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Research Article

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A study on a type of degenerate poly-Dedekind sums

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Abstract: Dedekind sums and their generalizations are defined in terms of Bernoulli functions and their generalizations. As a new generalization of the Dedekind sums, the degenerate poly-Dedekind sums, which are obtained from the Dedekind sums by replacing Bernoulli functions by degenerate poly-Bernoulli functions of arbitrary indices are introduced in this article and are shown to satisfy a reciprocity relation.

Keywords: degenerate poly-Dedekind sums, degenerate polylogarithm functions, degenerate poly-Bernoulli polynomials

MSC 2020: 11F20, 11B68, 11B83

1 Introduction and preliminaries

As is well known, the Euler polynomials and Bernoulli polynomials are given by

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (\text{see [1,2]}), \tag{1}$$

when $x = 0$, $E_n = E_n(0)$ are the Euler numbers; $B_n = B_n(0)$ are the Bernoulli numbers.

Apostol considered generalized Dedekind sums by replacing the first Bernoulli function appearing in Dedekind sums by any Bernoulli functions and derived a reciprocity relation for them

$$S_p(h,m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m}\overline{B}_p\left(\frac{h\mu}{m}\right) \quad (\text{see [3–9]}), \tag{2}$$

where $\overline{B}_p(x) = B_p(x-[x])$ are the Bernoulli functions, where $[x]$ denotes the greatest integer function not exceeding x.

As one generalization of the Apostol's generalized Dedekind sums, the poly-Dedekind sums associated with the type 2 poly-Bernoulli functions were introduced (see [10,11]). The poly-Dedekind sums associated with the poly-Bernoulli functions were recently introduced,

$$S_p^{(k)}(h,m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m}\overline{B}_p^{(k)}\left(\frac{h\mu}{m}\right) \quad (\text{see [12]}), \tag{3}$$

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when $k = 1$, we have $B_p^{(1)}(x) = B_p(x)$, where $\overline{B}_p^{(k)}(x) = B_p^{(k)}(x - [x])$ are the poly-Bernoulli functions, $B_p^{(k)}(x)$ are the poly-Bernoulli polynomials given by

$$\frac{Li_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{p=0}^{\infty} B_p^{(k)}(x)\frac{t^p}{p!} \quad (\text{see [12]}), \tag{4}$$

when $x = 0$, $B_p^{(k)} = B_p^{(k)}(0)$ are the poly-Bernoulli numbers.

Kim first introduced the Dedekind-type DC sums

$$T_p(h, m) = 2\sum_{\mu=0}^{m-1}(-1)^{\mu}\frac{\mu}{m}\overline{E}_p\left(\frac{h\mu}{m}\right), \quad (h, m \in \mathbb{N}) \quad (\text{see [13,14]}), \tag{5}$$

where $\overline{E}_p(x) = E_p(x - [x])$ are the Euler functions (see [3,13,14,19]).

Simsek (see [14]) found trigonometric representations of the Dedekind-type DC sums and their relations to Clausen functions, polylogarithm function, Hurwitz zeta function, generalized Lambert series (G-series), and Hardy-Berndt sums.

The poly-Dedekind-type DC sums, which are obtained by replacing the Euler functions appearing in Dedekind-type DC sums by any poly-Euler functions of arbitrary indices, were given by

$$T_p^{(k)}(h, m) = 2\sum_{\mu=1}^{m-1}(-1)^{\mu}\frac{\mu}{m}\overline{E}_p^{(k)}\left(\frac{h\mu}{m}\right) \quad (\text{see [15]}), \tag{6}$$

where $h, m, p \in \mathbb{N}$, $\overline{E}_p^{(k)}(x)$ are the poly-Euler functions, $\overline{E}_p^{(k)}(x) = E_p^{(k)}(x - [x])$, $E_p^{(k)}(x)$ are the poly-Euler polynomials given by

$$\frac{Li_k(1 - e^{-2t})}{t(e^t + 1)}e^{xt} = \sum_{p=0}^{\infty} E_p^{(k)}(x)\frac{t^p}{p!} \quad (\text{see [1,2,12]}), \tag{7}$$

when $x = 0$, $E_p^{(k)} = E_p^{(k)}(0)$ are the poly-Euler numbers.

