Research Article

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# An equation for complex fractional diffusion created by the Struve function with a *T*-symmetric univalent solution

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**Abstract:** A $T$-symmetric univalent function is a complex valued function that is conformally mapping the unit disk onto itself and satisfies the symmetry condition $\phi^{[T]}(\zeta) = [\phi(\zeta^T)]^{1/T}$ for all $\zeta$ in the unit disk. In other words, it is a complex function that preserves the unit disk's shape and orientation and is symmetric about the unit circle. They are used in the study of geometric function theory and the theory of univalent functions. In recent effort, we extend the class of fractional anomalous diffusion equations in a symmetric complex domain. we aim to present the analytic univalent solution for such a class using special functions technique. Our analysis and comparative findings are further supported by the geometric simulations for the univalent solution such as the convexity and starlikeness of the diffusion. As a consequence of illustration of a list of conditions yielding the univalent solutions (normalize analytic function in the open unit disk), the normalization of diffusion shape is achieved.

**Keywords:** fractional calculus, subordination and superordination, analytic function, inequalities, univalent function, open unit disk, fractional differential equation

**MSC 2020:** 30C55, 30C45

## 1 Introduction

Many physical processes, including electron transfer in semiconductors, nuclear proliferation, and pollutant transport across porous surfaces, have been reported to exhibit anomalous diffusion in recent decades [1–3]. Chemistry, biology, environmental science, and even economics all place a premium on the research of anomalous diffusion [4]. In the case of normal diffusion, the mean square displacement can be expressed as proportional to $\tau$. Nevertheless, it has also been discovered that in the case of anomalous diffusion, the mean square displacement may be expressed as proportional to $\tau^\beta$, $\beta \neq 1$. The anomalous diffusion procedures have been demonstrated to display the traits of long-range correlation and history dependency through experimental statement and hypothetical analysis [5]. Expending fractional derivatives, these anomalous diffusion events are often described and predicted [6]. Complex diffusion demonstrating as yet has certain puzzling unresolved open investigation questions, such as the diffusion processes that alter with time evolution, 3-D position, or system factors, the diffusion processes in the in-homogeneous intermediate, the multi-

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scale diffusion process, the anomalous diffusion processes under an oscillating exterior field, etc. [7–9]. Investigators have derived to the conclusion that the fractional differential equation model is inadequate to represent these intricate anomalous diffusion processes [10]. Anomalous behavior of univalent diffusion has many applications in physics and chemical processes, especially in melts and magmas [11], composite nano-channels [12], high magnetic fields' effects on biologically active molecules' diffusion [13], thermal diffusion [14], and molecularly imprinted impedimetric sensor [15]. Other applications can be located in [16]–[18]. The complex fractional diffusion equation, is a type of partial differential equation that describes the diffusion of a substance in a medium with a fractional order of differentiation. The equation typically has the form:

$$\partial u(\zeta,\tau)/\partial\tau = -(-\Delta)^{\alpha}u(\zeta,\tau),$$

where $u(\zeta,\tau)$ is the concentration of the substance at position $\zeta$ and time $\tau$, $\Delta$ is the Laplacian operator, and $\alpha$ is the fractional order of differentiation. The fractional order can be any value between 0 and 2, with $\alpha = 1$ corresponding to the standard diffusion equation. The CFDE can be used to model a wide range of phenomena, including anomalous diffusion in complex systems. In diffusion equations, the outcome of being univalent is significant. Since the solutions to the diffusion equations are known to be incorrect for infinite layers as they are not univalent functions, the diffusion peaks must unavoidably move faster than the troughs and eventually reach these levels.

In this study, we suggest a class of CFDEs in the open unit disk. We aim to deliver the analytic solutions for the class using the concept of special functions (generalized Struve-H function) and the geometric function theory. The diffusion shape is recognized for the univalent solution, specifically the convex and starlike solutions. Inequalities of differential subordination are illustrated. Differential subordination is a concept

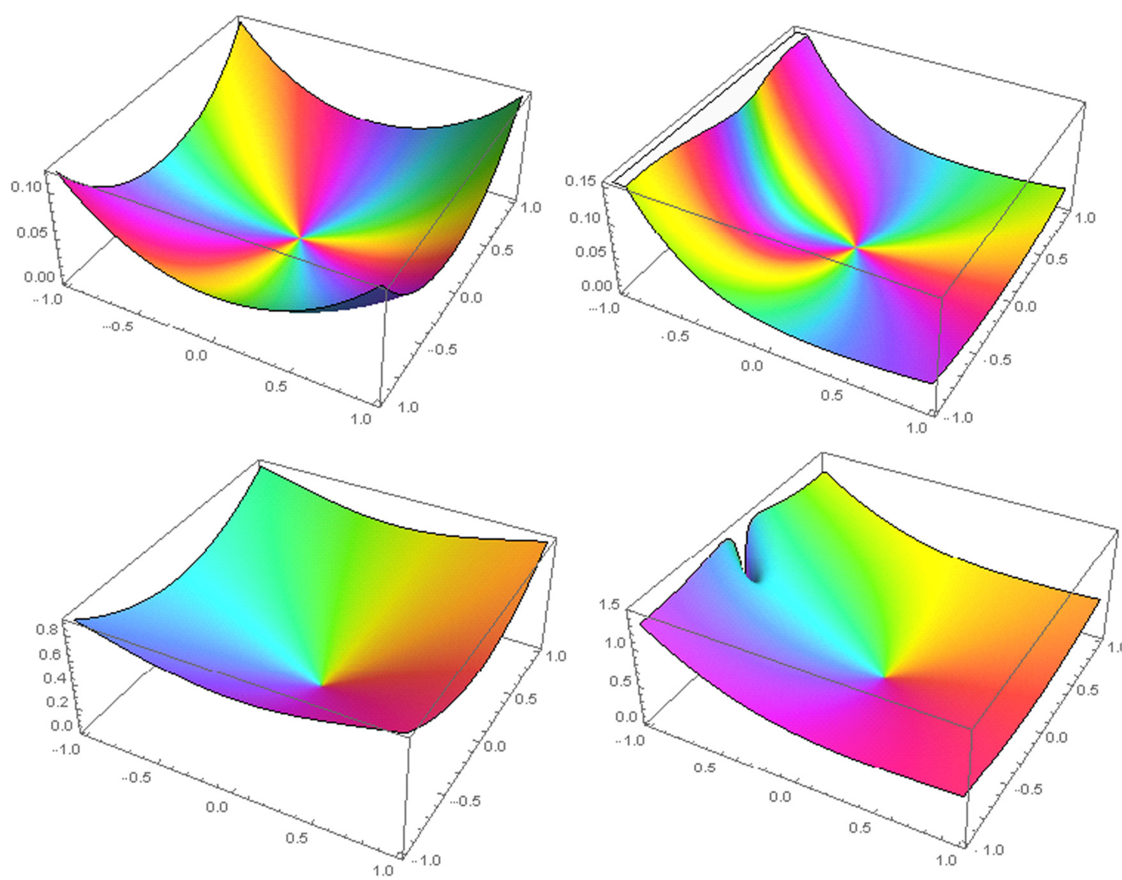


**Figure 1:** Upper graphs are for $\mathbf{H}_2(\zeta)$ and $\mathbf{H}_2(\log[1+\zeta])$ in the open unit disk $\Delta, |\zeta| < 1$, respectively. The lower graphs are for $\mathbf{H}_0(\zeta)$, and $\mathbf{H}_0(\log[1+\zeta])$ in $\Delta$, respectively.

in complex analysis that deals with the relationship between analytic functions. It is a method of comparing the growth rates of two analytic functions using their derivatives. The concept of differential subordination is used to compare the growth rates of different analytic functions and to find bounds on their coefficients and other properties.

We organize the effort as follows: Section 2 imposes the techniques that will be used, Section 3 contains the outcomes of the main conditions that yield the analytic solutions with the diffusion shape, and Section 4 deals with the main conclusions and discoveries in this work.

