Research Article

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Mixed-type SP-iteration for asymptotically nonexpansive mappings in hyperbolic spaces

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Abstract: In this article, we introduce and study some strong convergence theorems for a mixed-type SP-iteration for three asymptotically nonexpansive self-mappings and three asymptotically nonexpansive non-self-mappings in uniformly convex hyperbolic spaces. In addition to that, we provide an illustrative example. The findings here expand and improve upon some of the relevant conclusions found in the published literature.

Keywords: SP-iteration, asymptotically nonexpansive self-mapping, asymptotically nonexpansive nonself-mapping, common fixed point, hyperbolic space

MSC 2020: 47H10, 47H09, 46B20

1 Introduction and preliminaries

When it comes to approximating the fixed points of nonlinear mappings in Hilbert and Banach spaces, iterative techniques are a very effective instrument, see [1–15]. Takahashi [16], who was the first person to study the fixed points for nonexpansive mappings in the context of convex metric spaces, was the one who first proposed the idea of convex metric space. One kind of metric space that has a structure that is convex is called the hyperbolic space. Since numerous convex structures have been imposed on hyperbolic spaces, the term "hyperbolic space" has been defined in a variety of different ways (see [17–19]). Banach and CAT(0) spaces are hyperbolic spaces established in the study by Kohlenbach [18]. See [17–22] for more discussion of hyperbolic spaces.

In this whole work, we will be doing our research in the hyperbolic space that was first described by Kohlenbach [18] as follows.

$(\mathcal{X}, \zeta, \mathcal{H})$ is said to be a hyperbolic space if $(\mathcal{X}, \zeta)$ is a metric space and $\mathcal{H} : \mathcal{X} \times \mathcal{X} \times [0, 1] \to \mathcal{X}$ is a function satisfying

(i) $\zeta(t, \mathcal{H}(u, v, \alpha)) \leq (1 - \alpha)\zeta(t, u) + \alpha\zeta(t, v)$,
(ii) $\zeta(\mathcal{H}(u, v, \alpha), \mathcal{H}(u, v, \beta)) = |\alpha - \beta|\zeta(u, v)$,
(iii) $\mathcal{H}(u, v, \alpha) = \mathcal{H}(v, u, 1 - \alpha)$,
(iv) $\zeta(\mathcal{H}(u, t, \alpha), \mathcal{H}(v, s, \alpha)) \leq (1 - \alpha)\zeta(u, v) + \alpha\zeta(t, s), \forall u, v, s, t \in \mathcal{X}, \alpha, \beta \in [0, 1]$.

Example 1.1. [23] Suppose that $\mathcal{X}$ is a real Banach space with norm $\|\cdot\|$ and the function $\zeta : \mathcal{X} \times \mathcal{X} \times [0, \infty)$ defined by $\zeta(u, v) = \|u - v\|$. Then, $(\mathcal{X}, \zeta, \mathcal{H})$ is a hyperbolic space with $\mathcal{H} : \mathcal{X} \times \mathcal{X} \times [0, 1] \to \mathcal{X}$ defined by $\mathcal{H}(u, v, \alpha) = (1 - \alpha)u + \alpha v$.

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Assume that $(\mathcal{X}, \zeta, \mathcal{H})$ is a hyperbolic space and $\mathcal{K} \subseteq \mathcal{X}$. We have that $\mathcal{K}$ is convex if $\mathcal{H}(u, v, \alpha) \in \mathcal{K}\ \forall u, v \in \mathcal{K}, \alpha \in [0, 1]$. Recall that $(\mathcal{X}, \zeta, \mathcal{H})$ is said to be

(i) strictly convex [16] if there exists a unique element $t \in \mathcal{X}$ such that $\zeta(t, u) = \alpha\zeta(u, v)$ and $\zeta(t, v) = (1 - \alpha)\zeta(u, v), \forall u, v \in \mathcal{X}, \alpha \in [0, 1]$.

(ii) uniformly convex [24] if $\exists \phi \in (0, 1]$ such that

$$\zeta\left(\mathcal{H}\left(u, v, \frac{1}{2}\right), u\right) \leq (1 - \phi)\psi$$

whenever $\zeta(u, s) \leq \psi$, $\zeta(v, s) \leq \psi$ and $\zeta(u, v) \geq \varepsilon\psi$ for all $u, v, s \in \mathcal{X}$, $\psi > 0$, and $\varepsilon \in (0, 2]$.

For given $\psi > 0$ and $\varepsilon \in (0, 2]$, a mapping $\gamma : (0, \infty) \times (0, 2] \to (0, 1]$ such that $\phi = \gamma(\psi, \varepsilon)$ is say to be modulus of uniform convexity. For a fixed ε, we call γ is monotone if γ decreases with ψ. It is worth noting that a uniformly convex hyperbolic space is strictly convex [25].

Suppose that $(\mathcal{X}, \zeta)$ is a metric space, $\emptyset \neq \mathcal{K}$ is a subset of $\mathcal{X}$, $\mathcal{F}(\mathcal{T}) = \{u \in \mathcal{K} : \mathcal{T}u = u\}$, and $\zeta(u, \mathcal{F}(\mathcal{T})) = \inf\{\zeta(u, t) : t \in \mathcal{F}(\mathcal{T})\}$.

Take into account the common assumption that $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is

(i) nonexpansive if

$$\zeta(\mathcal{T}u, \mathcal{T}v) \leq \zeta(u, v), \quad \forall u, v \in \mathcal{K}.$$

(ii) asymptotically nonexpansive if there exists $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\zeta(\mathcal{T}^n u, \mathcal{T}^n v) \leq k_n \zeta(u, v), \quad \forall u, v \in \mathcal{K}, n \geq 1. \tag{1}$$

(iii) uniformly $\mathcal{L}$-Lipschitzian if there exists $\mathcal{L} > 0$ such that

$$\zeta(\mathcal{T}^n u, \mathcal{T}^n v) \leq \mathcal{L}\zeta(u, v), \quad \forall u, v \in \mathcal{K}, n \geq 1.$$

As a result, any nonexpansive mapping with $k_n = 1\ \forall n \geq 1$ is an asymptotically nonexpansive mapping. Furthermore, every asymptotically nonexpansive mapping implies a uniformly $\mathcal{L}$-Lipschitzian mapping with $\mathcal{L} = \sup_{n\in\mathbb{N}}\{k_n\}$. It should be noted that a subset $\mathcal{K}$ of $\mathcal{X}$ is called a retract (see [22,26]) if there exists a continuous mapping $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ such that $\mathcal{P}u = u, \forall u \in \mathcal{K}$.