In this article, as another generalization, we consider degenerate poly-Dedekind sums, which are obtained by replacing the first Bernoulli function appearing in Dedekind sums by degenerate poly-Bernoulli functions of arbitrary indices. We derive a reciprocity relation for them.

For the rest of this section, we recall some necessary facts that are needed throughout this article. For any $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad e_\lambda(t) = e_\lambda^1(t) \quad (\text{see [1,11,16–18]}), \tag{8}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x - \lambda)\cdots(x - (n - 1)\lambda)$, $(n \geq 1)$.

In [19], the degenerate Bernoulli polynomials are defined by

$$\frac{t}{e_\lambda(t) - 1}e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}, \tag{9}$$

when $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are the degenerate Bernoulli numbers.

We observe that

$$\begin{aligned}\sum_{l=0}^{n-1} e_\lambda^l(t) &= \frac{t}{t(e_\lambda(t) - 1)}(e_\lambda^n(t) - 1)\\ &= \frac{1}{t}\sum_{j=0}^{\infty}(B_{j,\lambda}(n) - B_{j,\lambda})\frac{t^j}{j!}\\ &= \sum_{j=0}^{\infty}\frac{B_{j+1,\lambda}(n) - B_{j+1,\lambda}}{j + 1}\frac{t^j}{j!}.\end{aligned} \tag{10}$$

On the other hand,

$$\sum_{l=0}^{n-1} e_\lambda^l(t) = \sum_{l=0}^{n-1} e_{\lambda/l}(lt) = \sum_{j=0}^{\infty}\sum_{l=0}^{n-1} l^j (1)_{j,\lambda/l}\frac{t^j}{j!}. \tag{11}$$

By (10) and (11), we have

$$\sum_{l=0}^{n-1} l^j(1)_{j,\lambda/l} = \frac{B_{j+1,\lambda}(n) - B_{j+1,\lambda}}{j+1}. \tag{12}$$

The type 2 degenerate Stirling numbers are defined by (see [20,21])

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad (n \geq 0). \tag{13}$$

The degenerate polylogarithmic function of index k is defined by Kim and Kim to be (see [20])

$$\mathrm{Li}_{k,\lambda}(x) = \sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k}x^n, \quad (k \in \mathbb{Z}), \quad |x|<1. \tag{14}$$

When $k = 1$,

$$\mathrm{Li}_{1,\lambda}(x) = -\log_\lambda(1-x) = \sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{n!}x^n, \tag{15}$$

where $\log_\lambda(x)$ is the inverse to $e_\lambda(x)$.

In [16], the degenerate poly-Bernoulli polynomials of index k are defined by

$$\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{16}$$

Note that $B_{n,\lambda}^{(1)}(x) = \beta_{n,\lambda}(x)$, when $x = 0$, $B_{n,\lambda}^{(k)} = B_{n,\lambda}^{(k)}(0)$ are the degenerate poly-Bernoulli numbers.

From (16), we note that

$$B_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda}^{(k)}(x)_{n-l,\lambda} = \sum_{l=0}^{n}\binom{n}{l}B_{n-l,\lambda}^{(k)}(x)_{l,\lambda}, \quad (n \geq 0). \tag{17}$$

2 Reciprocity relations for the degenerate poly-Dedekind sums

By replacing poly-Bernoulli functions by degenerate poly-Bernoulli functions of arbitrary indices from the poly-Dedekind sums, the degenerate poly-Dedekind sums are given in the following,

$$S_{p,\lambda}^{(k)}(h,m) = \sum_{\mu=1}^{m-1}\frac{\mu}{m}\overline{B}_{p,\lambda}^{(k)}\left(\frac{h\mu}{m}\right), \tag{18}$$

where $h, m, p \in \mathbb{N}$, $\overline{B}_{p,\lambda}^{(k)}(x)$ are the degenerate poly-Bernoulli functions, $\overline{B}_{p,\lambda}^{(k)}(x) = B_{p,\lambda}^{(k)}(x-[x])$.

