DE GRUYTER Demonstratio Mathematica 2023; 56: 20230111

Research Article

Azzh Saad Alshehry, Rashid Abu-Dawwas*, and Majd Al-Rashdan

Some notes on graded weakly 1-absorbing primary ideals

https://doi.org/10.1515/dema-2023-0111
received November 2, 2022; accepted August 13, 2023

Abstract: A proper graded ideal P of a commutative graded ring R is called graded weakly 1-absorbing primary if whenever x, y, z are nonunit homogeneous elements of R with $0 \neq xyz \in P$, then either $xy \in P$ or z is in the graded radical of P. In this article, we explore more results on graded weakly 1-absorbing primary ideals.

Keywords: graded prime ideal, graded primary ideal, graded 1-absorbing primary ideal, graded weakly primary ideal, graded weakly 1-absorbing primary ideal

MSC 2020: 13A02, 13A15, 16W50

1 Introduction

In dispersion through this article, G is a group and R is a commutative ring with nonzero unity 1 unless specified differently. If $R = \oplus_{g\in G}R_g$ with the property $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$, then R is aforementioned to be a graded ring (gr-R). The aspects of R_g are called homogeneous of degree g. If $s \in R$, then s can be expressed uniquely as $\sum_{g\in G}s_g$, where s_g is the component of s in R_g, and $s_g = 0$ is represented by the symbol. The set of all homogeneous aspects of R is $\bigcup_{g\in G}R_g$ and is denoted by $h(R)$. The component R_e is a subring of R and $1 \in R_e$. Let R be a gr-R and P be an ideal of R. Then, P is aforementioned to be a graded ideal (gr-I) if $P = \oplus_{g\in G}(P \cap R_g)$, i.e., for $p \in P$, $p_g \in P$ for all $g \in G$. An ideal of a gr-R is not necessarily gr-I. For a G-gr-R R and a gr-I P of R, R/P is a G-gr-R with $(R/P)_g = (R_g + P)/P$ for all $g \in G$. For further phrasing, see [1].

A proper gr-I P of R is aforementioned to be a graded prime ideal (gr-p-I) if $xy \in P$ implies either $x \in P$ or $y \in P$, for all $x, y \in h(R)$ [2]. It is clear that if P is a prime ideal of R and it is a gr-I, then P is a gr-p-I of R. Indeed, the example below demonstrates that a gr-p-I is not necessarily a prime ideal:

Example 1.1. Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then, R is gr-R by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Consider the gr-I $P = pR$ of R, where p is a prime number with $p = c^2 + d^2$, for some $c, d \in \mathbb{Z}$. We show that P is a gr-p-I of R. Let $xy \in P$ for some $x, y \in h(R)$.

<u>Case 1</u>: Assume that $x, y \in R_0$. In this instance, if $x, y \in \mathbb{Z}$, where p divides xy, then either p divides x or p divides y, which implies that $x \in P$ or $y \in P$.

<u>Case 2</u>: Assume that $x, y \in R_1$. In such a case, $x = ia$ and $y = ib$ for some $a, b \in \mathbb{Z}$ such that p divides $xy = -ab$, and then p divides a or p divides b in $\mathbb{Z}$, which suggests that p divides $x = ia$ or p divides $y = ib$ in R. Then, there is that $x \in P$ or $y \in P$.

*** Corresponding author: Rashid Abu-Dawwas,** Department of Mathematics, Yarmouk University, Irbid 21163, Jordan, e-mail: rrashid@yu.edu.jo
Azzh Saad Alshehry: Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia, e-mail: asalshihry@pnu.edu.sa
Majd Al-Rashdan: Department of Mathematics, Yarmouk University, Irbid 21163, Jordan, e-mail: 2020105011@ses.yu.edu.jo

Case 3: Consider that $x \in R_0$ and $y \in R_1$. In this instance, $x \in \mathbb{Z}$ and $y = ib$ for some $b \in \mathbb{Z}$ such that p divides $xy = ixb$ in R, i.e., $ixb = p(\alpha + i\beta)$ for some $\alpha, \beta \in \mathbb{Z}$. Then, we obtain $xb = p\beta$, i.e., p divides xb in $\mathbb{Z}$, and again p divides x or p divides b, which implies that p divides x or p divides $y = ib$ in R. Thus, $x \in P$ or $y \in P$.

