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Research Article

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Results on the modified degenerate Laplace-type integral associated with applications involving fractional kinetic equations

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Abstract: Recently, integral transforms are a powerful tool used in many areas of mathematics, physics, engineering, and other fields and disciplines. This article is devoted to the study of one important integral transform, which is called the modified degenerate Laplace transform (MDLT). The fundamental formulas and properties of the MDLT are obtained. Furthermore, as an application of the acquired MDLT, we solved a simple differential equation and fractional-order kinetic equations. The outcomes covered here are general in nature and easily reducible to new and known outcomes.

Keywords: degenerate functions, modified degenerate Laplace transforms, fractional kinetic equations

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1 Introduction

The study of integral transforms is a growing field of research that has become a fundamental tool in solving several fractional differential equations in mathematical physics, mathematical statistics, mathematical modeling, control theory, mathematical biology, and other scientific fields (see, e.g., [1–6]). One particular integral transform that frequently appears in recent studies and applications is the Laplace transform (see, e.g., [7–11]). In recent years, various generalizations of this transform have been proposed by many authors, for instance, Akel et al. [12], Ortigueira and Machado [13], Jarad and Abdeljawad [14], Ganie and Jain [15], and others [16,17].

Nowadays, application of the Laplace transform on fractional kinetic equations involving different special functions such as the Mittag-Leffler function, the Galué Struve function, the generalized Galué type Struve function, the Hurwitz-Lerch zeta function, the extended τ-Gauss hypergeometric function, the (p, q)-extended τ-hypergeomtric and confluent hypergeometric functions have been discussed by several researchers, e.g., the reader may refer to recent works [4,18–25].

Gaining insight from the recently mentioned works, in this study, we discuss further properties and applications of the modified degenerate Laplace integral transform (MDLIT) [26,27], which is a generalization of the Laplace transform. This article is organized as follows: in Section 2, we investigated the convergence properties, the transition theorem, and the convolution theorem of the MDLIT; the composition formulas for the differential and integral operators with the MDLIT are considered in Section 3; as clarification of the applications to the general theory of differential equations and fractional differential equations, a simple

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ordinary differential equation, and fractional kinetic equations are solved by the MDLIT method in Section 4; and finally, we append a conclusion on the outcomes in Section 5.

2 The MDLIT and some properties

In recent years, there have been a wide variety of studies involving various degenerate functions and polynomials with various applications (see, e.g., [26–35]). Especially, Kim and Kim [28] defined the degenerate Laplace integral transform for $\lambda \in (0, \infty)$ as follows:

$$\mathcal{F}(\varphi) = \mathbb{L}_{\lambda}(\varphi) = \mathfrak{L}_{\lambda}\{f(\eta);\ \varphi\} = \int_{0}^{\infty}(1 + \lambda\eta)^{-\frac{\varphi}{\lambda}} f(\eta)\mathrm{d}\eta, \tag{1}$$

provided that the improper integral converges and $(1 + \lambda\eta)^{-\frac{\varphi}{\lambda}}$ is the kernel of the transformation. This was motivated by the degenerate exponential function of two variables, which is defined as follows:

$$e_{\lambda}^{\eta} = (1 + \lambda\eta)^{\frac{1}{\lambda}}, \quad \lambda \in (0, \infty). \tag{2}$$

It is clear that $\lim_{\lambda\to 0} e_{\lambda}^{\eta} = e^{\eta}$. The authors also investigated some properties and formulas related to the degenerate Laplace transformation in [28].

Within this framework, Kim et al. [26] introduced the MDLIT of a function $f(\eta)$ by the form

$$\mathcal{ML}_{\lambda}(\varphi) = \mathbb{ML}_{\lambda}\{f(\eta);\ \varphi\} = \int_{0}^{\infty}(1 + \lambda\varphi)^{-\frac{\eta}{\lambda}} f(\eta)\mathrm{d}\eta,\ \ (\lambda \in (0, \infty), \mathrm{Re}(\eta) \ge 0). \tag{3}$$

Note that $\lim_{\lambda\to 0}(1 + \lambda\varphi)^{-\frac{\eta}{\lambda}} = e^{-\varphi\eta}$ and also that equation (3) becomes the following classical Laplace transform:

$$\mathcal{ML}(\varphi) = \mathbb{ML}\{f(\eta);\ \varphi\} = \int_{0}^{\infty} e^{-\varphi\eta} f(\eta)\mathrm{d}\eta. \tag{4}$$

In addition, we can state the linearity of the MDLIT (equation (3)) as follows:

$$\mathbb{ML}_{\lambda}\{\alpha_1 f(\eta) + \alpha_2 g(\eta);\ \varphi\} = \alpha_1 \mathbb{ML}_{\lambda}\{f(\eta);\ \varphi\} + \alpha_2 \mathbb{ML}_{\lambda}\{f(\eta);\ \varphi\},$$

where α_2 and α_2 are coefficients independent of η.

