Research Article

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Discrete complementary exponential and sine integral functions

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Abstract: Discrete analogues of the sine integral and complementary exponential integral functions are investigated. Hypergeometric representation, power series, and Laplace transforms are derived for each. The difficulties in extending these definitions to other common trigonometric integral functions are discussed.

Keywords: discrete special function, generalized hypergeometric, sine integral, exponential integral, difference equation

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1 Introduction

In recent years, discrete special functions have been studied by a variety of authors in different contexts. The type of discrete analogue we focus on here is described in [1,2] as a "shadow" of its classical counterpart. By this, it is meant that the discrete analogue of a continuous (i.e., defined on the real line) function $\tilde{f} : A \subset \mathbb{R} \to \mathbb{R}$ with Laplace transform $F(z) = \mathcal{L}_{\mathbb{R}}\{f\}(z)$ is a function $f : B \subset \mathbb{Z} \to \mathbb{R}$ whose "discrete Laplace transform" is also F.

The original set of elementary shadow functions were defined for arbitrary time scales, which introduced notation such as e_p for the dynamic exponential and $\sin_p$ for the dynamic sine. The recent study of further special functions in integers or natural numbers (both often referred to as "discrete" time scales) started with the discrete Bessel function [3], which, e.g., was later applied to understand qualitative properties of solutions of discrete wave equations [4]. The discrete Bessel functions were then later greatly generalized to discrete hypergeometric functions [5,6], which are relevant to our work here. Generalization of the elementary special functions on time scales to a broader framework appears in [7].

Many classical special functions are defined as an indefinite integral, but this has not carried through to discrete special functions. We shall investigate later the discrete analogues of two common special functions often defined in terms of integrals and observe some of the difficulties that arise when attempting to find the integral formulation of a discrete special function.

The sine integral and complementary exponential integral functions are well studied special functions. We will focus on the discrete analogues of these functions in particular, because they are defined as antiderivatives of elementary functions. While the theory of discrete special functions has been quite successful in finding analogues of functions from their power series, analogues of functions classically defined by indefinite integrals have not been as simple to find. This article will reveal some of the difficulty – we shall show that the discrete analogue of the sine integral does retain its definition as an antiderivative in the discrete analogue,

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but the discrete complementary exponential integral fails to retain it. Moreover, the other related special functions such as the cosine integral and the broader class of exponential integrals do not appear to have straightforward discrete analogues at all, likely due to their use of complex analysis that has no current discrete analogue in resolving integrals across singularities.

2 Preliminaries and definitions

We make significant use of the forward difference operator $\Delta f(t) = f(t+1) - f(t)$, which can be rearranged to obtain

$$f(t+1) = \Delta f(t) + f(t). \tag{1}$$

We borrow the notation from the time scales calculus [8], most significantly

$$\int_a^b f(\tau)\Delta\tau := \sum_{k=a}^{b-1} f(k).$$

The falling powers $t^{\underline{k}}$ are the discrete analogue of the power functions and are defined by the formula $t^{\underline{k}} := \frac{\Gamma(t+1)}{\Gamma(t-k+1)}$. The $t^{\underline{k}}$ obey a power rule of the form $\Delta t^{\underline{k}} = kt^{\underline{k-1}}$ and an analogue of the integral for monomials

$$\int_a^b t^{\underline{k}}\Delta t = \frac{b^{\underline{k+1}} - a^{\underline{k+1}}}{k+1}. \tag{2}$$

The shift lemma for discrete calculus is

$$t^{\underline{n}}(t-n)^{\underline{m}} = t^{\underline{n+m}}. \tag{3}$$

We will use the notation $\mathscr{L}_{\mathbb{R}}$ for the classical Laplace transform, and the discrete Laplace transform is given by $\mathscr{L}_{\mathbb{Z}}\{f\}(z) = \sum_{k=0}^{\infty} \frac{f(k)}{(1+z)^{k+1}}$, which is a scaled and shifted $\mathcal{Z}$-transform (see [9,10] for a thorough investigation). If $X(t) = \int_s^t x(\tau)\Delta\tau$, then [10, Theorem 6.4]

$$\mathscr{L}_{\mathbb{Z}}\{X\}(z;\, s) = \frac{1}{z}\mathscr{L}\{x\}(z;\, s). \tag{4}$$