# 2 Convoluted fractional integral operator

This section covers many concepts that will be used in the conclusion.

## 2.1 Struve function

The Struve function is a mathematical function that is used to solve a variety of scientific and technical issues, specifically in the areas of sounds, diffraction theory, and electromagnetic. $\mathbf{H}_\rho(\zeta)$, where $\rho$ is the function's order and $\zeta$ is the complex argument, is the symbol used to represent it. The differential equation's answer, or the Struve function, is explained as follows:

$$\zeta^2 d^2\mathbf{H}_\rho/\mathrm{d}\zeta^2 + \zeta \mathrm{d}\mathbf{H}_\rho/\mathrm{d}\zeta + (\zeta^2 - \rho^2)\mathbf{H}_\rho = 2\pi i.$$

The complex $\rho$-indexed functions with the same name, developed and introduced by Hermann Struve in 1882, Struve functions, denoted by $H_\rho(\zeta)$, and have the power series form (Figure 1):

$$\mathbf{H}_\rho(\zeta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\rho+\frac{3}{2}\right)}\left(\frac{\zeta}{2}\right)^{2n+\rho+1},$$

where

$$\mathbf{H}_0(\zeta) = \frac{2\zeta}{\pi} - \frac{2\zeta^3}{9\pi} + \frac{2\zeta^5}{225\pi} - \frac{2\zeta^7}{11{,}025\pi} + \frac{2\zeta^9}{893{,}025\pi} + O(\zeta^{11}).$$

The generalization formula of the aforementioned series is given by:

$$\mathbf{H}_0(\log[1+\zeta]) = \frac{2\zeta}{\pi} - \frac{\zeta^2}{\pi} + \frac{4\zeta^3}{9\pi} - \frac{\zeta^4}{6\pi} + \frac{\zeta^5}{50\pi} + \frac{11\zeta^6}{180\pi} - \frac{3{,}541\zeta^7}{33{,}075\pi} + \frac{5{,}029\zeta^8}{37{,}800\pi} + \frac{525{,}769\zeta^9}{35{,}72{,}100\pi} + O(\zeta^{11}),$$

where the normalized formula is specified by:

$$\mathbb{H}_0(\zeta) := \left(\frac{\pi}{2}\right)\mathbf{H}_0(\log[1+\zeta]) = \zeta - \frac{\zeta^2}{2} + \cdots = \zeta + \sum_{n=2} h_n\zeta^n.$$

In this place, we note that the Struve function is used to determine the analytic solution of different classes of fractional differential equations. For example, it is involved in the fractional kinetic equation, fractional heat equation, and fractional integro-differential equation.

For this function, generalization happens because of unique situations and constraints: We can learn more about the Struve function's special situations and constraining characteristics by investigating its generalization. Knowing how the function behaves in particular circumstances can provide us important details about the underlying physical events. Additionally, it enhanced convergence and accuracy: utilizing the normalized and generalized forms of the Struve function in numerical computations may result in higher accuracy and quicker convergence rates. Particularly in computations and modeling on computers, this is important.

## 2.2 Fractional integration

Using the complex classical fractional integral [19]:

$$L^{\beta}\varphi(\zeta) = \frac{1}{\Gamma(\beta)}\int_{0}^{\zeta}\varphi(\xi)(\zeta-\xi)^{\beta-1}\mathrm{d}\xi,$$

where

$$L^{\beta}\zeta^{k} = \frac{\Gamma(1+k)}{\Gamma(1+k+\beta)}\zeta^{k+\beta},$$

we obtain the fractional integral series:

$$\begin{aligned} L^{\beta}\mathbb{H}_0(\zeta) &= L^{\beta}(\zeta) + \sum_{n=2} h_n L^{\beta}(\zeta^n) \\ &= \left(\frac{1}{\Gamma(2+\beta)}\right)\zeta^{\beta+1} + \sum_{n=2} h_n\left(\frac{\Gamma(n+1)}{\Gamma(n+1+\beta)}\right)\zeta^{\beta+n}. \end{aligned}$$

A normalization implies that

$$\mathbb{H}_0^{\beta}(\zeta) \coloneqq \Gamma(2+\beta)\zeta^{-\beta}L^{\beta}\mathbb{H}_0(\zeta) = \zeta + \sum_{n=2} h_n\left(\frac{\Gamma(2+\beta)\Gamma(n+1)}{\Gamma(n+1+\beta)}\right)\zeta^{n}. \tag{2.1}$$

The fractional normalized function $\mathbb{H}_0^{\beta}(\zeta)$ will be the solution of the fractional complex diffusion equation. Our aim is to illustrate a list of conditions that gives the univalency of $\mathbb{H}_0^{\beta}(\zeta)$. In this place, we note that the Struve function has many applications in optics studies describing the optical diffraction and diffusion processes [20–22].

In general, the normalized $\mathbf{H}_{\rho}(\zeta)$ can be realized using the transformation operator:

$$\mathcal{H}_{\rho}(\zeta) = 2^{\rho}\sqrt{\pi}\,\Gamma\left(\rho+\frac{3}{2}\right)\zeta^{-(\rho+1)/2}\mathbf{H}_{\rho}(\sqrt{\zeta}) = \sum_{n=0}^{\infty}\left(\frac{(-1/4)^{n}}{(3/2)_n\left(\rho+\frac{3}{3}\right)_n}\right)\zeta^{n},$$

where $(c)_n = \Gamma(c+n)/\Gamma(c)$. The fractional integral form is structured as follows:

$$\begin{aligned} L^{\beta}(\mathcal{H}_{\rho}(\zeta)) &= \sum_{n=0}^{\infty}\left(\frac{(-1/4)^{n}}{(3/2)_n\left(\rho+\frac{3}{3}\right)_n}\right)L^{\beta}(\zeta^{n}) \\ &= \sum_{n=0}^{\infty}\left(\frac{(-1/4)^{n}}{\left(\frac{3}{2}\right)_n\left(\rho+\frac{3}{2}\right)_n}\right)\left(\frac{\Gamma(n+1)}{\Gamma(n+1+\beta)}\right)\zeta^{n+\beta} \\ &= \sum_{n=0}^{\infty}\left(\frac{(-1/4)^{n}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\rho+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+n\right)\Gamma\left(\rho+\frac{3}{2}+n\right)}\right)\left(\frac{\Gamma(n+1)}{\Gamma(n+1+\beta)}\right)\zeta^{n+\beta}. \end{aligned}$$

Introducing the fractional integral transform:

$$\begin{aligned}
\mathcal{H}^{\beta}_{\rho}(\zeta) &\coloneqq \zeta^{1-\beta}\Gamma(1+\beta)L^{\beta}(\mathcal{H}_{\rho}(\zeta))\\
&= \sum_{n=0}^{\infty}\left(\frac{(-1/4)^{n}\Gamma(1+\beta)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\rho+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+n\right)\Gamma\left(\rho+\frac{3}{2}+n\right)}\right)\left(\frac{\Gamma(n+1)}{\Gamma(n+1+\beta)}\right)\zeta^{n+1}\\
&= \zeta+\sum_{n=1}^{\infty}\left(\frac{(-1/4)^{n}\Gamma(1+\beta)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\rho+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+n\right)\Gamma\left(\rho+\frac{3}{2}+n\right)}\right)\left(\frac{\Gamma(n+1)}{\Gamma(n+1+\beta)}\right)\zeta^{n+1}\\
&= \zeta+\sum_{n=2}^{\infty}\left(\frac{(-1/4)^{n-1}\Gamma(1+\beta)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\rho+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+(n-1)\right)\Gamma(\rho+\frac{3}{2}+(n-1))}\right)\left(\frac{\Gamma(n)}{\Gamma(n+\beta)}\right)\zeta^{n}\\
&= \zeta+\sum_{n=2}^{\infty}\left(\frac{(-1/4)^{n-1}\Gamma(1+\beta)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\rho+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+n\right)\Gamma(\rho+\frac{1}{2}+n)}\right)\left(\frac{\Gamma(n)}{\Gamma(n+\beta)}\right)\zeta^{n},
\end{aligned}$$