Furthermore, if we refer to the kernel function $\mathcal{K}(\varphi, \eta) = (1 + \lambda\varphi)^{-\frac{\eta}{\lambda}}$, then we have the following relation:

$$\mathcal{D}_{1,\mu}^{\beta,\lambda}(t) = \int_{0}^{\infty} \varphi^{\beta-1}\mathcal{K}(\varphi, 1)\ e^{\frac{-t}{\varphi^{\mu}}}\mathrm{d}\varphi. \tag{5}$$

As an example, the MDLIT of some basic functions can be obtained immediately from the following Axiom.

Axiom 2.1 The following formulas hold true:

$$\mathbb{ML}_{\lambda}\{\eta^{\varrho-1};\ \varphi\} = \left(\frac{\lambda}{\log(1 + \lambda\varphi)}\right)^{\varrho}\Gamma(\varrho)(\lambda \in (0, 1],\ \ \varphi, \varrho \in \mathbb{C}, \mathrm{Re}(\varphi) > 0, \mathrm{Re}(\varrho) > 0), \tag{6}$$

$$\mathbb{ML}_{\lambda}\{e^{\eta};\ \varphi\} = \frac{\lambda}{\log(1 + \lambda\varphi) - \lambda}\ (\mathrm{Re}(\varphi) > 0,\ \ \lambda \in (0, 1]), \tag{7}$$

$$\mathbb{ML}_{\lambda}\{\sin(\eta);\ \varphi\} = \frac{\lambda^2}{(\log(1 + \lambda\varphi))^2\ + \lambda^2}, (\mathrm{Re}(\varphi) > 0,\ \ \lambda \in (0, 1]), \tag{8}$$

and

$$\mathbb{ML}_\lambda\{\cosh(\eta);\ \varphi\} = \frac{\frac{\log(1+\lambda\varphi)}{\lambda}}{\left(\frac{\log(1+\lambda\varphi)}{\lambda}\right)^2 - 1},\ \ (\mathrm{Re}(\varphi) > 0,\ \ \lambda \in (0,1]). \tag{9}$$

Remark 2.1. For $\lambda \to 0$ in Axiom 2.1 and using the transform (4), we obtain the corresponding results for the classical Laplace transform as follows:

$$\mathbb{ML}\{\eta^{\varrho-1};\ \varphi\} = \frac{1}{\varphi^{\varrho}}\Gamma(\varrho)\ (\varrho, \varphi \in \mathbb{C}, \mathrm{Re}(\varrho) > 0, \mathrm{Re}(\varphi) > 0), \tag{10}$$

$$\mathbb{ML}\{e^{\eta};\ \varphi\} = \frac{1}{\varphi - 1}(\varphi \in \mathbb{C}, \mathrm{Re}(\varphi) > 1), \tag{11}$$

$$\mathbb{ML}\{\sin(\eta);\ \varphi\} = \frac{1}{\varphi^2 + 1}, (\varphi \in \mathbb{C}, \mathrm{Re}(\varphi) > 0), \tag{12}$$

and

$$\mathbb{ML}\{\cosh(\eta);\ \varphi\} = \frac{1}{\varphi^2 - 1}, (\varphi \in \mathbb{C}, \mathrm{Re}(\varphi) > 0). \tag{13}$$

Now, we present the convergence property, translation theorem, and convolution theorem of the MDLIT defined in equation (3), which were not discussed by Kim et al. [26].

2.1 Convergence property

The convergence property for the MDLIT is given in this section by Theorem 2.1 below. First, we present the following lemma, which is needed in the proof of the theorem.