Some interesting results concerning fixed-point iteration processes for nonexpansive nonself mappings can be found in [27–30].

Let $\mathcal{T} : \mathcal{K} \to \mathcal{X}$ be a mapping and $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ be a nonexpansive retraction. A mapping $\mathcal{T}$ is said to be asymptotically nonexpansive nonself-mapping (see [31]) if $\exists\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\zeta(\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}u, \mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}v) \leq k_n\zeta(u, v)\ \forall u, v \in \mathcal{K}, n \geq 1. \tag{2}$$

The identity map from $\mathcal{K}$ onto itself is denoted by $(\mathcal{P}\mathcal{T})^0$. We can see that if $\mathcal{T}$ is a self-mapping, then $\mathcal{P}$ is the identity mapping, therefore equation (2) becomes equation (1).

Goebel and Kirk [32] developed the class of asymptotically nonexpansive self-mappings in 1972, which is a significant generalization of the nonexpansive self-mappings class. Schu [33] presented the modified Mann iteration algorithm as follows:

$$u_{n+1} = (1 - \beta_n)u_n + \beta_n\mathcal{T}^n u_n, \ \ n \geq 1. \tag{3}$$

Since then, Schu's iteration process (3) has been frequently utilized to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert or Banach spaces [29–38].

Chidume et al. [31] proposed the notion of asymptotically nonexpansive nonself-mappings in 2003. They also investigated the iterative process

$$u_{n+1} = \mathcal{P}((1 - \beta_n)u_n + \beta_n\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}u_n). \tag{4}$$

If $\mathcal{T}$ is a self-mapping, then $\mathcal{P}$ is the identity mapping, and equation (4) becomes equation (3).

Wang [39] suggested the following iteration approach for two asymptotically nonexpansive nonself-mappings in 2006:

$$\begin{aligned} v_n &= \mathcal{P}((1-\alpha_n)u_n + \alpha_n \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n), \\ u_{n+1} &= \mathcal{P}((1-\beta_n)u_n + \beta_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n), \; n \geq 1, \end{aligned} \tag{5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1).

Thianwan [40] introduced the projection-type Ishikawa iteration for two asymptotically nonexpansive nonself-mappings as follows:

$$\begin{aligned} v_n &= \mathcal{P}((1-\alpha_n)u_n + \alpha_n \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n), \\ u_{n+1} &= \mathcal{P}((1-\beta_n)v_n + \beta_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n), \; n \geq 1, \end{aligned} \tag{6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in [0,1).

Guo et al. [41] investigated the iteration scheme

$$\begin{aligned} v_n &= \mathcal{P}((1-\alpha_n)\mathcal{S}_2^n u_n + \alpha_n \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n), \\ u_{n+1} &= \mathcal{P}((1-\beta_n)\mathcal{S}_1^n u_n + \beta_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n), \;\; n \geq 1, \end{aligned} \tag{7}$$

where $\mathcal{S}_1$ and $\mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ are asymptotically nonexpansive self-mappings, $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{X}$ are asymptotically nonexpansive nonself-mappings, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0,1) to approximate common fixed points of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$, and $\mathcal{T}_2$ under appropriate conditions.

Questions regarding hyperbolic groups, one of the principal subjects of study in geometric group theory, have primarily driven and dominated the study of hyperbolic spaces. The nonlinear class of hyperbolic spaces provides a comprehensive abstract theoretical framework with a rich geometrical structure for metric fixed point theory. Approximation methods and fixed point theory have been extended to hyperbolic spaces (see [39–48] and references therein).

Very recently, Jayashree and Eldred [49] introduced and studied the following mixed-type iteration scheme in a uniformly convex hyperbolic space and proved some strong convergence theorems for mixed-type asymptotically nonexpansive mappings:

$$\begin{aligned} v_n &= \mathcal{P}(\mathcal{H}(\mathcal{S}_2^n u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n, \alpha_n)), \\ u_{n+1} &= \mathcal{P}(\mathcal{H}(\mathcal{S}_1^n u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n, \beta_n)), \;\; n \geq 1, \end{aligned} \tag{8}$$

where $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ are two asymptotically nonexpansive self-mappings, $\mathcal{T}_1$ and $\mathcal{T}_2 : \mathcal{K} \to \mathcal{X}$ are two asymptotically nonexpansive nonself-mappings, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0,1). Several articles have studied fixed points using two-step mixed-type iterative schemes in a uniformly convex hyperbolic space (see [50]).

Another three-step iteration process was introduced by Phuengrattana and Suantai [51], which is formulated as follows: $u_1 \in \mathcal{K}$,

$$\begin{aligned} w_n &= (1-\alpha_n)u_n + \alpha_n \mathcal{S}u_n, \\ v_n &= (1-\beta_n)w_n + \beta_n \mathcal{S}w_n, \\ u_{n+1} &= (1-\gamma_n)v_n + \gamma_n \mathcal{S}v_n, \;\; n \geq 1, \end{aligned} \tag{9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in [0,1]. Such iterative method is called SP-iteration. They proved some convergence theorems for the SP-iteration process. In addition, the SP-iteration is equivalent to that of iterative schemes due to Mann [52], Ishikawa [53], and Noor [54] and converges faster than the others for the class of continuous and nondecreasing functions.

Elastoviscoplasticity, liquid crystal, and eigenvalue problems were all solved by Glowinski and Le Tallec [55] using a three-step iterative method. They demonstrated that compared to the two-step and one-step iterative techniques, and the three-step approximation method performs better.

Haubruge et al. [56] investigated the convergence analysis of the three-step iterative schemes of Glowinski and Le Tallec [55]. They applied these three-step iterations to obtain new splitting-type algorithms for solving variational inequalities, separable convex programming, and minimization of a sum of convex functions. Under certain conditions, they also demonstrated that three-step iterations result in highly parallelized algorithms.

Thus, it is evident that three-step schemes play an important role in solving numerous problems in the pure and applied sciences.

Motivated by the above recent results, we suggest a mixed-type SP-iteration for three asymptotically nonexpansive self and nonself mappings in the setting of uniformly convex hyperbolic spaces.