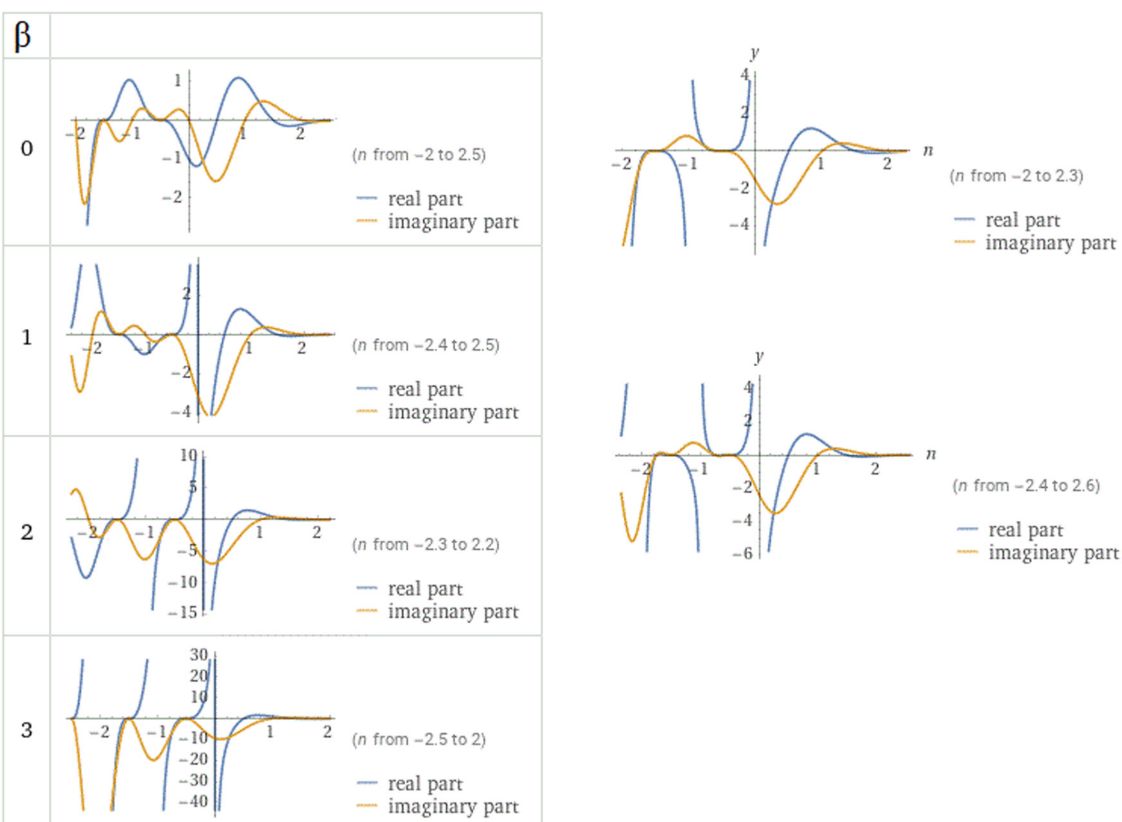


**Figure 2:** Behaviour of the coefficients of the operator $[\mathcal{H}^{\beta}_{0} * \psi]$, when $\psi$ is the Koebe function. The right plot is for $\beta = 1/2$ and $3/4$, respectively.

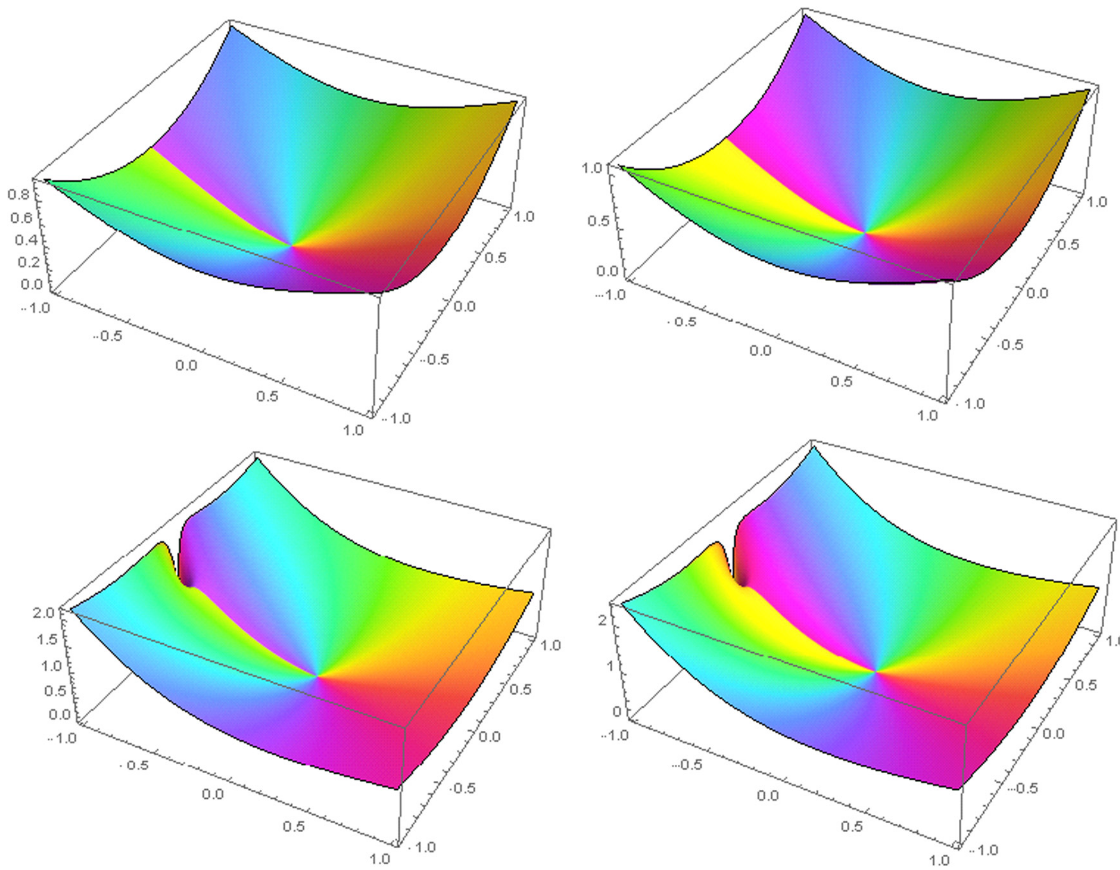


**Figure 3:** Upper graphs are $\mathcal{H}_0^{1/2}(\zeta)$ and $\mathcal{H}_0^{1/4}(\zeta)$, respectively. Lower graphs are $\frac{\pi}{2}\mathcal{H}_0^{1/2}(\log[1+\zeta])$ and $\frac{\pi}{2}\mathcal{H}_0^{1/4}(\log[1+\zeta])$, respectively.

we obtain the normalized fractional Struve function:

$$\mathcal{H}_\rho^\beta(\zeta) = \zeta + \sum_{n=2}^{\infty} \hbar_n(\beta,\rho)\zeta^n, \tag{2.2}$$

where (Figure 2)

$$\begin{aligned}\hbar_n(\beta,\rho) &:= \left(\frac{(-1/4)^{n-1}\Gamma(1+\beta)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\rho+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+n\right)\Gamma(\rho+\frac{1}{2}+n)}\right)\left(\frac{\Gamma(n)}{\Gamma(n+\beta)}\right)\\ &= \frac{\sqrt{\pi}(-1)^{n-1}2^{1-2n}\Gamma(\beta+1)\Gamma(n)\Gamma(\rho+3/2)}{\Gamma(n+1/2)\Gamma(\beta+n)\Gamma(n+\rho+1/2)}.\end{aligned}$$

For example, when $\beta = 1$ and $\rho = 0$, we have the next limit of the connections when $n \to \infty$,

$$\lim_{n\to\infty} \frac{((-1)^{-1+n}4^{-n}\pi\Gamma(n))}{(\Gamma(1/2+n)^2\Gamma(1+n))} = 0.$$

Structure (2.2) is a fractional normalization of the Struve function. Normalization is a popular procedure in mathematics, particularly in the study of functions, to ensure consistency. Dimensionless numbers produced by normalizing the Struve function can facilitate comparison and analysis of outcomes from various physical settings. As a consequence, one can investigate the geometric properties of the fractional integral operator associated with a special function in the complex domain.