Note that

$$S_{p,\lambda}^{(1)}(h,m) = \sum_{\mu=1}^{m-1}\frac{\mu}{m}\overline{B}_{p,\lambda}^{(1)}\left(\frac{h\mu}{m}\right) = \sum_{\mu=1}^{m-1}\frac{\mu}{m}\overline{\beta}_{p,\lambda}\left(\frac{h\mu}{m}\right). \tag{19}$$

From (16), we note that

$$\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1} = \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}\frac{t^n}{n!}. \tag{20}$$

So we have

$$\begin{aligned}\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t)) &= \left(\sum_{l=0}^{\infty} B_{l,\lambda}^{(k)}\frac{t^l}{l!}\right)(e_\lambda(t)-1)\\ &= \left(\sum_{l=0}^{\infty} B_{l,\lambda}^{(k)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}-1\right)\\ &= \sum_{n=1}^{\infty}(B_{n,\lambda}^{(k)}(1)-B_{n,\lambda}^{(k)})\frac{t^n}{n!}.\end{aligned}\tag{21}$$

On the other hand, we also have

$$\begin{aligned}\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t)) &= \sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{(m-1)!m^k}(1-e_\lambda(-t))^m\\ &= \sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^{k-1}}\frac{(-1)^m}{m!}(e_\lambda(-t)-1)^m\\ &= \sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}(-1)^m}{m^{k-1}}\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)(-1)^n\frac{t^n}{n!}\\ &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{n-1}}{m^{k-1}}S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.\end{aligned}\tag{22}$$

Therefore, by (21) and (22), we obtain the following theorem.

Theorem 1. *For $n\in\mathbb{N}$, we have*

$$B_{n,\lambda}^{(k)}(1)-B_{n,\lambda}^{(k)} = \sum_{m=1}^{n}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{n-1}}{m^{k-1}}S_{2,\lambda}(n,m).$$

Note that, for $k=1$, we have the following corollary.

Corollary 1. *For $n\in\mathbb{N}$, we have*

$$\sum_{m=1}^{n}\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{n-1}S_{2,\lambda}(n,m) = \delta_{n,1},$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For $d\in\mathbb{N}$, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^x(t)\\ &= \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{(e_{\lambda/d}(\mathrm{d}t)-1)}\sum_{i=0}^{d-1}e_\lambda^{i+x}(t)\\ &= \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{\mathrm{d}t}\sum_{i=0}^{d-1}e_{\lambda/d}^{\frac{x+i}{d}}(\mathrm{d}t)\frac{\mathrm{d}t}{e_{\lambda/d}(\mathrm{d}t)-1}\\ &= \sum_{j=0}^{\infty}d^{j-1}\sum_{i=0}^{d-1}\beta_{j,\lambda/d}\left(\frac{x+i}{d}\right)\frac{t^j}{j!}\frac{1}{t}\sum_{l=1}^{\infty}\frac{(-\lambda)^{l-1}(1)_{l,1/\lambda}}{l^k}\frac{1}{(l-1)!}(1-e_\lambda(-t))^l\\ &= \sum_{j=0}^{\infty}d^{j-1}\sum_{i=0}^{d-1}\beta_{j,\lambda/d}\left(\frac{x+i}{d}\right)\frac{t^j}{j!}\frac{1}{t}\sum_{l=1}^{\infty}\frac{(-1)\lambda^{l-1}(1)_{l,1/\lambda}}{l^{k-1}}\frac{1}{l!}(e_\lambda(-t)-1)^l\\ &= \sum_{j=0}^{\infty}d^{j-1}\sum_{i=0}^{d-1}\beta_{j,\lambda/d}\left(\frac{x+i}{d}\right)\frac{t^j}{j!}\frac{1}{t}\sum_{m=1}^{\infty}\sum_{l=1}^{m}\frac{(-1)^{m-1}\lambda^{l-1}(1)_{l,1/\lambda}S_{2,\lambda}(m,l)}{l^{k-1}}\frac{t^m}{m!}\\ &= \sum_{j=0}^{\infty}d^{j-1}\sum_{i=0}^{d-1}\beta_{j,\lambda/d}\left(\frac{x+i}{d}\right)\frac{t^j}{j!}\sum_{m=0}^{\infty}\sum_{l=1}^{m+1}\frac{(-1)^m\lambda^{l-1}(1)_{l,1/\lambda}S_{2,\lambda}(m+1,l)}{l^{k-1}(m+1)}\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{i=0}^{d-1}\sum_{l=1}^{n-j+1}\binom{n}{j}d^{j-1}\beta_{j,\lambda/d}\left(\frac{x+i}{d}\right)\frac{(-1)^{n-j}\lambda^{l-1}(1)_{l,1/\lambda}S_{2,\lambda}(n-j+1,l)}{l^{k-1}(n-j+1)}\right)\frac{t^n}{n!}.\end{aligned}\tag{23}$$