Let $(\mathcal{X}, \zeta, \mathcal{H})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{X}$. Suppose that $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ is a nonexpansive retraction, $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ are three asymptotically nonexpansive self-mappings, and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to \mathcal{X}$ are three asymptotically nonexpansive nonself-mappings. The set of common fixed point of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$ is denoted by $\Omega \coloneqq F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(\mathcal{S}_3) \cap F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \cap F(\mathcal{T}_3)$. The iteration procedure that follows is a translation of the SP-iteration presented in the study by Phuengrattana and Suantai [51] from Banach spaces to hyperbolic spaces:

$$\begin{cases} u_1 \in \mathcal{K}, \\ w_n = \mathcal{P}(\mathcal{H}(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} u_n, \alpha_n)), \\ v_n = \mathcal{P}(\mathcal{H}(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, \beta_n)), \\ u_{n+1} = \mathcal{P}(\mathcal{H}(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \gamma_n)), \quad n \leq 1, \end{cases} \tag{10}$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $[0, 1)$.

We will require the following essential lemmas to prove our main convergence theorems.

Lemma 1.2. [42] *Assume $\{s_n\}, \{b_n\}$, and $\{c_n\}$ be sequences nonnegative real numbers such that*

$$s_{n+1} \leq (1 + b_n)s_n + c_n \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.

Lemma 1.3. [57] *Assume that $\{u_n\}$ and $\{v_n\}$ be sequences of a uniformly convex hyperbolic space $(\mathcal{X}, \zeta, \mathcal{H})$ such that, for $\mathcal{R} \in [0, \infty)$, $\lim_{n\to\infty} \sup \zeta(u_n, a) \leq \mathcal{R}$, $\lim_{n\to\infty} \sup \zeta(v_n, a) \leq \mathcal{R}$, and*

$$\lim_{n\to\infty} \zeta(\mathcal{H}(u_n, v_n, \mu_n), a) = \mathcal{R},$$

where $\mu_n \in [a, b]$ with $0 < a \leq b < 1$, then $\lim_{n\to\infty} \zeta(u_n, v_n) = 0$.

2 Main results

In this section, we consider a uniformly convex hyperbolic space $(\mathcal{X}, \zeta, \mathcal{H})$ and prove a strong convergence theorem for $\mathcal{X}$, using the iterative scheme given in equation (10). The following lemmas are needed.

Lemma 2.1. *Let $\emptyset \neq \mathcal{K}$ be a closed convex subset of a uniformly convex hyperbolic space $(\mathcal{X}, \zeta, \mathcal{H})$. Suppose that $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ are three asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\}, \{k_n^{(3)}\} \subset [1, \infty)$, and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to \mathcal{X}$ are three asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\}, \{l_n^{(3)}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2, 3$, respectively, and $\Omega \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are real sequence in $[0, 1)$. From $u_1 \in \mathcal{K}$, define the sequence $\{u_n\}$ using equation (10). Then, $\lim_{n\to\infty} \zeta(u_n, p)$ exists $\forall p \in \Omega$.*

Proof. Let $p \in \Omega$ and setting $h_n = \max\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\}$. From equation (10), we have

$$\begin{aligned} \zeta(w_n, p) &= \zeta(\mathcal{P}(\mathcal{H}(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n, \alpha_n)), p) \\ &\le \zeta(\mathcal{H}(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n, \alpha_n), p) \\ &\le (1-\alpha_n)\zeta(\mathcal{S}_3^n u_n, p) + \alpha_n \zeta(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n, p) \\ &\le (1-\alpha_n)h_n\zeta(u_n, p) + \alpha_n h_n \zeta(u_n, p) \\ &= h_n \zeta(u_n, p) \end{aligned} \tag{11}$$

and

$$\begin{aligned} \zeta(v_n, p) &= \zeta(\mathcal{P}(\mathcal{H}(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n, \beta_n)), p) \\ &\le \zeta(\mathcal{H}(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n, \beta_n), p) \\ &\le (1-\beta_n)\zeta(\mathcal{S}_2^n w_n, p) + \beta_n \zeta(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n, p) \\ &\le (1-\beta_n)h_n\zeta(w_n, p) + \beta_n h_n \zeta(w_n, p) \\ &= h_n \zeta(w_n, p) \\ &\le h_n^2 \zeta(u_n, p). \end{aligned} \tag{12}$$

Using equation (12), we have

$$\begin{aligned} \zeta(u_{n+1}, p) &= \zeta(\mathcal{P}(\mathcal{H}(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n, \gamma_n)), p) \\ &\le \zeta(\mathcal{H}(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n, \gamma_n), p) \\ &\le (1-\gamma_n)\zeta(\mathcal{S}_1^n v_n, p) + \gamma_n \zeta(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n, p) \\ &\le (1-\gamma_n)h_n\zeta(v_n, p) + \gamma_n h_n \zeta(v_n, p) \\ &= h_n \zeta(v_n, p) \\ &\le h_n^3 \zeta(u_n, p) \\ &= (1 + (h_n^3 - 1))\zeta(u_n, p). \end{aligned} \tag{13}$$

Since $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2, 3$, we have $\sum_{n=1}^{\infty}(h_n^{(3)} - 1) < \infty$. Using Lemma 1.2, $\lim_{n\to\infty}\zeta(u_n, p)$ exists. □

Lemma 2.2. *Let $\emptyset \ne \mathcal{K}$ be a closed convex subset of a uniformly convex hyperbolic space $(\mathcal{X}, \zeta, \mathcal{H})$. Suppose that $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ are three asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\}, \{k_n^{(3)}\} \subset [1, \infty)$, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to \mathcal{X}$ are three asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\}, \{l_n^{(3)}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2, 3$, respectively, and $\Omega \ne \emptyset$. Assume $\{u_n\}$ be a sequence defined by equation (10) and the following conditions hold:*
(i) $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$, $\exists \varepsilon \in (0, 1)$,
(ii) $\zeta(u, \mathcal{T}_i v) \le \zeta(\mathcal{S}_i u, \mathcal{T}_i v)$, $\forall u, v \in \mathcal{K}$, $i = 1, 2, 3$.

Then, $\lim_{n\to\infty}\zeta(u_n, \mathcal{S}_i u_n) = \lim_{n\to\infty}\zeta(u_n, \mathcal{T}_i u_n) = 0$ for $i = 1, 2, 3$.