This fractional function plays an important role to describe the analytic solution of the CFDE (Figure 3). Moreover, Formula (2.2) indicates that the generalized function $\mathcal{H}_\rho^\beta(\zeta)$ belongs to the class of normalized functions in the open unit disk denoted by $\Lambda$ and having the general structure:

$$\psi(\zeta) = \zeta + \sum_{n=2}^{\infty} \psi_n \zeta^n, \tag{2.3}$$

where $\psi(0) = 0$ and $\psi'(0) = 1$. A convolution product implies that

$$[\mathcal{H}_\rho^{\beta} * \psi](\zeta) = \zeta + \sum_{n=2}^{\infty} \hbar_n(\beta, \rho)\psi_n \zeta^n \in \Lambda.$$

Therefore, we can proceed to study its geometric behaviors. Some properties of $[\mathcal{H}_\rho^{\beta} * \psi](\zeta)$ are given in the next outcomes.

**Proposition 2.1.** *Consider the operator* $[\mathcal{H}_\rho^{\beta} * \psi](\zeta)$. *If it satisfies the inequality*:

$$\sum_{n=2}^{\infty} n(n - (1 - \sqrt{6}))|\hbar_n(\beta, \rho)||\psi_n| < \sqrt{6},$$

*then* $[\mathcal{H}_\rho^{\beta} * \psi](\zeta)$ *is univalent in* $\Delta$.

**Proof.** A computation yields

$$\begin{aligned}\left|\frac{[\mathcal{H}_\rho^{\beta} * \psi]''(\zeta)}{[\mathcal{H}_\rho^{\beta} * \psi]'(\zeta)}\right| &= \left|\frac{\sum_{n=2}^{\infty} n(n-1)\hbar_n(\beta, \rho)\psi_n \zeta^{n-2}}{1 + \sum_{n=2}^{\infty} n\hbar_n(\beta, \rho)\psi_n \zeta^{n-1}}\right| \\ &\le \frac{\sum_{n=2}^{\infty} n(n-1)|\hbar_n(\beta, \rho)||\psi_n|}{1 - \sum_{n=2}^{\infty} n|\hbar_n(\beta, \rho)||\psi_n|} \\ &< \sqrt{6}.\end{aligned}$$

To prove that $[\mathcal{H}_\rho^{\beta} * \psi]$ is univalent, by the condition, inequality of the result yields that

$$\left|\frac{[\mathcal{H}_\rho^{\beta} * \psi]''(\zeta)}{[\mathcal{H}_\rho^{\beta} * \psi]'(\zeta)}\right| < \sqrt{6}.$$

Then, in virtue of the Umezawa lemma [23], we obtain the univalency of the operator $[\mathcal{H}_\rho^{\beta} * \psi]$. □

Note that since the Koebe function is an extreme function for univalency function, then we shall discuss some important properties of a special case of $[\mathcal{H}_\rho^{\beta} * \psi]$ when $\psi$ is the Koebe function.

**Proposition 2.2.** *Consider the operator* $[\mathcal{H}_\rho^{\beta} * \psi](\zeta)$. *If it satisfies the inequality*:

$$\frac{\zeta^2[\mathcal{H}_\rho^{\beta} * \psi]^{(3)}}{[\mathcal{H}_\rho^{\beta} * \psi]'(\zeta)} - \left(\frac{\zeta[\mathcal{H}_\rho^{\beta} * \psi]''(\zeta)}{[\mathcal{H}_\rho^{\beta} * \psi]'(\zeta)}\right)^2 + \frac{(2\zeta[\mathcal{H}_\rho^{\beta} * \psi]''(\zeta))}{([\mathcal{H}_\rho^{\beta} * \psi]'(\zeta))} + 1 < \chi + (1-\chi)\left(\frac{1+\zeta}{1-\zeta}\right)^{3/2}, \quad \chi \in [0, 1),$$

*then* $[\mathcal{H}_\rho^{\beta} * \psi]$ *is convex of order* $\chi$.

**Proof.** Let

$$\Phi(\zeta) = 1 + \left(\frac{\zeta[\mathcal{H}_\rho^{\beta} * \psi]''(\zeta)}{[\mathcal{H}_\rho^{\beta} * \psi]'(\zeta)}\right).$$

And let

$$\Psi(\zeta) = \frac{\Phi(\zeta) - \chi}{1 - \chi}.$$

Obviously, $\Psi(0) = 1$ and

$$\zeta\Psi'(\zeta) = \frac{\zeta\Phi'(\zeta)}{1 - \chi}.$$

Then, in view of Miller and Mocanu lemma [24], we have that the inequality

$$\Psi(\zeta) + \zeta\Psi'(\zeta) \prec \left(\frac{1 + \zeta}{1 - \zeta}\right)^{3/2}$$

gives the inequality:

$$\Psi(\zeta) \prec \frac{1 + \zeta}{1 - \zeta}.$$

But

$$\Phi(\zeta) + \zeta\Phi'(\zeta) = \frac{\zeta^2[\mathcal{H}_\rho^{\beta*}\psi]^{(3)}}{[\mathcal{H}_\rho^{\beta*}\psi]'(\zeta)} - \left(\frac{\zeta[\mathcal{H}_\rho^{\beta*}\psi]''(\zeta)}{[\mathcal{H}_\rho^{\beta*}\psi]'(\zeta)}\right)^2 + \frac{(2\zeta[\mathcal{H}_\rho^{\beta*}\psi]''(\zeta))}{([\mathcal{H}_\rho^{\beta*}\psi]'(\zeta))} + 1$$

$$\prec \chi + (1 - \chi)\left(\frac{1 + \zeta}{1 - \zeta}\right)^{3/2}, \quad \chi \in [0, 1)$$

Then,

$$\Phi(\zeta) \prec \frac{1 + (1 - 2\chi)\zeta}{1 - \zeta}$$

or $\Re(\Phi(\zeta)) > \chi$, which yields that $[\mathcal{H}_\rho^{\beta*}\psi]$ is convex of order $\chi$. □

## 3 Special case of $[\mathcal{H}_\rho^{\beta*}\psi](\zeta)$

The $T$-symmetric Koebe function is formulated as follows:

$$\kappa^{[T]}(\zeta) = \frac{\zeta}{(1 - \zeta^T)^{2/T}}, \quad \zeta \in \Delta.$$

It maps $\Delta$ into the set $\cup_{m=1}^{T}\{r \exp(i\pi(2m - 1)/T)\}$. The integral of this function is given by the hypergeometric function:

$$\int \kappa^{[T]}(\zeta)\mathrm{d}\zeta = \frac{1}{2}\zeta^2 {}_2F_1\left(\frac{2}{T}, \frac{2}{T}, \frac{(T + 2)}{T}, \zeta^T\right) + c.$$

Moreover, the parametric $T$-symmetric Koebe function is suggested as follows:

$$\kappa_\vartheta^{[T]}(\zeta) = \frac{\zeta}{(1 - e^{i\vartheta}\zeta^T)^{2/T}}, \quad \zeta \in \Delta.$$

