s\ell}}{b_{\ell-1}}, \quad \ell = 2,\dots,m-1. \end{cases} \tag{9}$$

Proof. Recall that the diffusion matrix is positive definite, hence its eigenvalues are necessarily positive, the eigenvectors of the diffusion matrix associated with the eigenvalues λ_ℓ are defined as $V = (v_1,\dots,v_m)^T$. For an eigenpair (λ, V), the components in $A_m V = \lambda V$ are

$$\begin{cases} a_1 v_1 + b_1 v_2 = \lambda v_1, \\ b_{\ell-1}v_{\ell-1} + a_\ell v_\ell + b_\ell v_{\ell+1} = \lambda v_\ell, \quad 2 \le \ell \le m-1, \\ b_{m-1}v_{m-1} + a_m v_m = \lambda v_m. \end{cases}$$

If $v_m = 0$, then under the assumption $b_\ell \neq 0$ for all $\ell = 1,\dots,m-1$ we have that all v_ℓ are zero. We can therefore take $v_m = 1$ and $V = (v_1,\dots,v_{m-1})$ is a solution of upper triangular system

$$\begin{cases} b_{\ell-1}v_{\ell-1} + (a_\ell - \lambda)v_\ell + b_\ell v_{\ell+1} = 0, \quad 2 \le \ell \le m-2, \\ b_{m-2}v_{m-2} + (a_{m-1} - \lambda)v_{m-1} = -b_{m-1}, \\ b_{m-1}v_{m-1} = \lambda - a_m. \end{cases}$$

The solution of this system is given by

$$\begin{cases} v_{m-1} = \dfrac{\lambda - a_m}{b_{m-1}}, \\ v_{\ell-1} = -\dfrac{b_\ell v_{\ell+1} + (a_\ell - \lambda)v_\ell}{b_{\ell-1}}, \quad \ell = 2,\dots,m-1. \end{cases}$$

□

3.1 Formulation of the result

Since the initial conditions are in $\sum_{\mathfrak{S},3}$, under assumptions (A1) and (A2), the next proposition says that the classical solution of system (1) remains in $\sum_{\mathfrak{S},3}$ for all t in $[0, T_{\max})$.

Proposition 1. *Suppose that assumptions* (A1)–(A2) *are satisfied. Then, for any U_0 in $\sum_{\mathfrak{S},3}$, the classical solution U of system* (1) *on* $[0, T_{\max}) \times \Omega$ *remains in $\sum_{\mathfrak{S},3}$ for all t* in $[0, T_{\max})$.

Usually, to construct an invariant regions for systems such as (1) we make a linear change of variables u_i to obtain a new equivalent system with diagonal diffusion matrix for which standard techniques can be applied to deduce global existence [8–12].

Let $V_s = (v_{s1}, \ldots, v_{sm})^T$ be an eigenvector of the matrix A_m associated with its eigenvalue $(\lambda_s)_{s=1}^m$ where $\lambda_1 > \lambda_2 > \ldots > \lambda_m$. Multiplying the kth equation of (1) by $(-1)^{i_s} V_{sk}$, $i_s = 1, 2$ and $k = 1, \ldots, m$, and adding the resulting equations, we obtain

$$\begin{cases} \dfrac{\partial W}{\partial t} - \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)\Delta W = \Psi(W) & \text{on }]0, +\infty[\times \Omega, \\ \partial_\eta W = 0 \quad \text{or} \quad W = 0 & \text{on }]0, +\infty[\times \partial\Omega, \\ W(0, x) = W_0(x) & \text{on } \Omega, \end{cases} \tag{10}$$

where

$$\begin{cases} W = ((w_s)_{s=1}^m)^T, & w_s = \langle (-1)^{i_s} V_s, U \rangle, \\ \Psi = ((\Psi_s)_{s=1}^m)^T, & \Psi_s = \langle (-1)^{i_s} V_s, F \rangle, \\ W_0 = ((w_s^0)_{s=1}^m)^T, & w_s^0 = \langle (-1)^{i_s} V_s, U_0 \rangle. \end{cases}$$

Proposition 2. *System* (10) *admits a unique classical solution W on* $[0, T_{\max}) \times \Omega$, *where*

$$T_{\max}(\|w_1^0\|_\infty, \|w_2^0\|_\infty, \ldots, \|w_m^0\|_\infty)$$

denotes the eventual blow-up time. Furthermore, if $T_{\max} < +\infty$, then

$$\lim_{t \to T_{\max}} \sum_{s=1}^m \|w_s(t, .)\|_\infty = +\infty.$$

Therefore, if there exists a positive constant C such that

$$\sum_{s=1}^m \|w_s(t, .)\|_\infty \le C \quad \text{for all } t \in [0, T_{\max}),$$

then, $T_{\max} = +\infty$.