Proof. Let $p \in \Omega$ and setting $h_n = \max\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\}$. From Lemma 2.1, we can see that $\lim_{n\to\infty}\zeta(u_n, p)$ exists. Suppose that $\lim_{n\to\infty}\zeta(u_n, p) = c$, letting $n \to \infty$ in equation (13), we obtain

$$\lim_{n\to\infty} \zeta(\mathcal{H}(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n, \gamma_n), p) = c. \tag{14}$$

Using equation (12), we obtain $\zeta(\mathcal{S}_1^n v_n, p) \le h_n^3 \zeta(u_n, p)$. Using the lim sup on both sides of this inequality, we obtain

$$\limsup_{n\to\infty} \zeta(\mathcal{S}_1^n v_n, p) \le c. \tag{15}$$

Taking the lim sup in equation (12), we obtain $\lim_{n\to\infty}\sup\zeta(v_n, p) \le c$. Thus,

$$\lim_{n\to\infty}\sup\zeta(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n, p) \le \lim_{n\to\infty}\sup h_n\zeta(v_n, p) = c. \tag{16}$$

By equations (14), (15), (16), and Lemma 1.3, we obtain

$$\lim_{n\to\infty}\zeta(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) = 0. \tag{17}$$

Using condition (ii), we have

$$\lim_{n\to\infty}\zeta(v_n, (\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n)) \le \lim_{n\to\infty}\zeta(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n). \tag{18}$$

Using equation (18), we obtain

$$\lim_{n\to\infty}\zeta(v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) = 0. \tag{19}$$

From equation (13), we obtain

$$\begin{aligned}\zeta(u_{n+1}, p) &\le \zeta(\mathcal{H}(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n, \gamma_n), p)\\ &\le (1-\gamma_n)\zeta(\mathcal{S}_1^n v_n, p) + \gamma_n\zeta(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) + \gamma_n\zeta(\mathcal{S}_1^n v_n, p)\\ &= \zeta(\mathcal{S}_1^n v_n, p) + \gamma_n\zeta(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n)\\ &\le h_n\zeta(v_n, p) + \gamma_n\zeta(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n).\end{aligned} \tag{20}$$

Taking the lim inf into consideration on both sides of the inequality (20), using equation (17), $\sum_{n=1}^{\infty}(h_n - 1) < \infty$, and $\lim_{n\to\infty}\zeta(u_{n+1}, p) = c$, we have

$$\lim_{n\to\infty}\inf\zeta(v_n, p) \ge c. \tag{21}$$

Since $\lim_{n\to\infty}\sup\zeta(v_n, p) \le c$, by equation (21), we have

$$\lim_{n\to\infty}\zeta(v_n, p) = c.$$

Letting $n \to \infty$ in equation (12), we have

$$\lim_{n\to\infty}\zeta(\mathcal{H}(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n, \beta_n), p) = c. \tag{22}$$

In addition, using equation (11), we obtain $\zeta(\mathcal{S}_2^n w_n, p) \le h_n^2\zeta(u_n, p)$. Taking the lim sup on both sides of this inequality, we obtain

$$\lim_{n\to\infty}\sup\zeta(\mathcal{S}_2^n w_n, p) \le c. \tag{23}$$

Taking the lim sup in equation (11), we obtain $\lim_{n\to\infty}\sup\zeta(w_n, p) \le c$. Thus,

$$\lim_{n\to\infty}\sup\zeta(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n, p) \le \lim_{n\to\infty}\sup h_n\zeta(w_n, p) = c. \tag{24}$$

Using Lemma 1.3 and equations (22), (23), and (24), we obtain

$$\lim_{n\to\infty}\zeta(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) = 0. \tag{25}$$

Using condition (ii), we have

$$\lim_{n\to\infty}\zeta(w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) \le \lim_{n\to\infty}\zeta(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n),$$

and thus,

$$\lim_{n\to\infty}\zeta(w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) = 0. \tag{26}$$

From equation (12), we obtain

$$\begin{aligned}\zeta(v_n,p) &\le \zeta(\mathcal{H}(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n, \beta_n), p)\\ &\le (1-\beta_n)\zeta(\mathcal{S}_2^n w_n, p) + \beta_n\zeta(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) + \beta_n\zeta(\mathcal{S}_2^n w_n, p)\\ &= \zeta(\mathcal{S}_2^n w_n, p) + \beta_n\zeta(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n)\\ &\le h_n\zeta(w_n,p) + \beta_n\zeta(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n).\end{aligned} \tag{27}$$

Taking the lim inf into consideration on both sides of the inequality (27), using equation (25), $\sum_{n=1}^{\infty}(h_n-1)<\infty$ and $\lim_{n\to\infty}\zeta(v_n,p)=c$, we have

$$\lim_{n\to\infty}\inf\zeta(w_n,p)\ge c. \tag{28}$$

Since $\lim_{n\to\infty}\sup\zeta(w_n,p)\le\lim_{n\to\infty}\sup h_n\zeta(u_n,p)\le c$, by equation (28), we have

$$\lim_{n\to\infty}\zeta(w_n,p)=c.$$

Letting $n\to\infty$ in the inequality (11), we obtain

$$c=\lim_{n\to\infty}\zeta(w_n,p)\le\lim_{n\to\infty}\zeta(\mathcal{H}(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n, \alpha_n), p)\le\lim_{n\to\infty}\zeta(u_n,p)=c,$$

and so

$$\lim_{n\to\infty}\zeta(\mathcal{H}(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n, \alpha_n), p)=c. \tag{29}$$

Moreover, we obtain

$$\lim_{n\to\infty}\sup\zeta(\mathcal{S}_3^n u_n, p)\le\lim_{n\to\infty}\sup h_n\zeta(u_n,p)=c \tag{30}$$

and

$$\lim_{n\to\infty}\sup\zeta(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n, p)\le\lim_{n\to\infty}\sup h_n\zeta(u_n,p)=c. \tag{31}$$

Following equations (29), (30), (31) and Lemma 1.3, we obtain

$$\lim_{n\to\infty}\zeta(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n)=0. \tag{32}$$

Next, we show that

$$\lim_{n\to\infty}\zeta(u_n,\mathcal{T}_1u_n)=\lim_{n\to\infty}\zeta(u_n,\mathcal{T}_2u_n)=\lim_{n\to\infty}\zeta(u_n,\mathcal{T}_3u_n)=0.$$