So, P is a gr-p-I of R. On the other hand, P is not a prime ideal of R since $(c - id)(c + id) = c^2 + d^2 = p \in P$, $(c - id) \notin P$, and $(c + id) \notin P$.

Allow for P to be a gr-I of R. Then, the graded radical of P is denoted by $\text{Gr-rad}(P)$ and is defined as follows:

$$\text{Gr-rad}(P) = \left\{ s = \sum_{g \in G} s_g \in R : \forall g \in G, \exists n_g \in \mathbb{N} \quad \text{s.t. } s_g^{n_g} \in P \right\}.$$

Recall that Gr-rad(P) is every time a gr-I of R [2].

A proper gr-I P of R is aforementioned to be a graded primary ideal (gr-py-I) if $xy \in P$ suggests either $x \in P$ or $y \in \text{Gr-rad}(P)$, for all $x, y \in h(R)$ [3]. In this situation, $Q = \text{Gr-rad}(P)$ is a gr-p-I of R and P is allegedly graded Q-primary.

Since gr-p-I's and gr-py-I's are vital in commutative graded ring theory, numerous authors have looked into various generalizations of these gr-Is. Atani [4] proposed the idea of graded weakly prime ideals. A proper gr-I P of R is called a graded weakly prime ideal (gr-wp-I) whenever $x, y \in h(R)$ and $0 \neq xy \in P$, then $x \in P$ or $y \in P$. Atani [5] presented the impression of graded weakly primary ideals. A proper gr-I P of R is called a graded weakly primary ideal (gr-w-py-I) of R if whenever $x, y \in h(R)$ and $0 \neq xy \in P$, then $x \in P$ or $y \in \text{Gr-rad}(P)$. New generalizations of graded primary ideals and graded weakly primary ideals are, accordingly, the notions of graded 1-absorbing primary ideals and graded weakly 1-absorbing primary ideals proposed by Abu-Dawwas and Bataineh [6,7]. A proper gr-I P of R is called a graded 1-absorbing primary ideal (gr-1-ab-py-I) if whenever nonunit elements $x, y, z \in h(R)$ and $xyz \in P$, then $xy \in P$ or $z \in \text{Gr-rad}(P)$. A proper gr-I P of R is called a graded weakly 1-absorbing primary ideal (gr-w-1-ab-py-I) if whenever nonunit elements $x, y, z \in h(R)$ and $0 \neq xyz \in P$, then $xy \in P$ or $z \in \text{Gr-rad}(P)$. Certainly, every gr-py-I is gr-1-ab-py-I. The following example demonstrates that the converse is not true in general:

Example 1.2. [6] Consider $R = K[X, Y]$, where K is a field, and $G = \mathbb{Z}$. Then, R is gr-R by $R_n = \oplus_{i+j=n,i,j\geq 0} KX^iY^j$ for all $n \in \mathbb{Z}$. Recall that $\deg(X) = \deg(Y) = 1$. Consider the gr-I $P = \langle X^2, XY \rangle$ of R. Then, $\text{Gr-rad}(P) = \langle X \rangle$, and it is obvious that P is a gr-1-ab-py-I of R. On the contrary, P is not gr-py-I of R Example 2.11 in the study by Soheilnia and Darani [8].

It is recognizable that a gr-1-ab-py-I of R is gr-w-1-ab-py-I. However, since $\{0\}$ is always gr-w-1-ab-py-I, a gr-w-1-ab-py-I of R is not necessarily gr-1-ab-py-I, see Example 1.3.

Example 1.3. [7] Propose $R = \mathbb{Z}_6[i]$ and $G = \mathbb{Z}_2$. So, R is gr-R by $R_0 = \mathbb{Z}_6$ and $R_1 = i\mathbb{Z}_6$. Now, $P = \{0\}$ is a gr-w-1-ab-py-I of R. On the other hand, $2, 3 \in h(R)$ such that $2.2.3 \in P$ with neither $2.2 \in P$ nor $3 \in \text{Gr-rad}(P)$. Hence, P is not a gr-1-ab-py-I of R.