3.2 Compactness of the solution

Lemma 5. We consider the generator A of a semigroup of contractions $S(t)$ and a function $F : X \to X$ which is Lipschitz continuous on bounded sets. We consider the semilinear evolution equation

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} - Au = F(u(t)), \\ u(0) = u_0. \end{cases} \tag{5.1}$$

in classical form with $u \in C([0, T], D(A)) \cap C^1([0, T], X)$, and $u_0 \in X$ as well as the associated formulation

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))\mathrm{d}s.$$

$$\begin{cases} u \in C([0, T], D(A)) \cap C^1([0, T], X), \\ \dfrac{\mathrm{d}u}{\mathrm{d}t} - Au = F(u(t)), \\ u(0) = u_0, \end{cases} \tag{11}$$

admits a unique solution u verifying

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))\mathrm{d}s, \quad \forall t \in [0, T_{\max}[. \tag{12}$$

Proof. See Pazy [20]. □

In this section, we will give a compactness result of operator L defining the solution of problem (11) in the case where the initial value equals zero [$u(0) = 0$], i.e.,

$$L(F)(t) = u(t) = \int_0^t S(t - s)F(u(s))\mathrm{d}s, \quad \forall t \in [0, T]. \tag{13}$$

Theorem 1. [13–16,23] *If for all $t > 0$, the operators $S(t)$ are compact, then L is compact of $L^1([0, T], X)$ in $L^1([0, T], X)$.*

Proof. Step 1. To show that $S(\lambda)L : F \to S(\lambda)L(F)$ is compact in $L^1([0, T], X)$, it suffices to prove that the set $\{S(\lambda)L(F)(t);\ \|F\|_1 \leq 1\}$ is relatively compact in $L^1([0, T], X)$, $\forall t \in [0, T]$.

Since $S(t)$ is compact, the application $t \to S(t)$ is continuous of $]0, +\infty[$ in $\pounds(X)$; therefore,

$$\forall \varepsilon > 0, \forall \delta > 0, \exists \eta > 0, \forall 0 \leq h \leq \eta, \forall t \geq \delta,\ \|S(t + h) - S(t)\|_{\pounds(X)} \leq \varepsilon,$$

By choosing $\lambda = \delta$, we have for $0 \leq t \leq T - h$

$$\begin{aligned} &S(\lambda)u(t + h) - S(\lambda)u(t) \\ &= \int_0^{t+h} S(\lambda + t + h - s)F(u(s))\mathrm{d}s - \int_0^t S(\lambda + t - s)F(u(s))\mathrm{d}s \\ &= \int_t^{t+h} S(\lambda + t + h - s)F(u(s))\mathrm{d}s + \int_0^t (S(\lambda + t + h - s) - S(\lambda + t - s))F(u(s))\mathrm{d}s, \end{aligned}$$

where from

$$\|S(\lambda)u(t + h) - S(\lambda)u(t)\|_X \leq \int_t^{t+h} \|F(u(s))\|_X \mathrm{d}s + \varepsilon\int_0^t \|F(u(s))\|_X \mathrm{d}s$$

we define $v(t)$ by

$$v(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq T \\ 0 & \text{otherwise,} \end{cases}$$

therefore,

$$\|S(\lambda)v(t + h) - S(\lambda)v(t)\|_1 \leq (h + \varepsilon T)\|F(u(s))\|_1,$$

which implies that $\{S(\lambda)v, \|F\|_1 \leq 1\}$ is equi-integrable, then $\{S(\lambda)L(F)(t), \|F\|_1 \leq 1\}$ is relatively compact in $L^1([0, T], X)$, which means that $S(\lambda)L$ is compact.