Indeed, condition (ii) implies

$$\zeta(u_n,\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n)\le\zeta(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n). \tag{33}$$

By equations (32) and (33), it implies that

$$\lim_{n\to\infty}\zeta(u_n,\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n)=0. \tag{34}$$

Using equation (10), we have

$$\begin{aligned}\zeta(w_n,\mathcal{S}_3^n u_n) &\le (1-\alpha_n)\zeta(\mathcal{S}_3^n u_n, \mathcal{S}_3^n u_n)+\alpha_n\zeta(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n)\\ &= \alpha_n\zeta(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n).\end{aligned}$$

Following from equation (32),

$$\lim_{n\to\infty}\zeta(w_n,\mathcal{S}_3^n u_n)=0. \tag{35}$$

In addition, we have

$$\zeta(w_n,u_n)\le\zeta(w_n,\mathcal{S}_3^n u_n)+\zeta(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n)+\zeta(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n, u_n). \tag{36}$$

Using equations (32), (34), (35), and (36), we have

$$\lim_{n\to\infty} \zeta(w_n, u_n) = 0. \tag{37}$$

Furthermore,

$$\zeta(\mathcal{S}_2^n w_n, w_n) \leq \zeta(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n) + \zeta(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, w_n),$$

by using equations (25) and (26), we have

$$\lim_{n\to\infty} \zeta(\mathcal{S}_2^n w_n, w_n) = 0. \tag{38}$$

It follows from equations (10), (26), and (38) that

$$\begin{aligned} \zeta(v_n, w_n) &= \zeta(\mathcal{H}(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, \beta_n), w_n) \\ &\leq (1-\beta_n)\zeta(\mathcal{S}_2^n w_n, w_n) + \beta_n \zeta(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, w_n) \\ &\to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{39}$$

Then, from equations (37) and (39), we have

$$\zeta(v_n, u_n) \leq \zeta(v_n, w_n) + \zeta(w_n, u_n) \to 0 \quad (\text{as } n \to \infty). \tag{40}$$

By the condition (ii), we know that

$$\zeta(u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} u_n) \leq \zeta(\mathcal{S}_1^n u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} u_n), \tag{41}$$

since

$$\begin{aligned} \zeta(\mathcal{S}_1^n u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} u_n) &\leq \zeta(\mathcal{S}_1^n u_n, \mathcal{S}_1^n v_n) + \zeta(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n) + \zeta(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} u_n) \\ &\leq h_n \zeta(u_n, v_n) + \zeta(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n) + h_n \zeta(v_n, u_n). \end{aligned} \tag{42}$$

Using equations (17) and (40) in equation (42), we obtain

$$\lim_{n\to\infty} \zeta(\mathcal{S}_1^n u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} u_n) = 0. \tag{43}$$

By using equations (41) and (43), we obtain

$$\lim_{n\to\infty} \zeta(u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} u_n) = 0. \tag{44}$$

From equations (26) and (37), we have

$$\begin{aligned} \zeta(u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} u_n) &\leq \zeta(u_n, w_n) + \zeta(w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n) + \zeta(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} u_n) \\ &\leq \zeta(u_n, w_n) + \zeta(w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n) + h_n \zeta(w_n, u_n) \to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{45}$$

Using equations (25), (26), and (37), we have

$$\begin{aligned} \zeta(\mathcal{S}_2^n u_n, u_n) &\leq \zeta(\mathcal{S}_2^n u_n, \mathcal{S}_2^n w_n) + \zeta(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n) + \zeta(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, w_n) + \zeta(w_n, u_n) \\ &\leq h_n \zeta(u_n, w_n) + \zeta(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n) + \zeta(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, w_n) + \zeta(w_n, u_n) \\ &\to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{46}$$

It follows from equations (45) and (46) that

$$\zeta(\mathcal{S}_2^n u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} u_n) \leq \zeta(\mathcal{S}_2^n u_n, u_n) + \zeta(u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} u_n) \to 0 \quad (\text{as } n \to \infty). \tag{47}$$

Using equation (17), we have

$$\begin{aligned} \zeta(u_{n+1}, \mathcal{S}_1^n v_n) &= \zeta(\mathcal{H}(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \gamma_n), \mathcal{S}_1^n v_n) \\ &\leq (1-\gamma_n)\zeta(\mathcal{S}_1^n v_n, \mathcal{S}_1^n v_n) + \gamma_n \zeta(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \mathcal{S}_1^n v_n) \\ &= \gamma_n \zeta(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \mathcal{S}_1^n v_n) \\ &\to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{48}$$

By equations (17) and (48), we have

$$\zeta(u_{n+1}, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) \le \zeta(u_{n+1}, \mathcal{S}_1^n v_n) + \zeta(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) \to 0 \quad (\text{as } n \to \infty). \tag{49}$$

Using equations (39) and (49), we have

$$\begin{aligned}\zeta(u_{n+1}, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}w_n) &\le \zeta(u_{n+1}, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) + \zeta(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}w_n)\\ &\le \zeta(u_{n+1}, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) + h_n\zeta(v_n, w_n) \to 0 \quad (\text{as } n \to \infty).\end{aligned} \tag{50}$$

Moreover, from equations (43) and (44), we have

$$\begin{aligned}\zeta(\mathcal{S}_1^n u_n, u_n) &\le \zeta(\mathcal{S}_1^n u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}u_n) + \zeta(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}u_n, u_n)\\ &\to 0 \quad (\text{as } n \to \infty).\end{aligned} \tag{51}$$

Using equations (45) and (51), we have

$$\zeta(\mathcal{S}_1^n u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) \le \zeta(\mathcal{S}_1^n u_n, u_n) + \zeta(u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) \to 0 \quad (\text{as } n \to \infty). \tag{52}$$

It follows from equations (40) and (52) that

$$\begin{aligned}\zeta(\mathcal{S}_1^n v_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) &\le \zeta(\mathcal{S}_1^n v_n, \mathcal{S}_1^n u_n) + \zeta(\mathcal{S}_1^n u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n)\\ &\le h_n\zeta(v_n, u_n) + \zeta(\mathcal{S}_1^n u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n)\\ &\to 0 \quad (\text{as } n \to \infty).\end{aligned} \tag{53}$$