We propose using the parametric 1-symmetric Koebe function in the development of the wave equation. A member of the family of convex univalent functions, the Koebe function, is an extreme function. In order to translate $\Delta$ onto the complex plane, the Koebe function

$$\kappa(\zeta) = \frac{\zeta}{(1 - \zeta)^2}, \quad \zeta \in \Delta,$$

stretches a slit along the ray from the point with radius 1/4 to the point $\zeta = 0$. We will use the structure's rotate Koebe function in our most recent endeavor:

$$\kappa_{\vartheta}(\zeta) = \frac{\zeta}{(1 - e^{i\vartheta}\zeta)^2} = \zeta + \sum_{n=2}^{\infty} n e^{i(n-1)\vartheta}\zeta^n, \quad \zeta \in \Delta.$$

The $\alpha$-fractional integral formula can be realized as follows:

$$L^{\alpha}\kappa_{\vartheta}(\zeta) = \left(\frac{\Gamma(2)}{\Gamma(2+\alpha)}\right)\zeta^{1+\alpha} + \sum_{n=2}^{\infty}\left(\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}\right) n e^{i(n-1)\vartheta}\zeta^{n+\alpha}, \quad \zeta \in \Delta.$$

Therefore, the normalized expression is as follows:

$$\begin{aligned}\kappa_{\vartheta}^{\alpha}(\zeta) &:= \Gamma(2+\alpha)\zeta^{-\alpha}L^{\alpha}\kappa_{\vartheta}(\zeta)\\ &= \zeta + \sum_{n=2}^{\infty}\left(\frac{\Gamma(2+\alpha)\Gamma(n+1)}{\Gamma(n+1+\alpha)}\right) n e^{i(n-1)\vartheta}\zeta^n\\ &=: \zeta + \sum_{n=2}^{\infty}\kappa_n(\alpha,\vartheta)\zeta^n, \quad \zeta \in \Delta.\end{aligned}$$

Clearly, $\kappa_{\vartheta}^{\alpha} \in \Lambda$. The fractional convoluted operator combining $\mathcal{H}_{\rho}^{\beta}$ and $\kappa_{\vartheta}^{\alpha}$ can be realized as follows:

$$\left[\mathcal{H}_{\rho}^{\beta} * \kappa_{\vartheta}^{\alpha}\right](\zeta;\vartheta) = \zeta + \sum_{n=2}^{\infty}\hbar_n(\beta,\rho)\kappa_n(\alpha,\vartheta)\zeta^n, \quad \zeta \in \Delta. \tag{3.1}$$

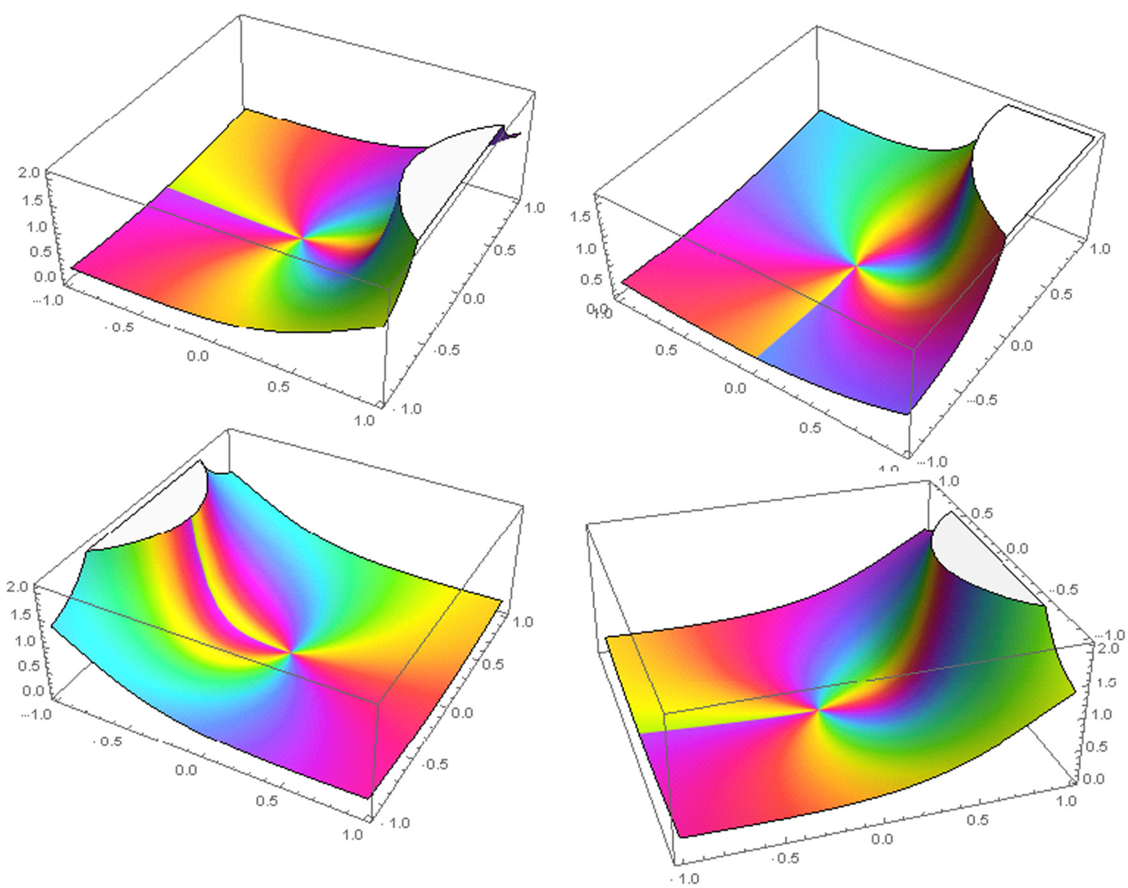

**Figure 4:** Graph of $\left[\mathcal{H}_{0}^{1/2} * \kappa_{\vartheta}^{1/2}\right](\zeta;\vartheta)$ when $\vartheta = 0, 1, \pi$, and $2\pi$, respectively.

We next continue to formulate the CFDE using the operator (3.1) (Figure 4). The formula can be expressed, as follows:

$$\frac{\partial^\alpha}{\partial\vartheta^\alpha}\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta) - \omega\frac{\partial^{2\beta}}{\partial\zeta^{2\beta}}\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta) = F(\zeta), \quad \alpha,\beta \in (0,1], \tag{3.2}$$

where $\partial^\alpha$ and $\partial^\beta$ are the classic fractional differential operators $\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)$ indicates the concentration of interest, $\omega$ is the generalized diffusion coefficient and $F$ is known as the nonlinear functional of the diffusion under consideration owing to $F(0) = 0$ and $F'(0) = 1$ (normalized function in $\Delta$). We aim to illustrate a list of sufficient conditions on $\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)$ and $F$ to be the univalent solution. The next section discusses the fundamental prerequisite for obtaining an analytically univalent solution satisfying the inequality:

$$\Re(\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)') > 0, \quad ' = d/\mathrm{d}\zeta.$$

The aforementioned inequality represents the class of bounded turning functions. This class has a connection with the symmetry property. A bounded turning function with the symmetric property is a mathematical function that satisfies the following properties:

- Boundedness: The function is limited to a certain range of values and does not go beyond that range. Turning: The function changes direction at certain points, meaning that it increases or decreases in value at certain intervals.
- Symmetry: The function is symmetric with respect to a certain point or axis, meaning that if you reflect the function over that point or axis, the result is identical to the original function.

Examples of functions that have these properties would be $\sin(\zeta)$ and $\cos(\zeta)$ functions.

Our results for the univalent solution of equation (3.2) for various assumptions regarding $F$ are presented in this section.