Step 2. We prove that $S(\lambda)L$ converges to L when λ tends to 0 in $L^1([0, T], X)$. We have

$$S(\lambda)u(t) - u(t) = \int_0^t S(\lambda + t - s)F(u(s))\mathrm{d}s - \int_0^t S(t - s)F(u(s))\mathrm{d}s.$$

So, for $t \geq \delta$, we have

$$\|S(\lambda)u(t) - u(t)\| \leq \int_\delta^t \|S(\lambda + s) - S(s)\|_{\pounds(X)} \|F(u(s))\|\mathrm{d}s + 2\int_{t-\delta}^t \|F(u(s))\|\mathrm{d}s.$$

We choose $0 < \lambda < \eta$, then

$$\|S(\lambda)u(t) - u(t)\| \leq \varepsilon\int_\delta^t \|F(u(s))\|\mathrm{d}s + 2\int_{t-\delta}^t \|F(u(s))\|\mathrm{d}s$$

and for $0 \le t < \delta$, we have

$$\|S(\lambda)u(t) - u(t)\| \le 2\int_0^t \|F(u(s))\| \mathrm{d}s.$$

Since $F \in L^1(0, T, X)$, we obtain

$$\|S(\lambda)u(t) - u(t)\| \le (\varepsilon T + 2\delta)\|F(u(s))\|_1.$$

So, if $\lambda \to 0$, then $S(\lambda)u \to u$ in $L^1([0, T], X)$.

The operator L is a uniform limit with compact linear operator between two Banach spaces, then L is compact in $L^1([0, T], X)$. □

Remark 1. The semigroup $S(t)$ generated by the operator Δ is compact in $L^1(\Omega)$.

4 Approximating problem

For all $n > 0$, we define the functions w_{s0}^n, $s = \overline{1, m}$, by:

$$w_{s0}^n = \min(w_{s0}, n),$$

from which it is clear that w_{s0}^n verifies (A3), i.e.,

$$w_{s0}^n \in L^1(\Omega),\ w_{s0}^n \ge 0,\ \forall s = \overline{1, m}.$$

Let us consider the following system:

$$\begin{cases} \dfrac{\partial w_{1_n}}{\partial t} - \lambda_1 \Delta w_{1_n} = \Psi_1(w_{1_n}, \dots, w_{m_n}) & \text{in }]0, +\infty[\, \times \Omega, \\ \vdots \\ \dfrac{\partial w_{m_n}}{\partial t} - \lambda_m \Delta w_{m_n} = \Psi_m(w_{1_n}, \dots, w_{m_n}) & \text{in }]0, +\infty[\, \times \Omega, \\ \dfrac{\partial w_{s_n}}{\partial \eta} = 0, \text{ or} w_{s_n} = 0,\ s = \overline{1, m} & \text{on }]0, +\infty[\, \times \partial\Omega, \\ w_{s_n}(0, x) = w_{s0}^n(x) \ge 0,\ s = \overline{1, m} & \text{in } \Omega. \end{cases} \tag{14}$$

4.1 Existence of a local solution and its positivity of the solution of problem (14)

We convert problem (14) to an abstract first-order system in the Banach space $X = (L^1(\Omega))^m$ of the form

$$\begin{cases} \dfrac{\partial \omega_n}{\partial t} = A\omega_n + \Psi(\omega_n) & \text{in } [0, T] \times \Omega, \\ \dfrac{\partial \omega_n}{\partial \eta} = 0 \,\text{or}\, \omega_n = 0 & \text{in } [0, T] \times \partial\Omega, \\ \omega_n(0, .) = \omega_{0_n}(.) \in X & \text{in } \Omega. \end{cases} \tag{15}$$

Here $\omega_n = (w_{s_n})_{s=1}^m$, the operator A is defined as $A = \operatorname{diag}(\lambda_1\Delta, \lambda_2\Delta, \dots, \lambda_m\Delta)$, where $D(A) := \{\omega_n \in X : \Delta\omega_n \in X\}$, the function Ψ is defined as $\Psi = (\Psi_s)_{i=1}^m$, and $\omega_{0_n} = (w_{s0}^n)_{s=1}^m$.

So system (15) can be returned to the shape of system (11), thus, if $(w_{1_n}, \dots, w_{m_n})$ is a solution of (15), it then verifies the integral equation

$$w_{s_n}(t) = S_s(t)w_{s0}^n + \int_0^t S_s(t - \tau)\Psi_s(w_{1_n}(\tau), \dots, w_{m_n}(\tau))\mathrm{d}\tau, \quad s = \overline{1, m}, \tag{16}$$

where $S_s(t)$ is the semigroup generated by the operator $\lambda_s\Delta$, $s = \overline{1, m}$.