It is well known [8, Theorem 3.90] that

$$\mathscr{L}_{\mathbb{Z}}\{t^{\underline{k}}\}(z;\, s) = \frac{k!}{z^{k+1}}. \tag{5}$$

The discrete exponential e_a is the unique solution of the initial value problem $\Delta y(t) = ay(t)$ and $y(0) = 1$. By the discrete Taylor's theorem,

$$e_a(t, 0) = \sum_{k=0}^{\infty} \frac{a^k t^{\underline{k}}}{k!}. \tag{6}$$

The discrete sine function $\sin_1$ is defined by $\sin_1(t) = \frac{e_i(t) - e_{-i}(t)}{2i}$, and it is straightforward to show from (6) that $\sin_1(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{\underline{2k+1}}}{(2k+1)!}$, so in particular,

$$\frac{\sin_1(t+1)}{t+1} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{\underline{2k}}}{(2k+1)!}. \tag{7}$$

The ${}_p\mathcal{F}_q$ generalized hypergeometric series is defined by:

$${}_p\mathcal{F}_q(a_1, \ldots, a_p;\, b_1, \ldots, b_q;\, z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!}, \tag{8}$$

where $(a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$ is called a Pochhammer symbol. The discrete analogue of (8) is [5, (23)] as follows:

$$ {}_pF_q(a_1,\dots,a_p;\, b_1,\dots,b_q;\, t, n, \xi) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\xi^k t^{\underline{nk}}}{k!}, $$

and it is related to the classical ${}_p\mathcal{F}_q$ [5, Proposition 2] by:

$$ {}_pF_q(\mathbf{a};\, \mathbf{b};\, t, n, \xi) = {}_{p+n}\mathcal{F}_q(\mathbf{a}, \mathbf{t};\, \mathbf{b};\, \xi(-n)^n), \tag{9} $$

where $\mathbf{t} = \left(-\frac{t}{n}, \dots, \frac{-t+n-1}{n}\right)$.

The classical sine integral function is defined by the integral

$$ \mathcal{S}i(x) = \int_0^x \frac{\sin(t)}{t}\mathrm{d}t, \tag{10} $$

and it follows that it obeys the series

$$ \mathcal{S}i(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^{2k-1}}{(2k-1)(2k-1)!}. \tag{11} $$

It has a classical hypergeometric representation:

$$ \mathcal{S}i(z) = z_1\mathcal{F}_2\left(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}; -\frac{z^2}{4}\right). \tag{12} $$

It solves the third-order differential equation

$$ zy''' + 2y'' + zy' = 0, \tag{13} $$

and it has the Laplace transform [11, 3.5.1]:

$$ \mathscr{L}_{\mathbb{R}}\{\mathcal{S}i\}(z) = \frac{1}{z}\arctan\left(\frac{1}{z}\right). \tag{14} $$

The related complementary exponential function $\mathcal{E}\mathrm{in}$ is defined by:

$$ \mathcal{E}\mathrm{in}(t) = \int_0^t \frac{1-e^{-\tau}}{\tau}\Delta\tau. \tag{15} $$

Another function, called the exponential integral $\mathcal{E}i$, is related to $\mathcal{E}\mathrm{in}$ by [12, 37:0:1]

$$ \mathcal{E}\mathrm{in}(t) = \gamma + \ln(|t|) - \mathcal{E}i(-t), \tag{16} $$

where γ is the Euler-Mascheroni constant. Its Laplace transform is given by [13, p. 1160]:

$$ \mathscr{L}_{\mathbb{R}}\{\mathcal{E}\mathrm{in}\}(z) = \frac{1}{z}\log\left(1 + \frac{1}{z}\right). \tag{17} $$