Using equations (37), (48), and (53), we have

$$\begin{aligned}\zeta(u_{n+1}, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) &\le \zeta(u_{n+1}, \mathcal{S}_1^n v_n) + \zeta(\mathcal{S}_1^n v_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) + \zeta(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n)\\ &\le \zeta(u_{n+1}, \mathcal{S}_1^n v_n) + \zeta(\mathcal{S}_1^n v_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) + h_n\zeta(u_n, w_n)\\ &\to 0 \quad (\text{as } n \to \infty).\end{aligned} \tag{54}$$

In addition, using equations (34), (37), (40), (48), and (51), we obtain

$$\begin{aligned}\zeta(u_{n+1}, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}w_n) &\le \zeta(u_{n+1}, \mathcal{S}_1^n v_n) + \zeta(\mathcal{S}_1^n v_n, \mathcal{S}_1^n u_n) + \zeta(\mathcal{S}_1^n u_n, u_n) + \zeta(u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n)\\ &\quad + \zeta(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}w_n)\\ &\le \zeta(u_{n+1}, \mathcal{S}_1^n v_n) + h_n\zeta(v_n, u_n) + \zeta(\mathcal{S}_1^n u_n, u_n) + \zeta(u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n) + h_n\zeta(u_n, w_n)\\ &\to 0 \quad (\text{as } n \to \infty).\end{aligned} \tag{55}$$

From $(\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-2}w_{n-1}, u_n \in \mathcal{K}$ $(i = 1, 2, 3)$, and $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ are three asymptotically nonexpansive non-self-mappings, we obtain

$$\begin{aligned}\zeta(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}w_{n-1}, \mathcal{T}_i u_n) &= \zeta(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-2}w_{n-1}, \mathcal{T}_i(\mathcal{P}u_n))\\ &\le \max\{l_1^{(1)}, l_1^{(2)}, l_1^{(3)}\}\zeta((\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-2}w_{n-1}, \mathcal{P}u_n)\\ &\le \max\{l_1^{(1)}, l_1^{(2)}, l_1^{(3)}\}\zeta(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-2}w_{n-1}, u_n).\end{aligned} \tag{56}$$

Using equations (50), (54), (55), and (56), for $i = 1, 2, 3$, we obtain

$$\lim_{n\to\infty} \zeta(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}w_{n-1}, \mathcal{T}_i u_n) = 0. \tag{57}$$

By using equations (26) and (54), we have

$$\zeta(u_{n+1}, w_n) \le \zeta(u_{n+1}, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) + \zeta(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n, w_n) \to 0 \quad (\text{as } n \to \infty). \tag{58}$$

Moreover, for $i = 1, 2, 3$, we have

$$\begin{aligned}\zeta(u_n, \mathcal{T}_i u_n) &\le \zeta(u_n, \mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n) + \zeta(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n, (\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}w_{n-1})) + \zeta(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}w_{n-1}, \mathcal{T}_i u_n)\\ &\le \zeta(u_n, \mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n) + \max\{\sup_{n\ge1} l_1^{(1)}, \sup_{n\ge1} l_2^{(2)}, \sup_{n\ge1} l_3^{(3)}\}\zeta(u_n, w_{n-1}) + \zeta(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}w_{n-1}, \mathcal{T}_i u_n).\end{aligned}$$

Therefore, it follows from equations (34), (44), (45), (57), and (58) that

$$\lim_{n\to\infty}\zeta(u_n,\mathcal{T}_1u_n)=\lim_{n\to\infty}\zeta(u_n,\mathcal{T}_2u_n)=\lim_{n\to\infty}\zeta(u_n,\mathcal{T}_3u_n)=0.$$

Lastly, we prove that

$$\lim_{n\to\infty}\zeta(u_n,\mathcal{S}_1u_n)=\lim_{n\to\infty}\zeta(u_n,\mathcal{S}_2u_n)=\lim_{n\to\infty}\zeta(u_n,\mathcal{S}_3u_n)=0.$$

In fact, for $i=1,2,3$, we have

$$\begin{aligned}\zeta(u_n,\mathcal{S}_iu_n)&\leq\zeta(u_n,\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n)+\zeta(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n,\mathcal{S}_iu_n)\\&\leq\zeta(u_n,\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n)+\zeta(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n,\mathcal{S}_i^nu_n).\end{aligned}$$

So, it follows from equations (32), (34), (43), (44), (45), and (47) that

$$\lim_{n\to\infty}\zeta(u_n,\mathcal{S}_1u_n)=\lim_{n\to\infty}\zeta(u_n,\mathcal{S}_2u_n)=\lim_{n\to\infty}\zeta(u_n,\mathcal{S}_3u_n)=0.$$ □

Example 2.3. [58] Suppose that $\mathcal{K}=[-1,1]$ is a subset of a real line $\mathcal{X}$ with $\zeta(u,v)=|u-v|$ and $\mathcal{H}:\mathcal{X}\times\mathcal{X}\times[0,1]\to\mathcal{X}$ be defined by $\mathcal{H}(u,v,\alpha):=\alpha u+(1-\alpha)v,\forall u,v\in\mathcal{X},\alpha\in[0,1]$. We have that $(\mathcal{X},d,\mathcal{H})$ is a complete uniformly hyperbolic space with a monotone modulus of uniform convexity and $\emptyset\neq\mathcal{K}\subseteq\mathcal{X}$ is a closed and convex. Let $\mathcal{S},\mathcal{T}:\mathcal{K}\to\mathcal{K}$ be two mappings defined by

$$\mathcal{T}u=\begin{cases}-2\sin\dfrac{u}{2}, & u\in[0,1],\\ 2\sin\dfrac{u}{2}, & u\in[-1,0)\end{cases}$$

and

$$\mathcal{S}u=\begin{cases}u, & u\in[0,1],\\ -u, & u\in[-1,0).\end{cases}$$

We have $F(\mathcal{T})=\{0\}$ and $F(\mathcal{S})=\{u\in\mathcal{K};0\leq u\leq1\}$. We prove that $\mathcal{T}$ is nonexpansive. Indeed, assume that $u,v\in[0,1]$ or $u,v\in[-1,0)$. Then,

$$\zeta(\mathcal{T}u,\mathcal{T}v)=|\mathcal{T}u-\mathcal{T}v|=2\left|\sin\frac{u}{2}-\sin\frac{v}{2}\right|\leq|u-v|=\zeta(u,v).$$