Theorem 2. *There exists $T_M > 0$ and ω_n a local solution of* (15) *for all $t \in [0, T_M]$.*

Proof. We know that $S_s(t)$ are semigroups of contraction and as Ψ is locally Lipschitz in ω_n in the space X, so we have $\exists T_M > 0$ and ω_n is a local solution of (15) on $[0, T_M]$. □

Lemma 6. *Let $(w_{1_n}, \dots, w_{m_n})$ be the solution of problem* (14) *such that*

$$w_{s0}^n(x) \geq 0, \quad \forall x \in \Omega.$$

Then,

$$w_{s_n}(t, x) \geq 0, \quad \forall (t, x) \in (0, T) \times \Omega.$$

Proof. Let $\bar{w}_{s_n} = e^{-\sigma t} w_{s_n}$, $\sigma > 0$, then

$$\frac{\partial w_{s_n}}{\partial t} = e^{\sigma t}\left(\frac{\partial \bar{w}_{s_n}}{\partial t} + \sigma \bar{w}_{s_n}\right), \quad \text{for all } 1 \leq s \leq m.$$

Consequently, by problem (15), we have $\bar{w}_{s_n}$, $1 \leq s \leq m$ is a solution of the system

$$\begin{cases} \dfrac{\partial \bar{w}_{1_n}}{\partial t} + \sigma \bar{w}_{1_n} - \lambda_1 \Delta \bar{w}_{1_n} = e^{-\sigma t} \Psi_1(\bar{w}_{1_n}, \dots, \bar{w}_{m_n}) & \text{in }]0, T[\times \Omega, \\ \qquad \vdots & \qquad \vdots \\ \dfrac{\partial \bar{w}_{m_n}}{\partial t} + \sigma \bar{w}_{m_n} - \lambda_m \Delta \bar{w}_{m_n} = e^{-\sigma t} \Psi_m(\bar{w}_{1_n}, \dots, \bar{w}_{m_n}) & \text{in }]0, T[\times \Omega, \\ \dfrac{\partial \bar{w}_{s_n}}{\partial \eta} = 0 \quad \text{or} \quad \bar{w}_{s_n} = 0, 1 \leq s \leq m & \text{on }]0, T[\times \partial\Omega, \\ \bar{w}_{s_n}(0, x) = w_{s0}^n(x) \geq 0, 1 \leq s \leq m & \text{in } \Omega. \end{cases} \tag{17}$$

Let $W_0 = (t_0, x_0)$ be the minimum of $\bar{w}_{1_n}$ on $]0, T[\times \Omega$. We will show that $\bar{w}_{1_n}(W_0) \geq 0$, which will imply that $\bar{w}_{1_n} \geq 0$ on $]0, T[\times \Omega$ and then $w_{1_n} \geq 0$ on $]0, T[\times \Omega$.

Suppose the contrary, namely, $\bar{w}_{1_n}(W_0) < 0$.

By the properties of the minimum, we can ensure that $W_0 \in]0, T] \times \Omega$ and

$$\frac{\partial \bar{w}_{1_n}}{\partial t}(W_0) = 0, \quad \Delta \bar{w}_{1_n}(W_0) \geq 0 \quad \text{if } 0 < t_0 < T,$$
$$\frac{\partial \bar{w}_{1_n}}{\partial t}(W_0) \leq 0, \quad \Delta \bar{w}_{1_n}(W_0) \geq 0 \quad \text{if } t_0 = T.$$

Hence, the first equation in (17) yields

$$\begin{aligned} \sigma \bar{w}_{1_n}(W_0) &= -\frac{\partial \bar{w}_{1_n}}{\partial t}(W_0) + \lambda_1 \Delta \bar{w}_{1_n}(W_0) + e^{-\sigma t_0} \Psi(\bar{w}_{1_n}(W_0), \dots, \bar{w}_{m_n}(W_0)) \\ &\geq e^{-\sigma t_0} \Psi(\bar{w}_{1_n}(W_0), \dots, \bar{w}_{m_n}(W_0)). \end{aligned}$$