Lemma 2.1. *If $f(\eta)$ is integrable over a finite interval (x, y), $0 < x < \eta < y$, and $\lambda \in (0, \infty)$ with $\varepsilon \in \mathbb{R}$ such that,*
(*i*) *for*

$$\int_y^z e^{-\varepsilon\eta} f(\eta)\mathrm{d}\eta y > 0,$$

resort to a finite limit as $z \to \infty$.
(*ii*) *for*

$$\int_\xi^x |f(\eta)|\mathrm{d}\eta \quad x > 0,$$

resort to a finite limit as $\xi \to 0^+$.

Then, the MDLIT $\mathbb{ML}_\lambda\{f(\eta);\ \varphi\}$ exists for $\mathrm{Re}\left(\frac{\log(1+\lambda\varphi)}{\lambda}\right) > \varepsilon$ for $\varphi \in \mathbb{C}$.

Proof. If x and ξ are arbitrary ($\xi < x$) and $\varepsilon > 0$, then we have

$$\left|\int_\xi^x (1+\lambda\varphi)^{-\frac{\eta}{\lambda}} f(\eta)\mathrm{d}\eta\right| \le \int_\xi^x e^{-\varepsilon\eta}|f(\eta)|\mathrm{d}\eta \le \int_\xi^x |f(\eta)|\mathrm{d}\eta, \tag{14}$$

and for $\varepsilon < 0$, we observe that

$$\left|\int_{\xi}^{x}(1+\lambda\varphi)^{-\frac{\eta}{\lambda}}f(\eta)\mathrm{d}\eta\right| \le \int_{\xi}^{x}e^{-\varepsilon\eta}|f(\eta)|\mathrm{d}\eta \le e^{-\varepsilon x}\int_{\xi}^{x}|f(\eta)|\mathrm{d}\eta. \tag{15}$$

From **(ii)**, the integral

$$\int_{0}^{x}(1+\lambda\varphi)^{-\frac{\eta}{\lambda}}f(\eta)\mathrm{d}\eta \tag{16}$$

exists for arbitrary positive values of x. Therefore, by using the given conditions in addition to the properties of the integral

$$\int_{0}^{z}(1+\lambda\varphi)^{-\frac{\eta}{\lambda}}f(\eta)\mathrm{d}\eta = \left(\int_{0}^{x}+\int_{a}^{y}+\int_{y}^{z}\right)(1+\lambda\varphi)^{-\frac{\eta}{\lambda}}f(\eta)\mathrm{d}\eta, \tag{17}$$

we arrive at the required result at $z \to \infty$ □

Theorem 2.1. *Assume that*

(*i*) *$f(\eta)$ is integrable over a finite limit (x, y), $0 < x < \eta < y$,*

(*ii*) *for arbitrary positive x, the integral $\int_{\xi}^{x}|f(\eta)|\mathrm{d}\eta$ resort to a finite limit as $\xi \to 0^{+}$,*

(*iii*) *$f(\eta) = O(e^{\varepsilon\eta})$, $\varepsilon > 0$ as $\eta \to \infty$, where $O(.)$ is the standard big O notation, which means $f(\eta)$ is of order not exceeding $e^{\varepsilon\eta}$.*

Then, the MDLIT defined in equation (3) *converges absolutely if* $\mathrm{Re}\left(\frac{\log(1+\lambda\varphi)}{\lambda}\right) > \varepsilon,\ \lambda > 0$.

Proof. By using Lemma 2.1 and fundamental integrals, we can easily obtain the proof of Theorem 2.1. The details are omitted. □

Corollary 2.1. *Under the conditions of the hypothesis in Theorem* 2.1 *and $\lambda \to 0$, the Laplace integral transform $\mathcal{ML}(\varphi)$ defined in equation* (4) *converges absolutely for* $\mathrm{Re}(\varphi) > \varepsilon$.

2.2 Transition and convolution theorems

The following theorem shows the rule for transition on the η-axis.