Assume that $u\in[0,1]$, $v\in[-1,0)$ or $u\in[-1,0)$, $v\in[0,1]$. Then,

$$\zeta(\mathcal{T}u,\mathcal{T}v)=|\mathcal{T}u-\mathcal{T}v|=2\left|\sin\frac{u}{2}+\sin\frac{v}{2}\right|=4\left|\sin\frac{u+v}{4}\cos\frac{u-v}{4}\right|\leq|u+v|\leq|u-v|=\zeta(u,v).$$

Hence, $\mathcal{T}$ is nonexpansive. That is, $\mathcal{T}$ is an asymptotically nonexpansive mapping with $k_n=1$, $\forall n\geq1$. Similarly, we can prove that $\mathcal{S}$ is an asymptotically nonexpansive mapping with $l_n=1$ $\forall n\geq1$. Then, to demonstrate that $\mathcal{S}$ and $\mathcal{T}$ fulfill condition (ii) of Lemma 2.2, we must examine the following cases:

Case (i). Let $u,v\in[0,1]$. We have

$$\zeta(u,\mathcal{T}v)=|u-\mathcal{T}v|=\left|u+2\sin\frac{v}{2}\right|=|\mathcal{S}u-\mathcal{T}v|=\zeta(\mathcal{S}u,\mathcal{T}v).$$

Case (ii). Let $u,v\in[-1,0)$. We have

$$\zeta(u,\mathcal{T}v)=|u-\mathcal{T}v|=\left|u-2\sin\frac{v}{2}\right|\leq\left|-u-2\sin\frac{v}{2}\right|=|\mathcal{S}u-\mathcal{T}v|=\zeta(\mathcal{S}u,\mathcal{T}v).$$

Case (iii). Let $u\in[-1,0)$ and $v\in[0,1]$. We have

$$\zeta(u,\mathcal{T}v)=|u-\mathcal{T}v|=\left|u+2\sin\frac{v}{2}\right|\leq\left|-u+2\sin\frac{v}{2}\right|=|\mathcal{S}u-\mathcal{T}v|=\zeta(\mathcal{S}u,\mathcal{T}v).$$

Case (iv). Let $u \in [0, 1]$ and $v \in [-1, 0]$. We have

$$\zeta(u, \mathcal{T}v) = |u - \mathcal{T}v| = \left|u - 2\sin\frac{v}{2}\right| = |\mathcal{S}u - \mathcal{T}v| = \zeta(\mathcal{S}u, \mathcal{T}v).$$

It follows that the condition (ii) in Lemma 2.2 is satisfied. Moreover, we take $\alpha_n = \frac{n}{2n+1}$, $\beta_n = \frac{n}{3n+1}$, and $\gamma_n = \frac{n}{4n+1}$ $\forall n \geq 1$. We have that the conditions of Lemma 2.2 are fulfilled. Consequently, a convergence of the sequence $\{u_n\}$ produced by equation (10) to the point $0 \in F(\mathcal{T}) \cap F(\mathcal{S})$ can be obtained.

Now, we provide some numerical examples to illustrate the convergence behavior of iteration (8) comparing with iteration (10). All program computations are performed on an HP Laptop Intel(R) Core(TM) i7-1165G7, 16.00 GB RAM. We choose the starting point at $u_1 = 1$, and the stop criterion is defined by $||u_n - 0|| < 10^{-15}$. The convergence performance of both iterations are shown in Table 1 and Figure 1.

Under the same condition settings shown in Example 2.3, by Table 1 and Figure 1, our proposed iteration (10) has a better performance in both the time taken by CPU-runtime to reach the convergence and the number of iterations when comparing with iteration (8).

The next step is to prove strong convergence theorems.

Theorem 2.4. *Let $\mathcal{K}, \mathcal{X}, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$ satisfy the hypotheses of Lemma 2.2, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[\varepsilon, 1-\varepsilon]$, $\exists \varepsilon \in (0, 1)$, and $\mathcal{S}_i$ and $\mathcal{T}_i$ for any $i = 1, 2, 3$ satisfy the condition (ii) in Lemma 2.2. Suppose that there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ $\forall r \in (0, \infty)$ such that*

$$f(\zeta(u, \Omega)) \leq \zeta(u, \mathcal{S}_1 u) + \zeta(u, \mathcal{S}_2 u) + \zeta(u, \mathcal{S}_3 u) + \zeta(u, \mathcal{T}_1 u) + \zeta(u, \mathcal{T}_2 u) + \zeta(u, \mathcal{T}_3 u)$$

$\forall u \in \mathcal{K}$, where $\zeta(u, \Omega) = \inf\{d(u, p) : p \in \Omega\}$. Then, the sequence $\{u_n\}$ defined by equation (10) converges strongly to a common fixed point of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$.

Table 1: Computational result for all settings in Example 2.3

	Iteration (8)	**Iteration (10)**
No of Iter.	26	10
CPU time (s)	0.0035	0.0027

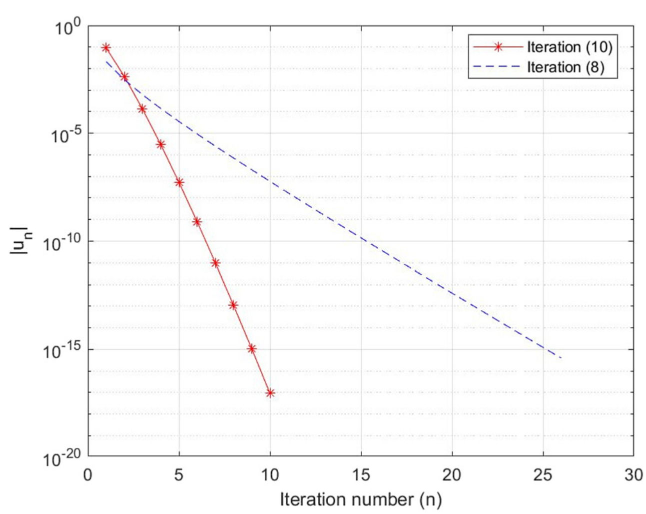

Figure 1: The value of $\{u_n\}$ generated by iterations (8) and (10).