Now, we use the structure of $\bar{w}_{1_n}(W_0)$ and hypothesis (5) to write

$$\Psi(\bar{w}_{1_n}(W_0), \dots, \bar{w}_{m_n}(W_0)) = \Psi(0, \dots, \bar{w}_{m_n}(W_0)) \geq 0.$$

This implies that $\bar{w}_{1_n}(W_0) \geq 0$, which is impossible by the hypotheses.
In the same way we find

$$w_{k_n}(t, x) \geq 0, \quad k = \overline{2, m},$$

then we obtain the positivity of $w_{s_n}(t, x) \geq 0$, $\forall (t, x) \in (0, T) \times \Omega$ and for all $s = \overline{1, m}$. □

4.2 Global existence of the solution of problem (14)

To prove the global existence of the solution of problem (14) for all nonnegative t, it is enough to find an estimate of the solution for everything $t \geq 0$.

For this, the following lemma shows the existence of an estimate of the solution of (14) in $L^1(\Omega)$.

Lemma 7. *Let* $(w_{1_n}, \dots, w_{m_n})$*, the solution of problem* (14)*, so there exists* $M(t)$*, which depends only on* t*, such that for all* $0 \leq t \leq T_M$*, we have*

$$\left\| \sum_{s=1}^{m} w_{s_n}(t) \right\|_{L^1(\Omega)} \leq M(t).$$

Proof. Let us add the equations of (14), we obtain

$$\frac{\partial}{\partial t} \sum_{s=1}^{m} w_{s_n} - \sum_{s=1}^{m} \lambda_s \Delta w_{s_n} = \sum_{s=1}^{m} \Psi_s(w_{1_n}, \dots, w_{m_n}).$$

By (7) we get

$$\frac{\partial}{\partial t} \sum_{s=1}^{m} w_{s_n} - \sum_{s=1}^{m} \lambda_s \Delta w_{s_n} \leq C_2 \sum_{s=1}^{m} w_{s_n}.$$

Let us integrate on Ω and apply the formula of Green, we find that

$$\frac{\partial}{\partial t} \int_\Omega \sum_{s=1}^{m} w_{s_n} \mathrm{d}x \leq C_2 \int_\Omega \sum_{s=1}^{m} w_{s_n} \mathrm{d}x,$$

so,

$$\frac{\frac{\partial}{\partial t} \int_\Omega \sum_{s=1}^{m} w_{s_n} \mathrm{d}x}{\int_\Omega \sum_{s=1}^{m} w_{s_n} \mathrm{d}x} \leq C_2,$$

integrating on $[0, t]$, we find

$$\ln \int_\Omega \sum_{s=1}^{m} w_{s_n} \mathrm{d}x \Big|_0^t \leq C_2 t,$$

thus

$$\ln \frac{\int_\Omega \sum_{s=1}^{m} w_{s_n}(t) \mathrm{d}x}{\int_\Omega \sum_{s=1}^{m} w_{s0}^n \mathrm{d}x} \leq C_2 t,$$

which implies that

$$\frac{\int_\Omega \sum_{s=1}^{m} w_{s_n}(t) \mathrm{d}x}{\int_\Omega \sum_{s=1}^{m} w_{s0}^n \mathrm{d}x} \leq \exp(C_2 t)$$

and for $w_{s0}^n \leq w_{s0}$, we have

$$\int_\Omega \sum_{s=1}^{m} w_{s_n}(t) \mathrm{d}x \leq \exp(C_2 t) \int_\Omega \sum_{s=1}^{m} w_{s0} \mathrm{d}x.$$

Let us put

$$M(t) = \exp(C_2 t) \left\| \sum_{s=1}^{m} w_{s0} \right\|_{L^1(\Omega)}.$$

Since w_{s_n} are positives, we obtain

$$\left\| \sum_{s=1}^{m} w_{s_n}(t) \right\|_{L^1(\Omega)} \leq M(t), \quad 0 \leq t \leq T_M. \qquad \square$$

We can conclude from this estimate that the solution $(w_{1_n}, \ldots, w_{m_n})$ given by the theory 2 is a global solution.

Now, we give the following lemma, which shows the existence of estimate of the solution $(w_{1_n}, \ldots, w_{m_n})$ of problem (14) in $L^1(Q)$.