**Proposition 3.1.** *Suppose that the solution of equation* (3.2) *satisfies the real inequality*:

$$\Re\left(\frac{\zeta\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)'}{\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta) - \left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](-\zeta;\,\vartheta)}\right) > 0, \tag{3.3}$$

*then* $[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha](\zeta;\,\vartheta)$ *is univalent in* $\Delta$.

**Proof.** Obviously, the solution $\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)$ fulfills $\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](0;\,\vartheta) = 0$ and $\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](0;\,\vartheta)' = 1$. Switching $-\zeta$ by $\zeta$ in (3.3), this implies that

$$\Re\left(\frac{\zeta\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](-\zeta;\,\vartheta)'}{\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta) - \left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](-\zeta;\,\vartheta)}\right) > 0. \tag{3.4}$$

Correlating the aforementioned real inequalities yields

$$\Re\left(\frac{\zeta(\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)' - \left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](-\zeta;\,\vartheta)')}{\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta) - \left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](-\zeta;\,\vartheta)}\right) > 0. \tag{3.5}$$

This brings the fact that $\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta) - \left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](-\zeta;\,\vartheta)$ is univalent in $\Delta$. Thus, the Kaplan theorem [25] shows that $\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)$ is the univalent solution of equation (3.2). □

The following results depend on various criteria for the solution of equation (3.2) to be univalent solvable.

**Proposition 3.2.** *Assume that $\phi$ is an analytic function in $\Delta$ with a non-negative real part. Moreover, suppose that the solution of equation* (3.2) *satisfies the real inequality*:

$$\Re\left(\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)' + \phi(\zeta)\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)''\right) > 0. \tag{3.6}$$

*Then, $[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha](\zeta;\,\vartheta)$ is the univalent solution in $\Delta$.*

**Proof.** Consider the real inequality (3.6). Define the admissible function $\Psi : \mathbb{C}^2 \to \mathbb{C}$ by the form:

$$\Psi(f, g) = f(\zeta) + \phi(\zeta)g(\zeta).$$

By substituting

$$f(\zeta) \coloneqq \left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)', \quad g(\zeta) \coloneqq \zeta\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)'',$$

we obtain

$$\Re\left(\Psi\left(\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)', \zeta\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)''\right)\right) > 0.$$

In virtue of [26]-Theorem 5, we have

$$\Re\left(\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)'\right) > 0,$$

which implies that $[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha](\zeta;\,\vartheta)$ is the univalent solution in $\Delta$. □

We proceed to show the univalency of the solution $[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha](\zeta;\,\vartheta)$ in $\Delta$.

**Proposition 3.3.** *Let $F(\zeta)$ be a bounded function in $\Delta$ such that*

$$\inf\Re\left(\frac{F(\zeta_1) - F(\zeta_2)}{\zeta_1 - \zeta_2}\right) > 0, \quad \zeta_1, \zeta_2 \in \Delta.$$

*Then,*

$$\left|\frac{\zeta}{\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)} - \frac{\zeta}{F(\zeta)}\right| \le \frac{2\inf\Re\left(\frac{F(\zeta_1) - F(\zeta_2)}{\zeta_1 - \zeta_2}\right)}{[\sup_{\zeta\in\Delta}(F(\zeta))]^2},$$

*which implies that $[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha](\zeta;\,\vartheta)$ is univalent in $\Delta$.*

**Proof.** Assume that $[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha](\zeta;\,\vartheta) = \zeta + \sum_{n=2}^\infty a_n\zeta^n$ and $F(\zeta) = \zeta + \sum_{n=2}^\infty b_n\xi^n$. Define a function $G : \Delta \to \Delta$, by:

$$G(\zeta) = \left[\frac{\zeta}{\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\vartheta)} - \frac{\zeta}{F(\zeta)}\right]''.$$

Obviously, $G$ is analytic in $\Delta$. Integration yields

$$\left[\frac{\zeta}{\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)} - \frac{\zeta}{F(\zeta)}\right]' = b_2 - a_2 + \int_0^\zeta G(t)\mathrm{d}t.$$

A computation gives

$$\left[\frac{\zeta}{\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)} - \frac{\zeta}{F(\zeta)}\right] = (b_2 - a_2)\zeta + \int_0^\zeta \mathrm{d}s\int_0^s G(t)\mathrm{d}t.$$

Consequently, we obtain

$$\left[\mathcal{H}_{\rho}^{\beta *}\kappa_{\vartheta}^{\alpha}\right](\zeta;\,\vartheta) = \frac{F(\zeta)}{1 + (b_2 - a_2)F(\zeta) + F(\zeta)(\sigma(\zeta)/\zeta)},$$

where

$$\sigma(\zeta) = \int_0^{\zeta} \mathrm{d}s \int_0^{s} G(t)\mathrm{d}t,$$

which leads to

$$\left(\frac{\sigma(\zeta)}{\zeta}\right)' = \frac{1}{\zeta^2}\int_0^{\zeta} t\sigma''(t)\mathrm{d}t = \frac{1}{\zeta^2}\int_0^{\zeta} tG(t)\mathrm{d}t.$$

The hypotheses of the proposition produce that

$$\left|\frac{\sigma(\zeta_2)}{\zeta_2} - \frac{\sigma(\zeta_1)}{\zeta_1}\right| = \left|\int_{\zeta_1}^{\zeta_2}\left(\frac{\sigma(\zeta)}{\zeta}\right)'\mathrm{d}\zeta\right| \le \left(\frac{2\inf\Re\left(\frac{F(\zeta_1)-F(\zeta_2)}{\zeta_1-\zeta_2}\right)}{[\sup_{\zeta\in\Delta}(F(\zeta))]^2}\right)\left(\frac{|\zeta_2-\zeta_1|}{2}\right),$$

where $\zeta_1 \neq \zeta_2$. Finally, we show that $\left[\mathcal{H}_{\rho}^{\beta *}\kappa_{\vartheta}^{\alpha}\right](\zeta_1;\,\vartheta) \neq \left[\mathcal{H}_{\rho}^{\beta *}\kappa_{\vartheta}^{\alpha}\right](\zeta_2;\,\vartheta)$ or

$$\left|\left[\mathcal{H}_{\rho}^{\beta *}\kappa_{\vartheta}^{\alpha}\right](\zeta_1;\,\vartheta) - \left[\mathcal{H}_{\rho}^{\beta *}\kappa_{\vartheta}^{\alpha}\right](\zeta_2;\,\vartheta)\right| > 0, \quad \zeta_1 \neq \zeta_2,$$

$$\begin{aligned}
&\left|\left[\mathcal{H}_{\rho}^{\beta *}\kappa_{\vartheta}^{\alpha}\right](\zeta_1;\,\vartheta) - \left[\mathcal{H}_{\rho}^{\beta *}\kappa_{\vartheta}^{\alpha}\right](\zeta_2;\,\vartheta)\right| \\
&= \frac{\left|F(\zeta_1) - F(\zeta_2) + F(\zeta_2)F(\zeta_1)\left(\frac{\sigma(\zeta_2)}{\zeta_2} - \frac{\sigma(\zeta_1)}{\zeta_1}\right)\right|}{\left|1 + (b_2-a_2)F(\zeta_1) + F(\zeta_1)\left(\frac{\sigma(\zeta_1)}{\zeta_1}\right)\right|\left|1 + (b_2-a_2)F(\zeta_2) + F(\zeta_2)\left(\frac{\sigma(\zeta_2)}{\zeta_2}\right)\right|} \\
&> \frac{|F(\zeta_1) - F(\zeta_2)| - \inf\Re\left(\frac{F(\zeta_1)-F(\zeta_2)}{\zeta_1-\zeta_2}\right)(\zeta_2-\zeta_1)}{\left|1 + (b_2-a_2)F(\zeta_1) + F(\zeta_1)\left(\frac{\sigma(\zeta_1)}{\zeta_1}\right)\right|\left|1 + (b_2-a_2)F(\zeta_2) + F(\zeta_2)\left(\frac{\sigma(\zeta_2)}{\zeta_2}\right)\right|} \\
&\ge 0.
\end{aligned}$$