Therefore, by (23), we obtain the following theorem.

Theorem 2. *For $k \in \mathbb{Z}$, $d \in \mathbb{N}$, $n \geq 0$, we have*

$$B_{n,\lambda}^{(k)}(x) = \sum_{j=0}^{n}\sum_{i=0}^{d-1}\sum_{l=1}^{n-j+1}\binom{n}{j}d^{j-1}\beta_{j,\lambda/d}\left(\frac{x+i}{d}\right)\frac{(-1)^{n-j}\lambda^{l-1}(1)_{l,1/\lambda}S_{2,\lambda}(n-j+1,l)}{l^{k-1}(n-j+1)}.$$

For $m, h \in \mathbb{N}$, by (18) and Theorem 2, we obtain

$$\begin{aligned}
hm^{p}S_{p,\lambda}^{(k)}(h,m) + mh^{p}S_{p,\lambda}^{(k)}(m,h) &= hm^{p}\sum_{\mu=0}^{m-1}\frac{\mu}{m}\overline{B}_{p,\lambda}^{(k)}\left(\frac{h\mu}{m}\right) + mh^{p}\sum_{\nu=0}^{h-1}\frac{\nu}{h}\overline{B}_{p,\lambda}^{(k)}\left(\frac{m\nu}{h}\right)\\
&= hm^{p}\sum_{\mu=0}^{m-1}\frac{\mu}{m}\sum_{j=0}^{p}h^{j-1}\binom{p}{j}\sum_{\nu=0}^{h-1}\sum_{l=1}^{p-j+1}\beta_{j,\lambda/h}\left(\frac{\mu}{m}+\frac{\nu}{h}\right)\\
&\quad\frac{(-1)^{p-j}\lambda^{l-1}(1)_{l,1/\lambda}S_{2,\lambda}(p-j+1,l)}{l^{k-1}(p-j+1)}\\
&\quad + mh^{p}\sum_{\nu=0}^{h-1}\frac{\nu}{h}\sum_{j=0}^{p}m^{j-1}\binom{p}{j}\sum_{\mu=0}^{m-1}\sum_{l=1}^{p-j+1}\beta_{j,\lambda/m}\left(\frac{\mu}{m}+\frac{\nu}{h}\right)\\
&\quad\frac{(-1)^{p-j}\lambda^{l-1}(1)_{l,1/\lambda}S_{2,\lambda}(p-j+1,l)}{l^{k-1}(p-j+1)}\\
&= \sum_{\mu=0}^{m-1}\frac{\mu}{m}\sum_{j=0}^{p}m^{p-j}(mh)^{j}\binom{p}{j}\sum_{\nu=0}^{h-1}\sum_{l=1}^{p-j+1}\beta_{j,\lambda/h}\left(\frac{\mu}{m}+\frac{\nu}{h}\right)\\
&\quad\frac{(-1)^{p-j}\lambda^{l-1}(1)_{l,1/\lambda}S_{2,\lambda}(p-j+1,l)}{l^{k-1}(p-j+1)}\\
&\quad + \sum_{\nu=0}^{h-1}\frac{\nu}{h}\sum_{j=0}^{p}h^{p-j}(mh)^{j}\binom{p}{j}\sum_{\mu=0}^{m-1}\sum_{l=1}^{p-j+1}\beta_{j,\lambda/m}\left(\frac{\mu}{m}+\frac{\nu}{h}\right)\\
&\quad\frac{(-1)^{p-j}\lambda^{l-1}(1)_{l,1/\lambda}S_{2,\lambda}(p-j+1,l)}{l^{k-1}(p-j+1)}\\