Mathematica reports that the general solution of the polynomial coefficient third-order linear differential equation

$$ t^2y'''(t) + (t^2+5t)y''(t) + (3t+4)y'(t) + y(t) = 0 \tag{18} $$

is $y(t) = \frac{c_2\mathcal{E}i(-t)}{t} + \frac{c_1}{t} + \frac{c_3\ln(t)}{t}$. By (16), the specific solution of (18) for $t > 0$ with parameters $c_2 = -1$, $c_1 = \gamma$, and $c_2 = 1$ is

$$ y(t) = \frac{\mathcal{E}\mathrm{in}(t)}{t}. \tag{19} $$

3 Discrete sine integral

We define the discrete analogue of (10) by:

$$\mathrm{Si}(t) = \int_0^t \frac{\sin_1(\tau+1)}{\tau+1}\Delta\tau. \tag{20}$$

The following theorem is an analogue of (11).

Proposition 3.1. *The following formula holds*:

$$\mathrm{Si}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{\underline{2k+1}}}{(2k+1)(2k+1)!}. \tag{21}$$

Proof. Using (7) and (2) to integrate from 0 to t, we obtain

$$\begin{aligned}\mathrm{Si}(t) &= \int_0^t \sum_{k=0}^{\infty} \frac{(-1)^k \tau^{\underline{2k}}}{(2k+1)!}\Delta\tau \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^t \tau^{\underline{2k}}\Delta\tau \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k t^{\underline{2k+1}}}{(2k+1)(2k+1)!},\end{aligned}$$

which completes the proof. □

We now represent the discrete sine integral in terms of discrete hypergeometric functions, analogous to (12).

Theorem 3.2. *The discrete sine integral has the following hypergeometric representations*:

$$\mathrm{Si}(t) = t\,{}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}; t-1, 2, -\frac{1}{4}\right) = t\,{}_3\mathcal{F}_2\left(\frac{1}{2}, \frac{-t+1}{2}, \frac{-t+2}{2}; \frac{3}{2}, \frac{3}{2}; -1\right),$$

and its defining series exists for $\mathrm{Re}(t) > -1$.

Proof. Compute

$$\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k\left(\frac{3}{2}\right)_k} = \frac{\frac{\Gamma\left(\frac{1}{2}+k\right)}{\Gamma\left(\frac{1}{2}\right)}}{\left[\frac{\Gamma\left(\frac{3}{2}+k\right)}{\Gamma\left(\frac{3}{2}\right)}\right]^2} = \frac{\Gamma\left(\frac{3}{2}\right)^2}{\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma\left(\frac{1}{2}+k\right)}{\Gamma\left(\frac{3}{2}+k\right)^2}.$$

Since $\Gamma\left(\frac{3}{2}+k\right) = \left(\frac{1}{2}+k\right)\Gamma\left(,\frac{1}{2}+k\right)$ and $\Gamma\left(\frac{1}{2}+k\right) = \frac{\Gamma\left(\frac{1}{2}\right)(2k-1)!}{2^{2k-1}(k-1)!}$, we obtain

$$\begin{aligned}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k\left(\frac{3}{2}\right)_k} &= \frac{\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+k\right)\left(\frac{1}{2}+k\right)^2}\\ &= \frac{1}{(2k+1)^2}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+k\right)}\\ &= \frac{2^{2k-1}}{(2k+1)^2}\left[\frac{(k-1)!}{(2k-1)!}\right]\frac{2k}{2k}\\ &= \frac{2^{2k}k!}{(2k+1)(2k+1)!}\end{aligned}$$

Since $t(t-1)^{\underline{2k}} = t^{\underline{2k+1}}$, the proof of the relation to ${}_1F_2$ is complete. The relationship to ${}_3\mathcal{F}_2$ follows from (9).

As a consequence of (9) and the well known convergence properties of ${}_pF_q$ [6, Corollary 4], it converges whenever $\mathrm{Re}(t-1) > \frac{1}{2} + \frac{1}{2} - \frac{3}{2} - \frac{3}{2}$, which simplifies to $\mathrm{Re}(t) > -1$, and the proof is completed □

We now prove the discrete analogue of (14), which demonstrates that Si is the shadow function associated with $\mathcal{S}i$ (Figure 1).