Theorem 2.2. *Let χ be a nonnegative real number, $\lambda \in (0, 1)$, and $\mathcal{F}(\varphi) = \mathbb{ML}_{\lambda}\{f(\eta);\ \varphi\}$, where $f(\eta) = 0$ for $\eta < 0$, then MDLIT of $f(\eta - \chi)$, where $f(\eta - \chi) = 0$ for $\eta < \chi$, is represented as follows*:

$$\mathbb{ML}_{\lambda}\{f(\eta-\chi);\ \varphi\} = [1+\lambda\varphi]^{\frac{-\chi}{\lambda}}\ \mathcal{F}(\varphi). \tag{18}$$

Proof. For this proof, we consider a function $f(\eta)$ such that $f(\eta) = 0$ for $\eta < 0$. Let $f(\eta - \chi)$ be the function obtained by transiting the η-axis by an amount $\chi > 0$. Note that $f(\eta - \chi) = 0$ for $\eta < \chi$. Thus, we obtain the MDLIT of $f(\eta - \chi)$ as follows. By changing the variable $\eta = \theta - \chi$ in the definition of MDLIT given in equation (3), we have

$$\mathcal{F}(\varphi) = \int_{\chi}^{\infty}f(\theta-\chi)[1+\lambda\varphi]^{-\frac{\theta-\chi}{\lambda}}\mathrm{d}\theta. \tag{19}$$

Multiplying both sides by $[1+\lambda\varphi]^{\frac{-\chi}{\lambda}}$, we have

$$\mathcal{F}(\varphi)[1+\lambda\varphi]^{\frac{-\chi}{\lambda}} = \int_{\chi}^{\infty} f(\theta-\chi)[1+\lambda\varphi]^{-\frac{\theta}{\lambda}}\mathrm{d}\theta. \tag{20}$$

Since $f(\theta-\chi)=0$ for $\theta<\chi,\ \chi\geq 0$, we obtain

$$\mathcal{F}(\varphi)[1+\lambda\varphi]^{\frac{-\chi}{\lambda}} = \int_{0}^{\infty} f(\theta-\chi)[1+\lambda\varphi]^{-\frac{\theta}{\lambda}}\mathrm{d}\theta = \mathbb{ML}_{\lambda}\{f(\theta-\chi);\,\varphi\}. \tag{21}$$

By changing the variable θ to η, we arrive at the required result (equation (18)). □

Next, we prove the rule for the convolution of two functions.

Theorem 2.3. (Convolution theorem) *For $\lambda\in(0,1]$ and let $\mathcal{F}(\varphi)$ and $\mathcal{G}(\varphi)$ be the MDLIT of functions $f(\eta)$ and $g(\eta)$, respectively, then the convolution $\mathcal{F}(\varphi)*\mathcal{G}(\varphi)$ is the MDLIT of the function $\int_0^{\eta} f(\eta-\chi)g(\chi)\mathrm{d}\chi$ as follows*:

$$\mathcal{F}(\varphi)*\mathcal{G}(\varphi) = \mathbb{ML}_{\lambda}\left\{\int_{0}^{\eta} f(\eta-\chi)g(\chi)\mathrm{d}\chi;\,\varphi\right\} = \mathbb{ML}_{\lambda}\{f(\eta);\,\varphi\}\mathbb{ML}_{\lambda}\{g(\eta);\,\varphi\}. \tag{22}$$

Proof. Consider the following product:

$$\mathcal{F}(\varphi)*\mathcal{G}(\varphi) = \int_{0}^{\infty} f(\theta)[1+\lambda\varphi]^{-\frac{\theta}{\lambda}}\mathrm{d}\theta \int_{0}^{\infty} g(\vartheta)[1+\lambda\varphi]^{-\frac{\vartheta}{\lambda}}\mathrm{d}\vartheta. \tag{23}$$

Since the integrals are uniformly convergent, we find that

$$\mathcal{F}(\varphi)*\mathcal{G}(\varphi) = \int_{0}^{\infty}\int_{0}^{\infty} f(\theta)g(\vartheta)[1+\lambda\varphi]^{-\frac{\theta+\vartheta}{\lambda}}\mathrm{d}\theta\wedge\mathrm{d}\vartheta,\ \ \theta>0,\ \ \vartheta>0, \tag{24}$$

where $\wedge$ denotes the wedge product. Setting $\eta=\theta+\vartheta$ and $\chi=\vartheta$, the differential in the (η,χ)-plane is $\mathrm{d}\theta\wedge\mathrm{d}\vartheta=\mathbf{J}\mathrm{d}\eta\wedge\mathrm{d}\chi=\mathrm{d}\eta\wedge\mathrm{d}\chi$, where $\mathbf{J}$ is the Jacobian. Hence, the new domain of integration is between the η-axis and the line $\eta=\chi$ in the first quadrant, thus we obtain

$$\mathcal{F}(\varphi)*\mathcal{G}(\varphi) = \int_{0}^{\infty}[1+\lambda\varphi]^{-\frac{\eta}{\lambda}}\left(\int_{0}^{\eta} f(\eta-\chi)g(\chi)\mathrm{d}\chi\right)\mathrm{d}\eta. \tag{25}$$

This completes the proof. □

3 Differential and integral operators

The composition representations for the MDLIT with differential and integral operators are established in the following theorem.