&= \sum_{\mu=0}^{m-1}\sum_{j=0}^{p}\sum_{\nu=0}^{h-1}\sum_{l=1}^{p-j+1}(\mu h)(mh)^{-1}m^{p-j}(mh)^{j}\binom{p}{j}\beta_{j,\lambda/h}\left(\frac{\mu}{m}+\frac{\nu}{h}\right)\\
&\quad\frac{(-1)^{p-j}\lambda^{l-1}(1)_{l,1/\lambda}S_{2,\lambda}(p-j+1,l)}{l^{k-1}(p-j+1)}\\
&\quad + \sum_{\mu=0}^{m-1}\sum_{j=0}^{p}\sum_{\nu=0}^{h-1}\sum_{l=1}^{p-j+1}(m\nu)(mh)^{-1}h^{p-j}(mh)^{j}\binom{p}{j}\beta_{j,\lambda/m}\left(\frac{\mu}{m}+\frac{\nu}{h}\right)\\
&\quad\frac{(-1)^{p-j}\lambda^{l-1}(1)_{l,1/\lambda}S_{2,\lambda}(p-j+1,l)}{l^{k-1}(p-j+1)}\\
&= \sum_{\mu=0}^{m-1}\sum_{j=0}^{p}\sum_{\nu=0}^{h-1}\sum_{l=1}^{p-j+1}\binom{p}{j}(mh)^{j-1}\frac{(-1)^{p-j}\lambda^{l-1}(1)_{l,1/\lambda}S_{2,\lambda}(p-j+1,l)}{l^{k-1}(p-j+1)}\\
&\quad\times\left((\mu h)m^{p-j}\beta_{j,\lambda/h}\left(\frac{\mu}{m}+\frac{\nu}{h}\right) + (m\nu)h^{p-j}\beta_{j,\lambda/m}\left(\frac{\mu}{m}+\frac{\nu}{h}\right)\right).
\end{aligned} \tag{24}$$

Therefore, by (24), we obtain the following reciprocity theorem for the degenerate poly-Dedekind sums associated with degenerate poly-Bernoulli functions with index k.

Theorem 3. *For $m, h, p \in \mathbb{N}$, $k \in \mathbb{Z}$, we have*

$$hm^pS_{p,\lambda}^{(k)}(h,m)+mh^pS_{p,\lambda}^{(k)}(m,h)=\sum_{\mu=0}^{m-1}\sum_{j=0}^{p}\sum_{\nu=0}^{h-1}\sum_{l=1}^{p-j+1}\binom{p}{j}(mh)^{j-1}\frac{(-1)^{p-j}\lambda^{l-1}(1)_{l,1/\lambda}S_{2,\lambda}(p-j+1,l)}{l^{k-1}(p-j+1)}$$
$$\times\left((\mu h)m^{p-j}\beta_{j,\lambda/h}\left(\frac{\mu}{m}+\frac{\nu}{h}\right)+(m\nu)h^{p-j}\beta_{j,\lambda/m}\left(\frac{\mu}{m}+\frac{\nu}{h}\right)\right).$$

When $k=1$, $\lambda\to 0$, by Corollary 1, we obtain the following reciprocity relation for the generalized Dedekind sums defined by Apostol.