Lemma 8. *For any solution* $(w_{1_n}, \ldots, w_{m_n})$ *of* (14), *there is a constant* $K(t)$, *which depends only on t, such that*

$$\left\| \sum_{s=1}^{m} w_{s_n}(t) \right\|_{L^1(Q)} \leq K(t) \left\| \sum_{s=1}^{m} w_{s0} \right\|_{L^1(\Omega)}.$$

Proof. To prove this lemma, we use the following results given by Bonafede and Schmitt [24].

So, we introduce $\theta \in C_0^\infty(Q)$, $\theta \geq 0$ and $\Phi \in C^{1,2}(Q)$ a nonnegative solution of the following system:

$$\begin{cases} -\Phi_t - \mathrm{d}\Delta\Phi = \theta & \text{on } Q, \\ \dfrac{\partial \Phi}{\partial \eta} = 0 \quad \text{or} \quad \Phi(t,.) = 0 & \text{on } [0, T[\times \partial\Omega, \\ \Phi(T,.) = 0 & \text{on } \Omega. \end{cases} \tag{18}$$

Furthermore, for all $q \in]1, +\infty[$, there exists a nonnegative constant c independent of θ, such that

$$\|\Phi\|_{L^{q'}(Q)} \leq c\, \|\theta\|_{L^q(Q)}.$$

According to Bonafede and Schmitt [24]:

$$\int_Q S_s(t) w_{s0}^n(x) \left(-\frac{\partial \Phi}{\partial t} - \mathrm{d}\Delta\Phi \right) \mathrm{d}t\mathrm{d}x = \int_\Omega w_{s0}^n(x) \Phi(0, x) \mathrm{d}x,$$

and that

$$\iint_Q \left(\int_0^t S_s(t-\tau) \Psi_s(w_{1_n}, \ldots, w_{m_n}) \mathrm{d}\tau \right) \left(-\frac{\partial \Phi}{\partial t} - \mathrm{d}\Delta\Phi \right) \mathrm{d}t\mathrm{d}x = \int_Q \Psi_s(w_{1_n}, \ldots, w_{m_n}) \Phi(\tau, x) \mathrm{d}\tau\mathrm{d}x,$$

where

$$\int_Q (S_s(t) w_{s0}^n(x)) \theta \mathrm{d}t\mathrm{d}x = \int_\Omega w_{s0}^n(x) \Phi(0, x) \mathrm{d}x \tag{19}$$

and

$$\iint_Q \left(\int_0^t S_s(t-\tau) \Psi_s(w_{1_n}, \ldots, w_{m_n}) \mathrm{d}\tau \right) \theta \mathrm{d}t\mathrm{d}x = \int_Q \Psi_s(w_{1_n}, \ldots, w_{m_n}) \Phi(\tau, x) \mathrm{d}\tau\mathrm{d}x. \tag{20}$$

Let us multiply equation (16) by θ, and let us integrate on Q, by using (19) and (20), we obtain

$$\begin{aligned} \int_Q w_{s_n}(t) \theta \mathrm{d}t\mathrm{d}x &= \int_Q S_s(t) w_{s0}^n(x) \theta \mathrm{d}t\mathrm{d}x + \iint_Q \left(\int_0^t S_s(t-\tau) \Psi_s(w_{1_n}, \ldots, w_{m_n}) \mathrm{d}\tau \right) \theta \mathrm{d}t\mathrm{d}x \\ &= \int_\Omega w_{s0}^n(x) \Phi(0, x) \mathrm{d}x + \int_Q \Psi_s(w_{1_n}, \ldots, w_{m_n}) \Phi(\tau, x) \mathrm{d}\tau\mathrm{d}x, \; s = \overline{1, m}, \end{aligned}$$

therefore,

$$\int_Q \sum_{s=1}^m w_{s_n}(t)\theta \mathrm{d}t\mathrm{d}x = \int_\Omega \sum_{s=1}^m w_{s0}^n(x)\Phi(0,x)\mathrm{d}x + \int_Q \sum_{s=1}^m \Psi_s(w_{1_n},\dots,w_{m_n})\Phi(\tau,x)\mathrm{d}\tau\mathrm{d}x,$$

according to (7) and as $w_{s0}^n \le w_{s0}$ we have

$$\int_Q \sum_{s=1}^m w_{s_n}(t)\theta \mathrm{d}t\mathrm{d}x \le \int_\Omega \sum_{s=1}^m w_{s0}(x)\Phi(0,x)\mathrm{d}x + \int_Q C_2 \sum_{s=1}^m w_{s_n}(t)\Phi(\tau,x)\mathrm{d}\tau\mathrm{d}x.$$