Hence, $[\mathcal{H}_{\rho}^{\beta *}\kappa_{\vartheta}^{\alpha}](\zeta;\,\vartheta)$ is the univalent solution in $\Delta$. □

**Proposition 3.4.** *One of the following inequalities implies that* $[\mathcal{H}_{\rho}^{\beta *}\kappa_{\vartheta}^{\alpha}](\zeta;\,\vartheta)$ *is univalent in* $\Delta$:

(*a*) $(1 - |\zeta|^2)\left|\frac{\zeta\left[\mathcal{H}_{\rho}^{\beta} * \kappa_{\vartheta}^{\alpha}\right](\zeta;\,\vartheta)''}{\left[\mathcal{H}_{\rho}^{\beta} * \kappa_{\vartheta}^{\alpha}\right](\zeta;\,\vartheta)'}\right| \le 1;$

(*b*) $\left|\frac{\zeta^2\left[\mathcal{H}_{\rho}^{\beta} * \kappa_{\vartheta}^{\alpha}\right](\zeta;\,\vartheta)'}{\left[\mathcal{H}_{\rho}^{\beta} * \kappa_{\vartheta}^{\alpha}\right](\zeta;\,\vartheta)^2} - 1\right| \le 1;$

(*c*) $(1 - |\zeta|^{2\varepsilon})\left|\frac{\zeta\left[\mathcal{H}_{\rho}^{\beta} * \kappa_{\vartheta}^{\alpha}\right](\zeta;\,\vartheta)''}{\left[\mathcal{H}_{\rho}^{\beta} * \kappa_{\vartheta}^{\alpha}\right](\zeta;\,\vartheta)'} + 1 - \varepsilon\right| \le \varepsilon,\ \varepsilon > 1/2;$

(*d*) $\left|1 + \frac{\zeta\left[\mathcal{H}_{\rho}^{\beta} * \kappa_{\vartheta}^{\alpha}\right](\zeta;\,\vartheta)''}{\left[\mathcal{H}_{\rho}^{\beta} * \kappa_{\vartheta}^{\alpha}\right](\zeta;\,\vartheta)'}\right| < \sqrt{7}.$

**Proof.** In view of the Becker theorem [27]-Corollary 4.1, we have that $[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha](\zeta;\,\vartheta)$ is univalent in $\Delta$. The second assumption implies that

$$\Re\left(\frac{\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)^2}{\zeta^2\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)'}\right) \ge 1/2.$$

Then, in view of [28]-Theorem 2, we obtain the univalency of $[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha](\zeta;\,\vartheta)$. The third inequality satisfies [29]-Corollary 3, when

$$g(\zeta) \coloneqq \left(\frac{\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)}{\zeta}\right)^2$$

and

$$h(\zeta) \coloneqq \frac{1}{\zeta} - \frac{\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)}{\zeta^2}.$$

Consider the fourth inequality to be true. Define a function:

$$\Xi(\zeta) = 1 + \frac{\zeta\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)''}{\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)'},$$

where $\Pi(\zeta) = \Xi^2(\zeta)$ such that $\Pi(\zeta) = 7$. A computation yields

$$\begin{aligned}\int_{|\zeta|=r<1} \Re(\Xi(\varrho))^2 \mathrm{d}\varrho &= \frac{1}{4}\left[\int_{|\zeta|=r<1} (\Pi(\zeta) + 2|\Pi(\zeta)| + \overline{\Pi(\zeta)})\mathrm{d}\varrho\right]\\ &< 7\pi + \frac{1}{4}\left[\int_{|\zeta|=r<1} (\Pi(\zeta) + \overline{\Pi(\zeta)})\mathrm{d}\varrho\right]\\ &= 8\pi.\end{aligned}$$

Schwarz's inequality implies that

$$\int_{|\zeta|=r<1} \Re|\Xi(\varrho)|d\varrho \le \left(\int_{|\zeta|=r<1} d\varrho\right)^{1/2}\left(\int_{|\zeta|=r<1} \Re(\Xi(\varrho))^2 d\varrho\right)^{1/2} < 4\pi.$$

Hence, according to [30]-Theorem 2(iii), page 197, the operator $[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha](\zeta;\,\vartheta)$ is univalent and convex in $\Delta$. □

**Proposition 3.5.** *Suppose that*

$$\min_{|\zeta|=r<1} \Re\left(\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)'\right) = \min_{|\zeta|=r<1}\left|\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)'\right|,$$

*and there is a constant* $c \in (0, 1/2]$ *such that*

$$\Re\left(\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)' + \frac{\zeta\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)''}{\left[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)'}\right) > 2c - 1.$$

*Then,* $[\mathcal{H}_\rho^{\beta*}\kappa_\vartheta^\alpha](\zeta;\,\vartheta)$ *is the univalent solution.*

**Proof.** Let $\zeta_0 \in \Delta$, with

$$\Re(\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)') = c.$$

By the condition of the result, we have

$$\Re(\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)') = \left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)' = c.$$

Then, in view of [31]-Theorem 3.1, we have

$$\frac{\zeta_0\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)''}{\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)'} \le c - 1.$$

Consequently, we obtain

$$\Re\left(\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)' + \frac{\zeta_0\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)''}{\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)'}\right) \le 2c - 1,$$

which is a contradiction; therefore, we have

$$\Re(\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\ \vartheta)') > c > 0.$$

Hence, $[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha](\zeta;\ \vartheta)$ is univalent in $\Delta$. □

The next result shows that the univalent solution is diffused in a starlikeness shape. Note that the normalized function $h$ is called starlike if it satisfies the real inequality:

$$\Re\left(\frac{\zeta h'(\zeta)}{h(\zeta)}\right) > 0.$$

**Proposition 3.6.** *Suppose that*

$$\min_{|\zeta|=r<1} \Re\left(\frac{\zeta\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\ \vartheta)'}{\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\ \vartheta)}\right) = \min_{|\zeta|=r<1} \left|\frac{\zeta\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\ \vartheta)'}{\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\ \vartheta)}\right|,$$

*and there is a constant $c > 0$ such that*

$$\Re\left(1 + \frac{\zeta\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\ \vartheta)''}{\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\ \vartheta)'}\right) > c.$$

*Then, the univalent solution $\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta;\ \vartheta)$ admits a starlike shape.*

**Proof.** Let $\zeta_0 \in \Delta$, with

$$\Re\left(\frac{\zeta_0\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)'}{\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)}\right) = c.$$

By the condition of the result, we have

$$\Re\left(\frac{\zeta_0\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)'}{\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)}\right) = \frac{\zeta_0\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)'}{\left[\mathcal{H}_\rho^{\beta *}\kappa_\vartheta^\alpha\right](\zeta_0;\ \vartheta)} = c.$$

But

$$1+\frac{\zeta_0\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right](\zeta_0;\,\vartheta)''}{\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right](\zeta_0;\,\vartheta)'}=\frac{\zeta_0\left(\frac{\zeta_0\left[\mathcal{H}_\rho^{\beta}*\kappa_\vartheta^\alpha\right](\zeta_0;\,\vartheta)'}{\left[\mathcal{H}_\rho^{\beta}*\kappa_\vartheta^\alpha\right](\zeta_0;\,\vartheta)}\right)'+\left(\frac{\zeta_0\left[\mathcal{H}_\rho^{\beta}*\kappa_\vartheta^\alpha\right](\zeta_0;\,\vartheta)'}{\left[\mathcal{H}_\rho^{\beta}*\kappa_\vartheta^\alpha\right](\zeta_0;\,\vartheta)}\right)^2}{\left(\frac{\zeta_0\left[\mathcal{H}_\rho^{\beta}*\kappa_\vartheta^\alpha\right](\zeta_0;\,\vartheta)'}{\left[\mathcal{H}_\rho^{\beta}*\kappa_\vartheta^\alpha\right](\zeta_0;\,\vartheta)}\right)}=c$$

Consequently, we obtain

$$\Re\left(1+\frac{\zeta\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)''}{\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)'}\right)=c,$$

which is a contradiction. Hence,

$$\Re\left(\frac{\zeta\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)'}{\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)}\right)>c,$$

which implies that $\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta)$ admits a starlike shape. □

The next lemma can be found in [32].