Proof. From Lemma 2.2, we have $\lim_{n\to\infty}\zeta(u_n, \mathcal{S}_i u_n) = 0 = \lim_{n\to\infty}\zeta(u_n, \mathcal{T}_i u_n)(i = 1, 2, 3)$. It follows from the hypothesis that

$$\lim_{n\to\infty} f(\zeta(u_n, \Omega)) \le \lim_{n\to\infty}(\zeta(u_n, \mathcal{S}_1 u_n) + \zeta(u_n, \mathcal{S}_2 u_n) + \zeta(u_n, \mathcal{S}_3 u_n) + \zeta(u_n, \mathcal{T}_1 u_n) + \zeta(u_n, \mathcal{T}_2 u_n) + \zeta(u_n, \mathcal{T}_3 u_n)) = 0.$$

Hence, $\lim_{n\to\infty} f(\zeta(u_n, \Omega)) = 0$. From $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$, $\forall r \in (0, \infty)$. Using Lemma 2.1, we have $\lim_{n\to\infty}\zeta(u_n, \Omega)$ exists. It follows that $\lim_{n\to\infty}\zeta(u_n, \Omega) = 0$. Next, we prove that $\{u_n\}$ is a Cauchy sequence in $\mathcal{K}$. Using equation (13), we have

$$\zeta(u_{n+1}, p) \le (1 + (h_n^3 - 1))\zeta(u_n, p)$$

$\forall n \ge 1$, where $h_n = max\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\}$ and $p \in \Omega$. For all $m, n > n \ge 1$, we obtain

$$\begin{aligned}\zeta(u_m, p) &\le (1 + (h_{m-1}^3 - 1))\zeta(u_{m-1}, p)\\ &\le e^{h_{m-1}^3 - 1}\zeta(u_{m-1}, p)\\ &\le e^{h_{m-1}^3 - 1}e^{h_{m-2}^3 - 1}\zeta(u_{m-2}, p)\\ &\vdots\\ &\le e^{\sum_{i=n}^{m-1}(h_i^3 - 1)}\zeta(u_n, p)\\ &\le M\zeta(x_n, z),\end{aligned}$$

where $M = e^{\sum_{i=1}^{\infty}(h_i^3 - 1)}$. So, for all $p \in \Omega$, we obtain

$$\zeta(u_n, u_m) \le \zeta(u_n, p) + \zeta(u_m, p) \le (1 + M)\zeta(u_n, p).$$

Taking the infimum over all $p \in \Omega$, we have

$$\zeta(u_n, u_m) \le (1 + M)\zeta(u_n, \Omega).$$

It follows from $\lim_{n\to\infty}\zeta(u_n, \Omega) = 0$ that $\{u_n\}$ is a Cauchy sequence. Since $\mathcal{K}$ is a closed subset in a complete hyperbolic space X, then $\{u_n\}$ converges strongly to some $p^* \in \mathcal{K}$. It is easy to see that $F(\mathcal{S}_1), F(\mathcal{S}_2), F(\mathcal{S}_3), F(\mathcal{T}_1), F(\mathcal{T}_2)$, and $F(\mathcal{T}_3)$ are closed, i.e., Ω is closed subset of $\mathcal{K}$. Since $\lim_{n\to\infty}\zeta(u_n, \Omega) = 0$ gives that $\zeta(p^*, \Omega) = 0$, we have $p^* \in \Omega$. The proof is completed. □

Theorem 2.5. *Considering the assumption in Lemma* 2.2 *and if one of* $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$, *and* $\mathcal{T}_3$ *is completely continuous, then the sequence* $\{u_n\}$ *defined by* (10) *converges strongly to a point in* Ω.

Proof. Let $\mathcal{S}_1$ be completely continuous. By Lemma 2.1, $\{x_n\}$ is bounded. This means that there is a subsequence $\{\mathcal{S}_1 u_{n_j}\}$ of $\{\mathcal{S}_1 u_n\}$ such that $\{\mathcal{S}_1 u_{n_j}\}$ converges strongly to some $\xi^* \in \mathcal{K}$. Moreover, by Lemma 2.2, we have

$$\lim_{j\to\infty}\zeta(u_{n_j}, \mathcal{S}_1 u_n) = \lim_{j\to\infty}\zeta(u_{n_j}, \mathcal{S}_2 u_n) = \lim_{j\to\infty}\zeta(u_{n_j}, \mathcal{S}_3 u_n) = 0 \quad \text{and}$$

$$\lim_{j\to\infty}\zeta(u_{n_j}, \mathcal{T}_1 u_n) = \lim_{j\to\infty}\zeta(u_{n_j}, \mathcal{T}_2 u_n) = \lim_{j\to\infty}\zeta(u_{n_j}, \mathcal{T}_3 u_n) = 0,$$

which implies that,

$$\zeta(u_{n_j}, \xi^*) \le \zeta(u_{n_j}, \mathcal{S}_1 u_{n_j}) + \zeta(\mathcal{S}_1 u_{n_j}, \xi^*) \to 0 \quad (\text{as } j \to \infty).$$

Hence, $\mathcal{S}_1 u_{n_j} \to \xi^* \in \mathcal{K}$. Consequently,

$$\zeta(\xi^*, \mathcal{S}_i\xi^*) = \lim_{j\to\infty}\zeta(u_{n_j}, \mathcal{S}_i u_{n_j}) = 0.$$

Since $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$ are continuous, for $i = 1, 2, 3$. By Lemma 2.2, we have

$$\zeta(\xi^*, \mathcal{T}_i\xi^*) = \lim_{j\to\infty}\zeta(u_{n_j}, \mathcal{T}_i u_{n_j}) = 0.$$

This implies that $\xi^* \in F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(\mathcal{S}_3) \cap F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \cap F(\mathcal{T}_3)$. Using Lemma 2.1, we obtain $\lim_{n\to\infty}\zeta(u_n, \xi^*)$ exists, and so $\lim_{n\to\infty}\zeta(u_n, \xi^*) = 0$. It follows that $\{u_n\}$ converges strongly to a common fixed point of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$. The proof is completed. □

3 Conclusions

Authors constructed a mixed-type SP-iteration to approximate a common fixed point of three asymptotically nonexpansive self and nonself-mappings in the setting of uniformly convex hyperbolic spaces. The mixed-type SP-iteration process (10) is a translation of the SP-iteration scheme from Banach spaces to hyperbolic spaces. Example 2.3 is also provided as an illustration. The authors established strong convergence results that outperformed delta and weak convergence results.

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