Using the Holder inequality, we deduce

$$\begin{aligned}\int_Q \sum_{s=1}^m w_{s_n}(t)\theta \mathrm{d}t\mathrm{d}x &\le \left\|\sum_{s=1}^m w_{s0}\right\|_{L^1(\Omega)} . \|\Phi(0,.)\|_{L^\infty(Q)} + C_2 \left\|\sum_{s=1}^m w_{s_n}(t)\right\|_{L^1(Q)} . \|\Phi\|_{L^\infty(Q)}\\
&\le \left(\left\|\sum_{s=1}^m w_{s0}\right\|_{L^1(\Omega)} + C_2\left\|\sum_{s=1}^m w_{s_n}(t)\right\|_{L^1(Q)}\right) . \|\Phi\|_{L^\infty(Q)}\\
&\le \max(1,C_2)\left(\left\|\sum_{s=1}^m w_{s0}\right\|_{L^1(\Omega)} + \left\|\sum_{s=1}^m w_{s_n}(t)\right\|_{L^1(Q)}\right) . \|\Phi\|_{L^\infty(Q)}\\
&\le k_1(t)\left(\left\|\sum_{s=1}^m w_{s0}\right\|_{L^1(\Omega)} + \left\|\sum_{s=1}^m w_{s_n}(t)\right\|_{L^1(Q)}\right) . \|\theta\|_{L^\infty(Q)},\end{aligned}$$

where $k_1(t) \ge \max(c, cC_2)$.

Since θ is arbitrary in $C_0^\infty(Q)$, this implies

$$\left\|\sum_{s=1}^m w_{s_n}(t)\right\|_{L^1(Q)} \le k_1(t)\left(\left\|\sum_{s=1}^m w_{s0}\right\|_{L^1(\Omega)} + \left\|\sum_{s=1}^m w_{s_n}(t)\right\|_{L^1(Q)}\right).$$

Taking $k(t) = \frac{k_1(t)}{1-k_1(t)}$ we find that

$$\left\|\sum_{s=1}^m w_{s_n}(t)\right\|_{L^1(Q)} \le k(t)\left\|\sum_{s=1}^m w_{s0}\right\|_{L^1(\Omega)}.$$

□

Now, we will prove the main result of this work: The existence of global solutions for the system (1) is equivalent to the existence of w_s, $s = \overline{1,m}$, illustrated by the following main theorem:

Theorem 3. *Suppose that the hypotheses* (A1)–(A4) *are satisfied, then system* (1) *has a nonnegative solution* w_s, $s = \overline{1,m}$, *in the sense*:

$$\begin{cases} w_s \in C([0,+\infty[, L^1(\Omega)), \quad s = \overline{1,m},\\ \Psi_s(w_1,\dots,w_m) \in L^1(Q) \quad \text{where } Q = (0,T)\times\Omega \text{ for all } T > 0,\\ w_s(t) = S_s(t)w_{s0} + \int_0^t S_s(t-\tau)\Psi_s(w_1(\tau),\dots,w_m(\tau))\mathrm{d}\tau, \quad s = \overline{1,m}, \forall t \in [0,T[, \end{cases} \tag{21}$$

where $S_s(t)$ are the semigroups of contractions in $L^1(\Omega)$ generated by $\lambda_s\Delta$, $s = \overline{1,m}$.

Proof. Let us define the application L by

$$L : (\omega_0, h) \to S(t)\omega_0 + \int_0^t S(t-\tau)h(\tau)\mathrm{d}\tau,$$

where $S(t)$ is the semigroup of contraction generated by the operator $\mathrm{d}\Delta$, according to the previous result of Theorem 1 and as $S(t)$ is compact, then the application L is adding two compact applications in $L^1(Q)$.

This is how L is compact from $(L^1(Q_T))^m$ in $L^1(Q_T)$.