**Lemma 3.7.** *For every univalent function $\phi(\zeta)$ in the open unit disk, there is a sequence of univalent polynomials $P_k(\zeta)$ in the open unit disk, with normalization $P_k(0)=\phi(0)$ satisfying the inclusion*:

$$P_k(\zeta)\subset\phi(\zeta).$$

The next proposition is a direct application of Lemma 3.7. The proposition shows that the univalent function $\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right]$ includes a finite sequence of univalent solutions.

**Proposition 3.8.** *Let the functional $\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right]$ be univalent* (*Proposition* 3.1–3.5). *Then, for every finite sequence of univalent solutions of CFDE, $P_k(\zeta)$ fulfilled the inequality relation*:

$$P_k(\zeta)\prec\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta),$$

*where $[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha](0)=P_k(0)=0$. Moreover, the univalent polynomial of order $m$, $[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha]_m(\zeta;\,\vartheta)$, generated by the operator $[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha](\zeta;\,\vartheta)$, satisfies the upper and lower bounds*:

$$-\left(\cot\left(\frac{\pi}{2(n+1)}\right)\right)^2\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right]_m(1;\,\vartheta)\le\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right]_m(-1;\,\vartheta)\le-\left(\tan\left(\frac{\pi}{2(n+1)}\right)\right)^2\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right]_m(1;\,\vartheta).$$

**Proof.** In view of Lemma 3.7, we have

$$P_k(\zeta)\subset\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right](\zeta;\,\vartheta),$$

where $[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha](0)=P_k(0)=0$. Then, by the definition of the subordination, if $\psi$ is subordinate to $\varphi$ when $\varphi$ is univalent, the subordination $\psi\prec\varphi$ is equivalent to the fact that $\psi(0)=\varphi(0)$ and $\psi(\Delta)\subset\varphi(\Delta)$ occur. Thus, we obtain the first result.

For the second part, we have

$$\left[\mathcal{H}_\rho^{\beta}{*}\kappa_\vartheta^\alpha\right]_m(1;\,\vartheta)=1+\hbar_2(\beta,\rho)\kappa_2(\alpha,\vartheta)+\hbar_3(\beta,\rho)\kappa_3(\alpha,\vartheta)+\cdots+\hbar_m(\beta,\rho)\kappa_m(\alpha,\vartheta)\coloneqq A_0+A_1.$$

Similarly, we have

$$\left[\mathcal{H}_{\rho}^{\beta *}\kappa_{\vartheta}^{\alpha}\right]_m(-1;\,\vartheta) = -1 + \hbar_2(\beta,\rho)\kappa_2(\alpha,\vartheta) - \hbar_3(\beta,\rho)\kappa_3(\alpha,\vartheta) + \cdots + (-1)^m \hbar_m(\beta,\rho)\kappa_m(\alpha,\vartheta) = -A_0 + A_1.$$

Therefore, a computation implies that

$$\frac{[\mathcal{H}_{\rho}^{\beta *}\kappa_{\vartheta}^{\alpha}]_m(-1;\,\vartheta)}{[\mathcal{H}_{\rho}^{\beta *}\kappa_{\vartheta}^{\alpha}]_m(1;\,\vartheta)} = \frac{-A_0 + A_1}{A_0 + A_1} = \frac{A_1/A_0 - 1}{A_1/A_0 + 1}.$$

In view of the Fejer inequality [33], we obtain

$$-\left(\cot\left(\frac{\pi}{2(n+1)}\right)\right)^2 \le \frac{A_1/A_0 - 1}{A_1/A_0 + 1} \le -\left(\tan\left(\frac{\pi}{2(n+1)}\right)\right)^2.$$

This completes the proof. □

**Example 3.9.** For example, consider the operator $[\mathcal{H}_{0}^{\beta *}\kappa_{\vartheta}^{\alpha}]$. Then, when $n = 2$, we obtain

$$A_0 = 1, \quad A_1 = -\frac{4}{(9(\alpha+2)(\beta+1))}.$$

Consequently, under one of the following conditions:

- $-26/9 < \alpha < -2,\ 0 < \beta \le \frac{(-9\alpha - 26)}{(9(\alpha+2))}$
- $-26/9 < \alpha < -2,\ \beta \ge \frac{(-9\alpha - 26)}{(9(\alpha+2))} > 0,$

and in view of Proposition 3.8, every univalent polynomial of order $m$ generated by $[\mathcal{H}_{0}^{\beta *}\kappa_{\vartheta}^{\alpha}]$ satisfies the upper and lower bounds (Figure 5)

$$-\left(\cot\left(\frac{\pi}{2(n+1)}\right)\right)^2 \le \frac{A_1/A_0 - 1}{A_1/A_0 + 1} \le -\left(\tan\left(\frac{\pi}{2(n+1)}\right)\right)^2 \Rightarrow -3 \le -\frac{8}{(9(\alpha+2)\beta + 9\alpha + 14)} - 1 \le -\frac{1}{3}.$$

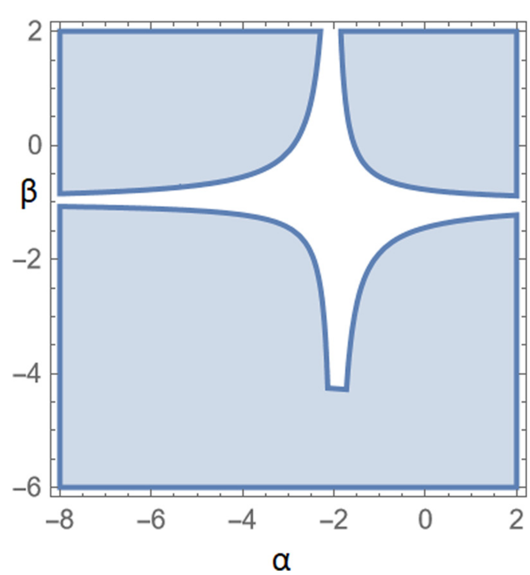


**Figure 5:** Relation between $\alpha$ and $\beta$ of the suggested operator $[\mathcal{H}_{0}^{\beta *}\kappa_{\vartheta}^{\alpha}]$ in Example 3.9.

# 4 Conclusion

A generalization and normalization of the Struve function is presented in this study to investigate univalent solutions of a class of CFDEs in the open unit disk (Propositions 3.3, 3.4, and 3.5). On the theory of the diffusion equation of a complex variable, the univalent solution is a particularly delicate property. Based on the theory of geometric functions, this characteristic leads to several geometric presentations for the solution. We have shown under some conditions, the univalent solution has a starlike diffusion behave (Proposition 3.6) and convex diffusion shape (Proposition 3.4-(d)). The computational and graphical presentations are given using the software MATHEMATICA 13.3.

As efforts for the future, one can make a generalization for the suggested operator $[\mathcal{H}_{\rho}^{\beta} * \kappa_{\vartheta}^{\alpha}]$ in view of different fractional calculus, including the ABC in a complex domain [34,35], the K-symbol fractional calculus, and the quantum calculus. The process of scaling a geometric function's values to fit inside a predetermined range or to have a uniform representation is known as normalization. This procedure is frequently used to simplify computations or analyses or to increase the function's comparability. It is crucial to select the normalization technique that best satisfies your unique data and analysis goals. Understanding the properties of your geometric function and the repercussions of the desired normalization technique is essential since different methods may have varied effects on the distribution and interpretation of the data.

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