Theorem 3.1. *For $\lambda\in(0,1]$ and $\mathcal{F}(\varphi)=\mathbb{ML}_{\lambda}\{f(\eta);\,\varphi\}$, we have*

(*i*)

$$\mathbb{ML}_{\lambda}\{f^{(m)}(\eta);\,\varphi\} = \left\{\frac{\log(1+\lambda\varphi)}{\lambda}\right\}^{m}\mathcal{F}(\varphi) - \sum_{k=1}^{m}\left\{\frac{\log(1+\lambda\varphi)}{\lambda}\right\}^{m-k} f^{(k-1)}(0^{+}), \tag{26}$$

provided that $f(\eta)$ *and its derivatives up to order m exist, and*

$$f(0^{+}) = \lim_{\varsigma\to 0} f^{(k-1)}(0^{+} + \varsigma).$$

(*ii*)

$$\mathbb{ML}_{\lambda}\{f^{(-m)}(\eta);\, \varphi\} = \left\{\frac{\lambda}{\log(1+\lambda\varphi)}\right\}^{m} \mathcal{F}(\varphi) - \sum_{k=1}^{m}\left\{\frac{\lambda}{\log(1+\lambda\varphi)}\right\}^{m-k+1} f^{(-k)}(0^{+}), \tag{27}$$

provided that $f(\eta)$ *and its integrals up to order m exist.*

(*iii*)

$$\mathfrak{D}^{m}\{\mathcal{F}(\varphi)\} = \left\{\frac{\lambda}{(1+\lambda\varphi)}\right\}^{m} \mathbb{ML}_{\lambda}\left\{(-1)^{m}\left(\frac{\eta}{\lambda}\right)_{m} f(\eta);\, \varphi\right\}, \tag{28}$$

where $\mathfrak{D}^{m} = \frac{\mathrm{d}^{m}}{\mathrm{d}\varphi^{m}}$, $\mathrm{Re}(\varphi) > 0$.

Proof. The result (i) is established in Theorem 9 in the study by Kim et al. [26].

To prove (ii), let $f(\eta)$ be an integrable function such that

$$\int_{0}^{\eta} f(\chi)\mathrm{d}\chi = \int f(\eta) - f^{(-1)}(0^{+}), \tag{29}$$

where $f^{(-1)}(0^{+}) = \int f(\eta)\mathrm{d}\eta\,|_{\eta=0}$. Applying integration by part and the transformation in equation (3), we obtain

$$\mathcal{F}(\varphi) = f^{(-1)}(0^{+}) + \frac{\log(1+\lambda\varphi)}{\lambda}\mathbb{ML}_{\lambda}\{f^{(-1)}(\eta);\, \varphi\}. \tag{30}$$

Applying integration by parts $m-1$ times and after minor simplification, we obtain the desired result in (ii).

To demonstrate (iii), by employing relation (3), we see that

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\mathcal{F}(\varphi) = \int_{0}^{\infty}\frac{\lambda}{1+\lambda\varphi}\left(-\frac{\eta}{\lambda}\right)(1+\lambda\varphi)^{-\frac{\eta}{\lambda}} f(\eta)\mathrm{d}\eta = -\frac{1}{1+\lambda\varphi}\,\mathbb{ML}_{\lambda}\{\eta f(\eta);\, \varphi\}.$$

Recursive application of this procedure eventually gives the asserted result in (iii). □

Many special cases can be obtained from Theorem 3.1, such as the following corollaries:

Corollary 3.1. *For all* $\lambda \to 0$ *in* (*i*) *of Theorem* 3.1 *and by virtue of the integral equation* (4), *we have*

$$\lim_{\lambda\to 0}\mathbb{ML}_{\lambda}\{f^{(m)}(\eta);\, \varphi\} = \varphi^{m}\mathcal{F}(\varphi) - \sum_{k=1}^{m}\varphi^{m-k} f^{(k-1)}(0^{+}) = \mathbb{ML}\{f^{(m)}(\eta);\, \varphi\}.$$