Therefore, there is a subsequence $(w^j_{1_n}, \ldots, w^j_{m_n})$ of $(w_{1_n}, \ldots, w_{m_n})$ and $(w_1, \ldots, w_m)$ of $(L^1(Q))^m$, such that:

$$(w^j_{1_n}, \ldots, w^j_{m_n}) \text{ converges toward } (w_1, \ldots, w_m).$$

Let us now show that $(w^j_{1_n}, \ldots, w^j_{m_n})$ is a solution of (16). We have

$$w^j_{s_n}(t,x) = S_s(t)w^j_{s0} + \int_0^t S_s(t-\tau)\Psi_s(w^j_{1_n}(\tau), \ldots, w^j_{m_n}(\tau))\mathrm{d}\tau,\ s = \overline{1,m}, \tag{22}$$

so, it is enough to show that $(w_1, \ldots, w_m)$ verifies (21), and it is clear that if $j \to +\infty$, we have the following limits:

$$\Psi_s(w^j_{1_n}, \ldots, w^j_{m_n}) \to \Psi_s(w_1, \ldots, w_m), \quad \text{a.e, } s = \overline{1,m} \tag{23}$$

and

$$w^j_{s0} \to w_{s0}, \quad s = \overline{1,m}.$$

Thus, to show that $(w_1, \ldots, w_m)$ verifies (21), it remains to show that

$$\Psi_s(w^j_{1_n}, \ldots, w^j_{m_n}) \to \Psi_s(w_1, \ldots, w_m), \quad s = \overline{1,m}, \text{ in } L^1(Q) \text{ when } j \to +\infty.$$

We integrate the equations of (14) on Q by taking into account that

$$-\lambda_s \int_Q \Delta w^j_{s_n}\mathrm{d}t\mathrm{d}x = 0,\ s = \overline{1,m}.$$

We have

$$\int_\Omega w^j_{s_n}(t)\mathrm{d}x - \int_\Omega w^j_{s0}\mathrm{d}x = \int_Q \Psi_s(w^j_{1_n}, \ldots, w^j_{m_n})\mathrm{d}t\mathrm{d}x,$$

where

$$-\int_Q \Psi_s(w^j_{1_n}, \ldots, w^j_{m_n})\mathrm{d}t\mathrm{d}x \le \int_\Omega w_{s0}\mathrm{d}x, \quad s = \overline{1,m}. \tag{24}$$

Let us put

$$\Upsilon_{s_n} = C_2 \sum_{s=1}^m w^j_{s_n} - \Psi_s(w^j_{1_n}, \ldots, w^j_{m_n}), \quad s = \overline{1,m},$$

it is clear that Υ_{s_n} is positive according to (7), from (24) we obtain

$$\int_Q \Upsilon_{s_n}\mathrm{d}t\mathrm{d}x \le C_2 \int_Q \sum_{s=1}^m w^j_{s_n}\mathrm{d}t\mathrm{d}x + \int_\Omega w_{s0}\mathrm{d}x.$$

Lemma 8 gives us

$$\int_Q \Upsilon_{s_n}\mathrm{d}t\mathrm{d}x < +\infty,$$

which implies

$$\int_Q |\Psi_s(w^j_{1_n}, \ldots, w^j_{m_n})|\mathrm{d}t\mathrm{d}x \le C_2 \int_Q \sum_{s=1}^m w^j_{s_n}\mathrm{d}t\mathrm{d}x + \int_Q \Upsilon_{s_n}\mathrm{d}t\mathrm{d}x < +\infty.$$

Let

$$\varphi_{s_n} = \Upsilon_{s_n} + C_2 \sum_{s=1}^m w^j_{s_n}, \quad s = \overline{1,m},$$

φ_{s_n} are in $L^1(Q)$ and positives and furthermore

$$|\Psi_s(w^j_{1_n}, \ldots, w^j_{m_n})| \leq \varphi_{s_n} \quad \text{a.e, } s = \overline{1, m}.$$

Let us combine this result with (23) and we apply the theorem of convergence dominated by Lebesgue, we obtain

$$\Psi_s(w^j_{1_n}, \ldots, w^j_{m_n}) \to \Psi_s(w_1, \ldots, w_m) \quad \text{in } L^1(Q).$$

By passing in the limit $j \to +\infty$ of (22) in $L^1(Q)$, we find

$$w_s(t) = S_s(t)w_{s0} + \int_0^t S_s(t-\tau)\Psi_s(w_1(\tau), \ldots, w_m(\tau))\,\mathrm{d}\tau, \quad s = \overline{1, m}.$$

Then, $(w_1, \ldots, w_m)$ verifies (21), consequently $(w_1, \ldots, w_m)$ is the solution of (1). □

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