Corollary 2. *For $m,h,p\in\mathbb{N}$, we have*

$$hm^pS_p(h,m)+mh^pS_p(m,h)=\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{h-1}(mh)^{p-1}(\mu h+m\nu)\overline{B}_p\left(\frac{\mu}{m}+\frac{\nu}{h}\right).$$

By (12) and (17), let $h=1$, we note that

$$\begin{aligned}S_{p,\lambda}^{(k)}(1,m)&=\sum_{\mu=1}^{m-1}\frac{\mu}{m}\overline{B}_{p,\lambda}^{(k)}\left(\frac{\mu}{m}\right)\\&=\sum_{\mu=1}^{m-1}\frac{\mu}{m}\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\left(\frac{\mu}{m}\right)_{p-\nu,\lambda}\\&=\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\sum_{\mu=1}^{m-1}\left(\frac{\mu}{m}\right)^{p-\nu+1}(1)_{p-\nu,\frac{m}{\mu}\lambda}\\&=\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\sum_{\mu=1}^{m-1}\left(\frac{\mu}{m}\right)^{p-\nu+1}\left((1)_{p-\nu+1,\frac{m}{\mu}\lambda}+(p-\nu)\frac{m}{\mu}\lambda(1)_{p-\nu,\frac{m}{\mu}\lambda}\right)\\&=\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\sum_{\mu=1}^{m-1}\left(\frac{\mu}{m}\right)^{p-\nu+1}(1)_{p-\nu+1,\frac{m}{\mu}\lambda}+\lambda\sum_{\nu=0}^{p}\binom{p}{\nu}(p-\nu)B_{\nu,\lambda}^{(k)}\sum_{\mu=1}^{m-1}\left(\frac{\mu}{m}\right)^{p-\nu}(1)_{p-\nu,\frac{m}{\mu}\lambda}\\&=\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{1}{p+2-\nu}(B_{p+2-\nu,\lambda}(m)-B_{p+2-\nu,\lambda})\\&\quad+\lambda\sum_{\nu=0}^{p}\binom{p}{\nu}(p-\nu)B_{\nu,\lambda}^{(k)}\frac{1}{p+1-\nu}(B_{p+1-\nu,\lambda}(m)-B_{p+1-\nu,\lambda}).\end{aligned}\tag{25}$$

By (17), we have

$$B_{p+2-\nu,\lambda}(m)-B_{p+2-\nu,\lambda}=\sum_{i=0}^{p+2-\nu}\binom{p+2-\nu}{i}B_{i,\lambda}(m)_{p+2-\nu-i,\lambda}-B_{p+2-\nu,\lambda}=\sum_{i=0}^{p+1-\nu}\binom{p+2-\nu}{i}B_{i,\lambda}(m)_{p+2-\nu-i,\lambda}.\tag{26}$$

In the same way as (26), we have

$$B_{p+1-\nu,\lambda}(m)-B_{p+1-\nu,\lambda}=\sum_{i=0}^{p-\nu}\binom{p+1-\nu}{i}B_{i,\lambda}(m)_{p+1-\nu-i,\lambda}.\tag{27}$$