Definition 1.4. [6,9] Let R be a G-graded ring and P be a graded ideal of R. Assume that $g \in G$, where $P_g \neq R_g$. Then,

- P is supposedly a g-1-absorbing primary ideal (g-1-ab-py-I) of R if whenever nonunit elements $x, y, z \in R_g$ such that $xyz \in P$, then $xy \in P$ or $z \in \text{Gr-rad}(P)$.
- P is presumably a g-weakly 1-absorbing primary ideal (g-w-1-ab-py-I) of R, whenever nonunit elements $x, y, z \in R_g$, where $0 \neq xyz \in P$, then $xy \in P$ or $z \in \text{Gr-rad}(P)$.
- P is repeatedly a g-prime ideal (g-p-I) of R if whenever $x, y \in R_g$, where $xy \in P$, then either $x \in P$ or $y \in P$.
- P is presumably a g-primary ideal (g-py-I) of R, if whenever $x, y \in R_g$, where $xy \in P$, then either $x \in P$ or $y \in \text{Gr-rad}(P)$.

- P is supposedly a g-weakly primary ideal (g-w-py-I) of R if whenever $x, y \in R_g$, where $0 \neq xy \in P$, then either $x \in P$ or $y \in \text{Gr-rad}(P)$.

In this article, we explore more outcomes on graded weakly 1-absorbing primary ideals. In fact, the study by Almahdi et al. [10] inspired quite a few of the outcomes. Among a number of outcomes, we proved that if R_e is a nonlocal ring and P is an e-w-1-ab-py-I of R that is not an e-w-py-I, then either $P_e^3 = 0$ or $P_e^2 = \langle s \rangle$ with s as an idempotent such that $\langle 1 - s \rangle$ is a maximal ideal of R_e (Theorem 2.4). In addition, we showed that if every nonzero gr-py-I of R is a gr-p-I and $\text{Gr-rad}(0)$ is a gr-m-I of R, then either $\text{Gr-rad}(0) = 0$ or $\text{Gr-rad}(0)$ is the unique nonzero proper gr-I of R (Proposition 2.8). In addition, we proved that if R is a HUN-ring and $\{0\}$ is a gr-py-I of R, then R is a gr-loc-R with gr-m-I $\text{Gr-rad}(0)$ (Theorem 2.12). Moreover, a nice characterization was introduced in Theorem 2.13. In addition, we showed that if R is a finitely generated gr-loc-R with gr-m-I X, R is a gr-D, and every gr-1-ab-py-I of R is a gr-w-py-I, then R is either HUN-ring or X is the unique nonzero gr-p-I of R (Theorem 2.14). Furthermore, we proved that if R is a first strongly gr-R, then every e-w-1-ab-py-I of R is an e-s-py-I if and only if $\text{Gr-rad}(0)$ is an e-p-I of R (Proposition 2.16). Finally, we showed that if R is a reduced first strongly gr-R, then every e-w-1-ab-py-I of R is an e-1-ab-py-I if and only if R_e is a domain (Proposition 2.19).

2 Results

Our results are presented in this paragraph.

Proposition 2.1. *Let P be a gr-I of R such that* $\text{Gr-rad}(P) = P$. *If P is a g-w-1-ab-py-I of R, then P is a g-p-I of R or $r^3 = 0$ for all $r \in P_g$.*

Proof. Suppose that $r \in P_g$ exists, where $r^3 \neq 0$. Let $x, y \in R_g$ in a manner that $xy \in P$. We may assume that x and y are nonunit. If $x^2y \neq 0$, then $x^2 \in P$ or $y \in \text{Gr-rad}(P)$. Hence, $x \in \text{Gr-rad}(P) = P$ or $y \in \text{Gr-rad}(P) = P$. Similarly, if $xy^2 \neq 0$, we arrive at the same result. Now, suppose that $x^2y = xy^2 = 0$. If $x^2P_g \neq 0$, then there exists $s \in P_g$ such that $x^2s \neq 0$, and $0 \neq x^2s = x^2(y + s) \in P$. If $y + s$ is a unit, then $x \in P$. Otherwise, $x^2 \in P$ or $y + s \in P$. Thus, $x \in P$ or $y \in P$. Similarly, if $y^2P_g \neq 0$, then $x \in P$ or $y \in P$. Suppose that $x^2P_g = y^2P_g = 0$. We have $(x^2 + r)^2(y^2 + r) = r^3 \in P$. If $x^2 + r$ (resp. $y^2 + r$) is a unit, then $y \in P$ (resp. $x \in P$). Otherwise, $(x^2 + r)^2 \in P$ or $y^2 + r \in P$. Thus, $x \in P$ or $y \in P$. Finally, we establish that P is a g-p-I of R. □

Corollary 2.2. *Let P be a gr-I of R in such a way that* $\text{Gr-rad}(P) = P$. *If P is a g-w-1-ab-py-I of R which is not a g-p-I, then $P_g \subseteq$* $\text{Gr-rad}(\{0\})$.