Corollary 3.2. *For all* $\lambda \to 0$ *in* (*ii*) *of Theorem* 3.1, *and by invoking of the integral in equation* (4), *we have*

$$\lim_{\lambda\to 0}\mathbb{ML}_{\lambda}\{f^{(-m)}(\eta);\, \varphi\} = \left(\frac{1}{\varphi}\right)^{m}\mathcal{F}(\varphi) - \sum_{k=1}^{m}\left(\frac{1}{\varphi}\right)^{m-k+1} f^{(k-1)}(0^{+}) = \mathbb{ML}\{f^{(-m)}(\eta);\, \varphi\}.$$

Remark 3.1. For $m = 1$ and $\lambda \in (0, 1]$ in (iii) of Theorem 3.1, we obtain a known result in Theorem 10 in the study by Kim et al. [26].

Remark 3.2. Many particular addenda of the similar results for the classical Laplace transform can be inserted from the previous results for $\lambda \to 0$, (see, e.g., [1,7]).

4 Applications

4.1 Ordinary differential equation

Example 4.1. *Second-order initial value problem*

Let us solve the following differential equation to determine $U(\eta)$:

$$z''(\eta) + z(\eta) = 2\exp(\eta), \quad z(0) = 1, \quad \text{and} \quad z'(0) = 2. \tag{31}$$

Suppose that the MDLIT of $z(\eta)$ be $Z(\varphi)$. Taking the MDLIT on both sides of equation (31), using the relation (26), and from Proposing 2.1, we arrive at

$$\mathbb{ML}_\lambda\{z''(\eta);\ \varphi\} + \mathbb{ML}_\lambda\{z(\eta);\ \varphi\} = 2\mathbb{ML}_\lambda\{e^\eta;\ \varphi\} = \left\{\frac{\log(1+\lambda\varphi)}{\lambda}\right\}^2 Z(\varphi) - \left\{\frac{\log(1+\lambda\varphi)}{\lambda}\right\} - 2 + Z(\varphi)$$
$$= \frac{2}{\left\{\frac{\log(1+\lambda\varphi)}{\lambda}\right\} - 1}.$$

After simplification, we have

$$Z(\varphi) = \frac{1}{\left\{\frac{\log(1+\lambda\varphi)}{\lambda}\right\} - 1} + \frac{1}{\left\{\frac{\log(1+\lambda\varphi)}{\lambda}\right\}^2 + 1}.$$

We thus obtain the solution in the form

$$Z(\eta) = \exp(\eta) + \sin(\eta). \tag{32}$$

Remark 4.1. The method presented here by the MDLIT can be used to solve any other differential equations.

4.2 Solve fractional kinetic equations

If an arbitrary reaction is characterized by a time-dependent $\mathcal{X} = \mathcal{X}(\eta)$, then the rate of change $\frac{d\mathcal{X}(\eta)}{d\eta}$ is given as follows:

$$\frac{d\mathcal{X}(\eta)}{d\eta} = -\phi + \psi,$$

where ϕ is the destruction rate and ψ is the production rate of $\mathcal{X}$.

Mathai and Haubold [36] established a functional differential equation involving the rate of change of reaction, the destruction rate, and the production rate as follows:

$$\frac{d\mathcal{X}(\eta)}{d\eta} = -\phi(\mathcal{X}_\eta) + \psi(\mathcal{X}_\eta), \tag{33}$$

where $\mathcal{X} = \mathcal{X}(\eta)$ is the rate of reaction, $\phi(\mathcal{X}_\eta)$ is the rate of destruction, $\psi(\mathcal{X}_\eta)$ is the rate of production, and $\mathcal{X}_\eta$ denotes the function defined by $\mathcal{X}_\eta(\eta^*) = \mathcal{X}(\eta) - \eta^*,\ \eta^* > 0$.