By (25), (26), and (27), we obtain

$$\begin{aligned}S_{p,\lambda}^{(k)}(1,m)&=\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{1}{p+2-\nu}(B_{p+2-\nu,\lambda}(m)-B_{p+2-\nu,\lambda})\\&\quad+\lambda\sum_{\nu=0}^{p}\binom{p}{\nu}(p-\nu)B_{\nu,\lambda}^{(k)}\frac{1}{p+1-\nu}(B_{p+1-\nu,\lambda}(m)-B_{p+1-\nu,\lambda})\\&=\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{1}{p+2-\nu}\sum_{i=0}^{p+1-\nu}\binom{p+2-\nu}{i}B_{i,\lambda}(m)_{p+2-\nu-i,\lambda}\\&\quad+\lambda\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{p-\nu}{p+1-\nu}\sum_{i=0}^{p-\nu}\binom{p+1-\nu}{i}B_{i,\lambda}(m)_{p+1-\nu-i,\lambda}.\end{aligned}\tag{28}$$

Now we assume that $p \geq 3$ is an odd positive integer, so that $B_{p,\lambda} = 0$. Then we have

$$\begin{aligned}
m^p S_{p,\lambda}^{(k)}(1,m) &= \sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{1}{p+2-\nu}\sum_{i=0}^{p+1-\nu}\binom{p+2-\nu}{i}B_{i,\lambda}m^p(m)_{p+2-\nu-i,\lambda}\\
&\quad+\lambda\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{p-\nu}{p+1-\nu}\sum_{i=0}^{p-\nu}\binom{p+1-\nu}{i}B_{i,\lambda}m^p(m)_{p+1-\nu-i,\lambda}\\
&=\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{1}{p+2-\nu}m^p(m)_{p+2-\nu,\lambda}+\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{1}{p+2-\nu}\sum_{i=1}^{p+1-\nu}\binom{p+2-\nu}{i}B_{i,\lambda}m^p(m)_{p+2-\nu-i,\lambda}\\
&\quad+\lambda\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{p-\nu}{p+1-\nu}m^p(m)_{p+1-\nu,\lambda}\\
&\quad+\lambda\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{p-\nu}{p+1-\nu}\sum_{i=1}^{p-\nu}\binom{p+1-\nu}{i}B_{i,\lambda}m^p(m)_{p+1-\nu-i,\lambda}\\
&=\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{1}{p+2-\nu}m^p(m)_{p+2-\nu,\lambda}+\sum_{i=1}^{p+1}\sum_{\nu=0}^{p+1-i}\binom{p}{\nu}\binom{p+2-\nu}{i}\frac{B_{\nu,\lambda}^{(k)}}{p+2-\nu}B_{i,\lambda}m^p(m)_{p+2-\nu-i,\lambda}\\
&\quad+\lambda\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{p-\nu}{p+1-\nu}m^p(m)_{p+1-\nu,\lambda}\\
&\quad+\lambda\sum_{i=1}^{p}\sum_{\nu=0}^{p-i}\binom{p}{\nu}\binom{p+1-\nu}{i}\frac{p-\nu}{p+1-\nu}B_{\nu,\lambda}^{(k)}B_{i,\lambda}m^p(m)_{p+1-\nu-i,\lambda}\\
&=\sum_{\nu=0}^{p}\binom{p}{\nu}\frac{B_{\nu,\lambda}^{(k)}}{p+2-\nu}m^p(m)_{p+2-\nu,\lambda}+\sum_{i=1}^{p-1}\sum_{\nu=0}^{p+1-i}\binom{p}{\nu}\binom{p+2-\nu}{i}\frac{B_{\nu,\lambda}^{(k)}}{p+2-\nu}B_{i,\lambda}m^p(m)_{p+2-\nu-i,\lambda}\\
&\quad+\binom{p+2}{p+1}\frac{B_{p+1,\lambda}}{p+2}m^p(m)_{1,\lambda}+\sum_{\nu=0}^{1}\binom{p}{\nu}\binom{p+1-\nu}{p}\frac{B_{\nu,\lambda}^{(k)}}{p+2-\nu}B_{p,\lambda}m^p(m)_{2-\nu,\lambda}\\
&\quad+\lambda\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{p-\nu}{p+1-\nu}m^p(m)_{p+1-\nu,\lambda}\\
&\quad+\lambda\sum_{i=1}^{p-1}\sum_{\nu=0}^{p-i}\binom{p}{\nu}\binom{p+1-\nu}{i}\frac{p-\nu}{p+1-\nu}B_{\nu,\lambda}^{(k)}B_{i,\lambda}m^p(m)_{p+1-\nu-i,\lambda}+\lambda\binom{p+1}{p}\frac{p}{p+1}B_{p,\lambda}m^p(m)_{1,\lambda}\\
&=\sum_{\nu=0}^{p}\binom{p}{\nu}\frac{B_{\nu,\lambda}^{(k)}}{p+2-\nu}m^p(m)_{p+2-\nu,\lambda}+\lambda\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{p-\nu}{p+1-\nu}m^p(m)_{p+1-\nu,\lambda}\\
&\quad+\sum_{i=1}^{p-1}\sum_{\nu=0}^{p+1-i}\binom{p}{\nu}\binom{p+2-\nu}{i}\frac{B_{\nu,\lambda}^{(k)}}{p+2-\nu}B_{i,\lambda}m^p(m)_{p+2-\nu-i,\lambda}\\
&\quad+\lambda\sum_{i=1}^{p-1}\sum_{\nu=0}^{p-i}\binom{p}{\nu}\binom{p+1-\nu}{i}\frac{p-\nu}{p+1-\nu}B_{\nu,\lambda}^{(k)}B_{i,\lambda}m^p(m)_{p+1-\nu-i,\lambda}+B_{p+1,\lambda}m^{p+1}.
\end{aligned}\tag{29}$$