Theorem 3.3. *The following formula holds*:

$$\mathscr{L}_{\mathbb{Z}}\{\mathrm{Si}\}(z) = \frac{1}{z}\arctan\left(\frac{1}{z}\right).$$

Proof. By (7) and (5), we compute

$$\mathscr{L}_{\mathbb{Z}}\left\{\frac{\sin_1(\cdot+1)}{\cdot+1}\right\}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\mathscr{L}\{t^{\underline{2k}}\}(z;\,0) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\left(\frac{1}{z}\right)^{2k+1} = \arctan\left(\frac{1}{z}\right). \tag{22}$$

Using (4) and (22), compute

$$\mathscr{L}_{\mathbb{Z}}\{\mathrm{Si}\}(z) = \frac{1}{z}\mathscr{L}_{\mathbb{Z}}\left\{\frac{\sin_1(\cdot+1)}{\cdot+1}\right\}(z) = \frac{1}{z}\arctan\left(\frac{1}{z}\right),$$

which completes the proof. □

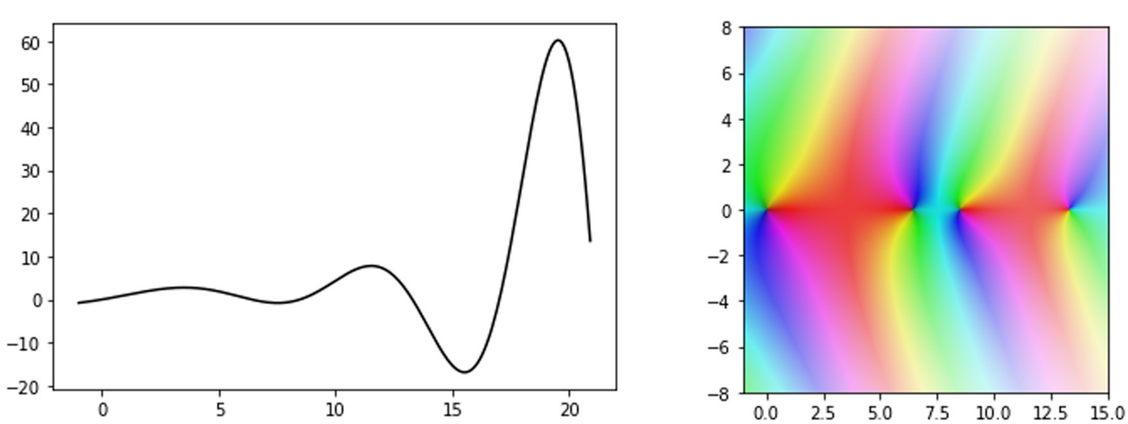

Figure 1: Plot of Si on the real line and its domain coloring in the complex plane.

We derive the difference equation analogue of (13) by manipulating (21) directly.

Theorem 3.4. *If* $y(t) = \operatorname{Si}(t)$, *then*

$$t\Delta^3 y(t-1) + 2\Delta^2 y(t) + t\Delta y(t-1) = 0. \tag{23}$$

Proof. First, compute $\Delta y(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{\underline{2k}}}{(2k+1)!}$,

$$\Delta^2 y(t) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k) t^{\underline{2k-1}}}{(2k+1)!} = -\sum_{k=0}^{\infty} \frac{(-1)^k (2k+2) t^{\underline{2k+1}}}{(2k+3)!}$$

and

$$\Delta^3 y(t) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k)(2k-1) t^{\underline{2k-2}}}{(2k+1)!} = -\sum_{k=0}^{\infty} \frac{(-1)^k (2k+2)(2k+1) t^{\underline{2k}}}{(2k+3)!}.$$

Now, substituting these into the left-hand side of (23) and taking (3) into account yield

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(-1)^k t^{\underline{2k+1}}}{(2k+1)!}\left[-\frac{(2k+2)(2k+1)}{(2k+2)(2k+3)} - \frac{2(2k+2)}{(2k+2)(2k+3)} + 1\right] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k t^{\underline{2k+1}}(2k+2)}{(2k+3)!}[-(2k+1) - 2 + (2k+3)] = 0, \end{aligned}$$

which completes the proof. □

Using (1) repeatedly in (23), we are able to remove the delay from all terms (see Corollary 4.5 below where we provide some details on how this is done).