A special case of equation (33), when spatial fluctuations or homogeneities in the quantity $\mathcal{X}(\eta)$ are neglected, is given by the following differential equation (Saxena et al. [18]):

$$\frac{d\mathcal{X}_i(\eta)}{d\eta} = -\varepsilon_i\, \mathcal{X}_i(\eta), \tag{34}$$

where initial condition $\mathcal{X}_\eta(\eta = 0) = \mathcal{X}_0$ is the number of density of species i at time $\eta = 0$, $\varepsilon_i > 0$. Solution of standard kinetic equation (34) is given in the form

$$\mathcal{X}_i(\eta) = \mathcal{X}_0 e^{-\varepsilon_i \eta}. \tag{35}$$

By omitting the index i and integrating standard kinetic equation (34), we have

$$\mathcal{X}(\eta) - \mathcal{X}_0 = -\varepsilon_0 {}_0\mathbb{D}_\eta^{-1}\mathcal{X}(\eta), \tag{36}$$

where ${}_0\mathbb{D}_\eta^{-1}$ is standard integral operator. Saxena et al. [19] obtained the fractional generalization of the standard kinetic equation (34) as follows:

$$\mathcal{X}(\eta) - \mathcal{X}_0 = -\varepsilon^\varsigma {}_0\mathbb{D}_\eta^{-\varsigma}\mathcal{X}(\eta), \tag{37}$$

where ${}_0\mathbb{D}_\eta^{-\varsigma}$ is Riemann-Liouville fractional integral operator defined by Mathai and Haubold [36] as follows:

$${}_0\mathbb{D}_\eta^{-\varsigma} f(\eta) = \frac{1}{\Gamma(\varsigma)}\int_0^\eta (\eta - \tau)^{\varsigma-1} f(\tau)\mathrm{d}\tau, \quad \mathrm{Re}(\varsigma) > 0. \tag{38}$$

Saxena et al. [18,19] introduced a modification to the fractional kinetic equation (37) as follows:

$$\mathcal{X}(\eta) - \eta^{\omega-1}\mathcal{X}_0 = -\varepsilon^\varsigma {}_0\mathbb{D}_\eta^{-\varsigma}\mathcal{X}(\eta), \quad \mathrm{Re}(\varsigma) > 0, \mathrm{Re}(\omega) > 0. \tag{39}$$

In the study by Mathai et al. [36] and and Mathai and Haubold [20], another modification of the fractional kinetic equation (37) is given as follows:

$$\mathcal{X}(\eta) - \eta^{\omega-1}\mathcal{X}_0 \mathbf{E}_{\varsigma,\omega}^{\delta}(-\varepsilon^\varsigma \eta^\varsigma) = -\varepsilon^\varsigma {}_0\mathbb{D}_\eta^{-\varsigma}\, \mathcal{X}(\eta), \quad \mathrm{Re}(\varsigma) > 0, \mathrm{Re}(\delta) > 0, \mathrm{Re}(\omega) > 0, \tag{40}$$

where $\mathbf{E}_{\varsigma,\omega}^{\delta}(.)$ is the generalized Mittag-Leffler function defined in the study by Prabhakar in [37].

Now, we use the MDLIT to solve the fractional extensions of the kinetic equations (37), (39), and (40).

The following lemma is required to prove the subsequent results.

Lemma 4.1. *Let* $\lambda \in (0, \infty)$ *and* $\varsigma \in \mathbb{C}$ *with* $\mathrm{Re}(\varsigma) > 0$, *then we have*

$$\mathbb{ML}_\lambda\{{}_0\mathbb{D}_\eta^{-\varsigma} f(\eta);\, \varsigma\} = \left\{\frac{\lambda}{\log(1 + \lambda\varphi)}\right\}^\varsigma \mathbb{ML}_\lambda\{f(\eta);\, \varsigma\}. \tag{41}$$

Proof. From (3) into (38), we have

$$\mathbb{ML}_\lambda\{{}_0\mathbb{D}_\eta^{-\varsigma} f(\eta);\, \varphi\} = \frac{1}{\varsigma}\int_0^\infty\int_0^\eta (1 + \lambda\varphi)^{\frac{-\eta}{\lambda}}(\eta - \tau)^{\varsigma-1} f(\tau)\mathrm{d}\tau\mathrm{d}\eta. \tag{42}$$

Setting $\varpi = \eta - \tau$ and after some computations, we arrive at

$$\mathbb{ML}_\lambda\{{}_0\mathbb{D}_\eta^{-\varsigma} f(\eta);\, \varphi\} = \frac{1}{\varsigma}\int_0^\infty (1 + \lambda\varphi)^{\frac{-\tau}{\lambda}} f(\tau)\int_0^\infty (1 + \lambda\varphi)^{\frac{-\varpi}{\lambda}}\, \varpi^{\varsigma-1}\mathrm{d}\tau\mathrm{d}\eta. \tag{43}$$