Proof. Apply Proposition 2.1. □

Lemma 2.3. *Let R be a gr-R. Then, R_e holds all homogeneous idempotent elements of R.*

Proof. Let $x \in h(R)$ be an idempotent. Then, $x \in R_g$ for some $g \in G$, and then $x = x^2 = x$. $x \in R_gR_g \subseteq R_{g^2}$. If $x = 0$, then it has been completed. Suppose that $x \neq 0$. Then, $0 \neq x \in R_g \cap R_{g^2}$, which suggests that $g^2 = g$, i.e., $g = e$. As a deduction, $x \in R_e$. □

Theorem 2.4. *Allow for R to be a gr-R such that R_e is a nonlocal ring. If P is an e-w-1-ab-py-I of R that is not an e-w-py-I, then*
- *$P_e^3 = 0$, or*
- *$P_e^2 = \langle s \rangle$ with s as an idempotent such that $\langle 1 - s \rangle$ is a maximal ideal of R_e.*

Proof. Let us say that (2) is not met. Since P is not an e-w-py-I, there exists $x, y \in R_e$ in such a way that $0 \neq xy \in P$, $x \notin P$, and $y \notin$ Gr-rad(P). Certainly, x and y are nonunits. Suppose that $vx \in P$ for all nonunit $v \in R_e$. Let u be a unit in R_e. If $v + u$ is a nonunit, then $(v + u)x \in P$, and so $ux \in P$, a contradiction since $x \notin P$. Hence, for each nonunit $v \in R_e$ and each unit $u \in R_e$, $v + u$ is a unit. Thus, by Lemma 1 in the study by Badawi and Celikel [11], R_e is a local ring, a contradiction. Because of that, there exists a nonunit $v \in R_e$ such that $vx \notin P$. If $vxy \neq 0$, then $vx \in P$ since $y \notin$ Gr-rad(P) and P is an e-w-1-ab-py-I, a contradiction. Hence, $vxy = 0$. Presume the existence of $p \in P_e$ such that $vxp \neq 0$. Then, $0 \neq vxp = vx(y + p) \in P$. If $y + p$ is a unit, then $vx \in P$, a contradiction. Hence, since $vx \notin P$, we obtain $y + p \in$ Gr-rad(P). Thus, $y \in$ Gr-rad(P), a contradiction. Consequently, $vxP_e = 0$. Consider the existence of $p \in P_e$ in such a way that $vyp \neq 0$. Then, $0 \neq vyp = v(x + p) \in P$. If $x + p = u$ is a unit, then $uy = xy + py \in P$, and so $y \in P$, a contradiction. Hence, $x + p$ is a nonunit and $v(x + p) \in P$. So, $vx \in P$, a contradiction. Consequently, $vyP_e = 0$. Suppose that there exist $p, q \in P$, where $vpq \neq 0$. Then, $0 \neq vpq = v(x + p)(y + q) \in P$. As above, $x + p$ and $y + q$ are nonunits. Hence, $v(x + p) \in P$. So, $vx \in P$, a contradiction. Therefore, $vP_e^2 = 0$. Assume there exists $p \in P_e$, where $xyp \neq 0$. Then, $0 \neq xyp = (v + p)xy \in P$. Suppose that $u = v + p$ is a unit. Then, $up^2 = p^3$. Hence, $(pu^{-1})^3 = (pu^{-1})^2$. Thus, $s = (pu^{-1})^2$ is an idempotent. For each $q, t \in P_e$, we have $qtu = qtp$ and $qpu = qp^2$. Thus, $qtu^2 = t(qpu) = tqp^2$. Hence, $qt = qts$. Then, $P_e^2 \subseteq \langle s \rangle \subseteq P_e^2$. Therefore, $P_e^2 = \langle s \rangle$. By assumption, $\langle 1 - s \rangle$ is not a maximal ideal of R_e. If $\langle s \rangle$ is a maximal ideal of R_e, then $P_e = P_e^2 = \langle s \rangle$, a contradiction since P is not an e-w-py-I. Thus, neither $\langle 1 - s \rangle$ nor $\langle s \rangle$ is a maximal ideal of R_e. Hence, $R_e \approx R_e/\langle s \rangle \times R_e/\langle 1 - s \rangle$ is a product of two nonfield rings. By Theorem 13 of the study by Badawi and Yetkin [12], P is an e-py-I, a contradiction. Subsequently, $v + p$ is a nonunit, and so $(v + p)x \in P$. Then, $vx \in P$, a contradiction. As a consequence, $xyP_e = 0$. Consider the existence of $p, q \in P_e$ in such a way that $xpq \neq 0$. Then, $0 \neq xpq = x(v + p)(y + q) \in P$. As above, $v + p$ and $y + q$ are nonunits. Hence, $x(v + p) \in P$. So, $vx \in P$, a contradiction. As a consequence, $xP_e^2 = 0$. Suppose that there exist $p, q \in P_e$ where $ypq \neq 0$. Consequently, $0 \neq ypq = (v + p)(x + q)y \in P$. As above, $v + p$ and $x + q$ are nonunits. Hence, $(v + p)(x + q) \in P$. So, $vx \in P$, a contradiction. As a consequence, $yP_e^2 = 0$. Let $p, q, t \in P_e$ in such a way that $pqt \neq 0$. Afterward, $(v + p)(x + q)(y + t) = pqt \neq 0$. As above, $v + p$, $x + q$, and $y + t$ are nonunits. Then, $(v + p)(x + q) \in P$ or $y + t \in$ Gr-rad(P). That is, $vx \in P$ or $y \in$ Gr-rad(P), a contradiction. Hence, $P_e^3 = 0$. □