So, by (29), we obtain the following theorem.

Theorem 4. *Let $p \geq 3$ be an odd positive integer. Then we have*

$$\begin{aligned}
m^p S_{p,\lambda}^{(k)}(1,m) &= \sum_{\nu=0}^{p}\binom{p}{\nu}\frac{B_{\nu,\lambda}^{(k)}}{p+2-\nu}m^p(m)_{p+2-\nu,\lambda}+\lambda\sum_{\nu=0}^{p}\binom{p}{\nu}B_{\nu,\lambda}^{(k)}\frac{p-\nu}{p+1-\nu}m^p(m)_{p+1-\nu,\lambda}\\
&\quad+\sum_{i=1}^{p-1}\sum_{\nu=0}^{p+1-i}\binom{p}{\nu}\binom{p+2-\nu}{i}\frac{B_{\nu,\lambda}^{(k)}}{p+2-\nu}B_{i,\lambda}m^p(m)_{p+2-\nu-i,\lambda}\\
&\quad+\lambda\sum_{i=1}^{p-1}\sum_{\nu=0}^{p-i}\binom{p}{\nu}\binom{p+1-\nu}{i}\frac{p-\nu}{p+1-\nu}B_{\nu,\lambda}^{(k)}B_{i,\lambda}m^p(m)_{p+1-\nu-i,\lambda}+B_{p+1,\lambda}m^{p+1}.
\end{aligned}$$

3 Conclusion

As a further generalization of Dedekind sums, we further study the reciprocity relations of degenerate poly-Dedekind sums. In this article, we introduced the degenerate poly-Dedekind sums, which are obtained from the Dedekind sums by replacing Bernoulli polynomials by degenerate poly-Bernoulli functions. We derived several explicit expressions and identities for the poly-Bernoulli polynomials and numbers in terms of the degenerate Stirling numbers of the second kind. We also investigated the poly-Bernoulli polynomials in terms of the degenerate Bernoulli polynomials and the degenerate Stirling numbers of the second kind. Meanwhile, we obtained the reciprocity relations for the degenerate poly-Dedekind sums associated with degenerate poly-Bernoulli functions. It can be further explored in connection with modular forms, ζ function, and trigonometric sums, just as in the cases of Apostol-Dedekind sums.

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