Corollary 3.5. *If* $y(t) = \operatorname{Si}(t)$, *then*

$$(t+3)\Delta^3 y(t) + 2\Delta^2 y(t) + (t+1)\Delta y(t) = 0.$$

4 Discrete complementary exponential integral

Define

$$\mathcal{E}\mathrm{in}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{\underline{k+1}}}{(k+1)(k+1)!}.$$

Remark 4.1. The integral (15) has no singularity at the origin since the power series for $1 - e^{-\tau}$ contains a factor of τ which cancels the τ in the denominator. It is natural to assume that $\mathcal{E}\mathrm{in}$ would have a representation as an integral similar to (20), but this does not seem possible. The start of such a derivation of might look like this:

$$\mathcal{E}\mathrm{in}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k \int_0^t \tau^{\underline{k}} \Delta\tau}{(k+1)!} = \int_0^t \sum_{k=0}^{\infty} \frac{(-1)^k \tau^{\underline{k}}}{(k+1)!} \Delta\tau = -\int_0^t \sum_{k=1}^{\infty} \frac{(-1)^k \tau^{\underline{k-1}}}{k!} \Delta\tau.$$

The series being integrated is almost (6), except the missing $k = 0$ term and the power on τ. This can be corrected for by multiplying the series by $\frac{\tau-1}{\tau-1}$ and adding and subtracting 1 to obtain

$$\mathcal{E}\mathrm{in}(t) \stackrel{?}{=} -\int_0^t \left[\frac{1}{\tau-1} \sum_{k=1}^{\infty} \frac{(-1)^k (\tau-1)^{\underline{k}}}{k!}\right] \Delta\tau \stackrel{?}{=} -\int_0^t \frac{-1 + e_{-1}(\tau, 0)}{\tau - 1} \Delta\tau,$$

which superficially appears to be an analogue of (15). However, there are two problems that have arisen: first is that the numerator equals −1 except when $\tau = 0$ since $e_{-1}(\tau, 0) = 0$ for any $\tau \neq 0$ and $e_{-1}(\tau, \tau) = 1$ [14, p. 1384]. Second, when computing the integral, a division by zero will occur when $\tau = 1$, which is also unavoidable in this case. A similar situation is handled at the origin in the classic theory by the Cauchy principal value, but we do not have an analogue for that in discrete calculus at this time. This suggests that there is no integral formulation for the $\mathcal{E}$in function on this time scale.

We now find the representation of $\mathcal{E}$in as a discrete hypergeometric series.

Lemma 4.2.

$$\mathcal{E}\mathrm{in}(t) = t_2F_2(1, 1;\ 2, 2;\ t, 1, -1). \tag{24}$$

Proof. Compute

$$\frac{1}{(k+1)(k+1)!} = \frac{k!k!}{(k+1)!(k+1)!k!} = \frac{(1)_k(1)_k}{(2)_k(2)_k k!}.$$

Hence

$$\mathcal{E}\mathrm{in}(t) = t\sum_{k=0}^{\infty}\frac{(1)_k(1)_k(-1)^k(t-1)^{\underline{k}}}{(2)_k(2)_k k!} = t_2F_2(1, 1;\ 2, 2;\ t-1, 1, -1),$$

which completes the proof. □

By combining (9) with (24), we obtain the the relationship with ${}_3\mathcal{F}_2$. Also by [6, Corollary 4], the analogue of (19), we will use to prove an analogue of (18).