Further simplification yields the proof of Lemma 4.1. □

Remark 4.2. For $\lambda \to 0$ in Lemma 4.1, we have the corresponding result of the classical Laplace transform defined in equation (4) as follows:

$$\lim_{\lambda\to 0}\mathbb{ML}_\lambda\{{}_0\mathbb{D}_\eta^{-\varsigma} f(\eta);\, \varphi\} = \mathbb{ML}\{{}_0\mathbb{D}_\eta^{-\varsigma} f(\eta);\, \varphi\} = \left(\frac{1}{\varphi}\right)^\varsigma \mathbb{ML}\{f(\eta);\, \varphi\}. \tag{44}$$

Now, taking the MDLIT on both sides of equation (37) and simplifying by using Lemma 4.1, we obtain

$$\mathbb{ML}_{\lambda}\{\chi(\eta);\ \varphi\} = \chi_0\ \frac{\lambda}{\log(1+\lambda\varphi)}\left\{1+\left[\frac{\varepsilon\lambda}{\log(1+\lambda\varphi)}\right]\right\}^{-1}. \tag{45}$$

Taking the inverse transform, we obtain the solution of equation (37) as follows:

$$\mathcal{X}(\eta) = \mathcal{X}_0\sum_{s=0}^{\infty}\frac{(-1)^s(\varepsilon\eta)^{s\varsigma}}{\Gamma(1+\varsigma s)} = \mathcal{X}_0\ \mathfrak{E}_{\varsigma}(-\varepsilon^{\varsigma}\ \eta^{\varsigma}), \tag{46}$$

where $\mathfrak{E}_{\varsigma}(.)$ is the Mittag-Leffler function defined in the study by Wiman [38].

By applying the MDLIT on both sides of equation (39) and simplifying, we have

$$\mathbb{ML}_{\lambda}\{\chi(\eta);\ \varphi\} = \chi_0\left[\frac{\lambda}{\log(1+\lambda\varphi)}\right]^{\omega}\Gamma(\omega)\left\{1+\left[\frac{\varepsilon\lambda}{\log(1+\lambda\varphi)}\right]\right\}^{-1}. \tag{47}$$

We thus arrive at the solution of equation (39) in the form

$$\mathcal{X}(\eta) = \mathcal{X}_0\sum_{s=0}^{\infty}\frac{(-1)^s(\varepsilon\eta)^{s\varsigma}}{\Gamma(1+\varsigma s)} = \mathcal{X}_0\eta^{\omega-1}\Gamma(\omega)\mathfrak{E}_{\varsigma}(-\varepsilon^{\varsigma}\eta^{\varsigma}), \tag{48}$$

where $\mathfrak{E}_{\varsigma}(.)$ is the Mittag-Leffler function defined in the study by Wiman [38].

By applying the MDLIT on both sides of equation (40) and after some simplifications, we obtain

$$\mathbb{ML}_{\lambda}\{\chi(\eta);\ \varphi\} = \chi_0\left[\frac{\lambda}{\log(1+\lambda\varphi)}\right]^{\omega}\left\{1+\left[\frac{\varepsilon\lambda}{\log(1+\lambda\varphi)}\right]\right\}^{-(\delta+1)}. \tag{49}$$

Therefore, the solution of equation (40) is

$$\mathcal{X}(\eta) = \mathcal{X}_0\sum_{s=0}^{\infty}\frac{(-1)^s(\varepsilon\eta)^{s\varsigma}}{\Gamma(1+\varsigma s)} = \mathcal{X}_0\eta^{\omega-1}\mathfrak{E}_{\varsigma,\omega}^{\delta+1}(-\varepsilon^{\varsigma}\eta^{\varsigma}), \tag{50}$$

where $\mathfrak{E}_{\varsigma,\omega}^{\delta}(.)$ is the generalized Mittag-Leffler function defined in in the study by Prabhakar [37].

5 Conclusion

In this work, we are thrilled to have developed various properties of the MDLIT given in equation (3), such as convergence properties, transition theorem, convolution theorem, differential operator, and integral operator. Not only can this transform be used to solve initial value problems and fractional kinetic equations but it is also applicable to a wide range of other equations, including integral equations and differential equations. The outcomes attained here are likely valuable in applied science, engineering, and technology concerns – an exciting prospect that has been motivated by recent literature [26] and [27].

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