Corollary 4.3. The following formula holds:

$$\frac{\mathcal{E}\mathrm{in}(t+1)}{t+1} = {}_3\mathcal{F}_2(1, 1, -t; 2, 2;\ 1).$$

We now derive a difference equation for $\mathcal{E}$in, analogous to (18).

Theorem 4.4. *The function* $y(t) = \frac{\mathcal{E}\mathrm{in}(t+1)}{t+1}$ *solves the difference equation*

$$t^2\Delta^3y(t-2) + t^2\Delta^2y(t-2) + 5t\Delta^2y(t-1) + 3t\Delta y(t-1) + 4\Delta y(t) + y(t) = 0. \tag{25}$$

Proof. Let $\theta = t\rho\Delta$, where $(\rho f)(t) = f(t-1)$ is called the shift operator. By [5, Theorem 7] and (24),

$$[t\rho\Delta(t\rho\Delta + 1)(t\rho\Delta + 1) + t\rho(t\rho\Delta + 1)(t\rho\Delta + 1)]y(t) = 0.$$

Now, factoring inside the brackets, dividing by t, and dropping the shift operator on the left reveal

$$(\Delta + 1)(t\rho\Delta + 1)(t\rho\Delta + 1)y(t) = 0.$$

Performing the first step of this calculation yields

$$(\Delta + 1)(t\rho\Delta + 1)(t\Delta y(t-1) + y(t)) = 0.$$

Since $\Delta(t\Delta y(t-1)) = \Delta y(t) + t\Delta^2y(t-1)$, we have

$$(\Delta + 1)([t\Delta y(t-1) + t^2\Delta^2y(t-2) + t\Delta y(t-1)] + [t\Delta y(t-1) + y(t)]) = 0,$$

which simplifies to:

$$(\Delta + 1)(t^2\Delta^2y(t-2) + 3t\Delta y(t-1) + y(t)) = 0. \tag{26}$$

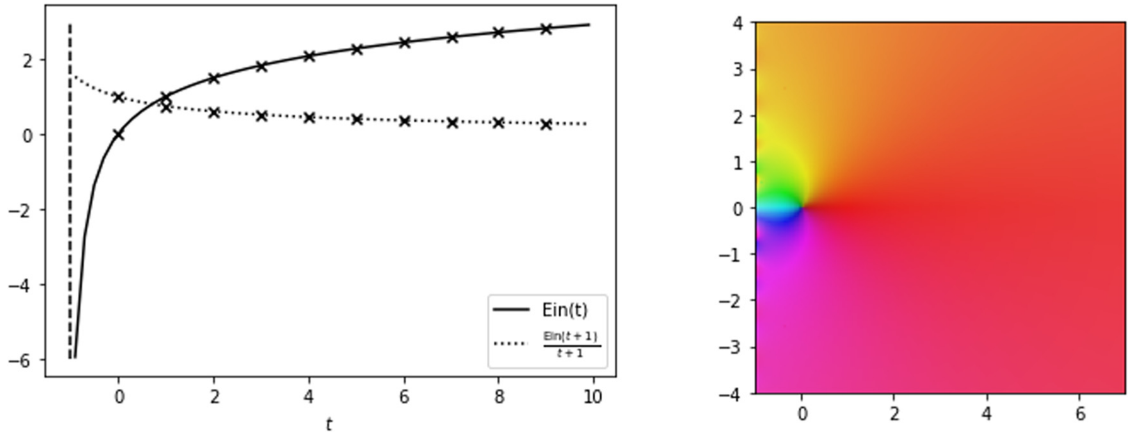

Figure 2: Plot of both $\mathcal{E}\mathrm{in}(t)$ and $\frac{\mathcal{E}\mathrm{in}(t+1)}{t+1}$ and the domain coloring of $\mathcal{E}\mathrm{in}$.

Finally, since $\Delta(t^2\Delta^2 y(t-2)) = 2t\Delta^2 y(t-1) + t^2\Delta^3 y(t-2)$, substituting $\Delta + t$ into (26) yields

$$2t\Delta^2 y(t-1) + t^2\Delta^3 y(t-2) + 3\Delta y(t) + 3t\Delta^2 y(t-1) + \Delta y(t) + t^2\Delta^2 y(t-2) + 3t\Delta y(t-1) + y(t) = 0,$$

which simplifies to (25), and the proof is completed □

Equation (25) can be rearranged by the repeated use of (1) to obtain the following result.

Corollary 4.5. *The function* $y(t) = \frac{\mathcal{E}\mathrm{in}(t+1)}{t+1}$ *solves the difference equation*:

$$(t+4)^2\Delta^3 y(t) + (t^2 + 11t + 27)\Delta^2 y(t) + 3(t+4)\Delta y(t) + y(t) = 0. \tag{27}$$

Proof. Replace t with $t+2$ in (25) to obtain

$$(t+2)^2\Delta^3 y(t) + (t+2)^2\Delta^2 y(t) + 5(t+2)\Delta^2 y(t+1) + 3(t+2)\Delta y(t+1) + 4\Delta y(t+2) + y(t+2) = 0.$$

Repeated use of (1) on $\Delta^2 y(t+1)$, $\Delta y(t+1)$, and $y(t+2)$ leads to:

$$\begin{aligned}(t+2)^2\Delta^3 y(t) &+ (t+2)^2\Delta^2 y(t) + 5(t+2)[\Delta^2 y(t) + \Delta^3 y(t)] + 3(t+2)[\Delta y(t) + \Delta^2 y(t)] + 4[\Delta^3 y(t) + 2\Delta^2 y(t)\\ &+ \Delta y(t)] + [\Delta^2 y(t) + 2\Delta y(t) + y(t)] = 0,\end{aligned}$$

which simplifies to (27), and the proof is completed □

We now derive the discrete analogue of (17), which shows that $\mathcal{E}\mathrm{in}$ is the shadow function associated with $\mathcal{E}\mathrm{in}$ (Figure 2).

Theorem 4.6. *The following formula holds*:

$$\mathscr{L}_{\mathbb{Z}}\{\mathcal{E}\mathrm{in}\}(z;\,0) = \frac{1}{z}\log\left(1 + \frac{1}{z}\right).$$

Proof. Using (5), calculate

$$\begin{aligned}
\mathscr{L}_{\mathbb{Z}}\{\mathcal{E}\mathrm{in}\}(z;\,0) &= \mathscr{L}\left\{\sum_{k=0}^{\infty} \frac{(-1)^k t^{\underline{k+1}}}{(k+1)(k+1)!}\right\}(z;\,0) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)} \mathscr{L}\left\{\frac{t^{\underline{k+1}}}{(k+1)!}\right\}(z;\,0) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)z^{k+2}} \\
&= \frac{1}{z} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)z^{k+1}} \\
&= \frac{1}{z} \log\left(1 + \frac{1}{z}\right),
\end{aligned}$$

which completes the proof. □

5 Conclusion

We have introduced the discrete analogues of the sine integral and complementary exponential integral functions. For each, we have expressed it as a series, found its relationship to hypergeometric functions, and computed its Laplace transform. Analytic continuation is likely a way to extend the domains of existence of Si and $\mathcal{E}$in beyond $\mathrm{Re}(t) > -1$, but doing so is beyond the scope of this article.

We have argued in Remark 4.1 that an integral form for $\mathcal{E}$in like (15) seems unattainable, but this might merely be a consequence of the discrete exponential e_{-1} vanishing. This suggests a future direction for research into generalizations of Si and $\mathcal{E}$in to time scales where e_{-1} does not vanish.

There are other related functions that were not considered here, of most importance would be analogues of the exponential integrals $\mathcal{E}i$ and $\mathcal{E}_n$, as well as the cosine integral $\mathcal{C}i$. All of these functions are defined by an integral that is computed via the Cauchy principal value; the theory needed to do similar work for discrete special functions does not exist, so developing it is of interest. Some possible directions might include contemporary theories of discrete complex analysis [15–17]. Another approach to understand $\mathcal{E}i$ might also be to investigate what the other independent solutions of (25) are.

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