A gr-R R is said to be strongly graded if $1 \in R_g R_{g^{-1}}$ for all $g \in G$, that is equivalent to $R_g R_h = R_{gh}$ for all $g, h \in G$ [1]. A gr-R R is said to be first strongly graded if $1 \in R_g R_{g^{-1}}$ for all $g \in \text{supp}(R, G) = \{g \in G : R_g \neq 0\}$ [13]. Undoubtedly, if R is strongly graded, then R is first strongly graded. The following example, however, demonstrates that the converse is not always true.

Example 2.5. Let $R = M_2(K)$ (the ring of all 2×2 matrices with entries from a field K) and $G = \mathbb{Z}_4$. Then R is gr-R by

$$R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \quad \text{and} \quad R_1 = R_3 = \{0\}.$$

R is first strongly graded since $I \in R_0 R_0$ and $I \in R_2 R_2$, but R is not strongly graded since $R_1 R_3 = 0 \neq R_0$.

Without any doubt, if R is strongly graded, then $\text{supp}(R, G) = G$. Besides, if R is first strongly graded, then $\text{supp}(R, G)$ is a subgroup of G. Actually, R is first strongly graded on the condition that $\text{supp}(R, G)$ is a subgroup of G and $R_g R_h = R_{gh}$ for all $g, h \in \text{supp}(R, G)$.

Theorem 2.6. *Let R be a first strongly gr-R such that R_e is a nonlocal reduced ring. Suppose that P is an e-w-1-ab-py-I of R. If P is not an e-w-py-I, then* Gr-rad$(P_e) = P_e$.

Proof. If $P_e^3 = 0$, then $P_e = 0$ and $P_g = R_g \cap P = R_e R_g \cap R_e P = R_e R_g \cap R_g R_{g^{-1}} P = R_g (R_e \cap R_{g^{-1}} P) \subseteq R_g (R_e \cap P) = R_g P_e = 0$ for all $g \in \text{supp}(R, G)$. Besides, for $g \notin \text{supp}(R, G)$, $R_g = 0$, which implies that $P_g = R_g \cap P = 0$. Hence, $P_g = 0$ for all $g \in G$, i.e., $P = 0$, which is an e-w-py-I, a contradiction. So, by Theorem 2.4, $P_e^2 = \langle s \rangle$ with s as an idempotent such that $\langle 1-s\rangle$ is a maximal ideal of R_e. We have that $R_e \approx R_e/\langle s\rangle \times R_e/\langle 1-s\rangle$ with the isomorphism $f(r) = (r+\langle s\rangle, r+\langle 1-s\rangle)$. Let $R_1 = R_e/\langle s\rangle$ and $K = R_e/\langle 1-s\rangle$. Then, without any doubt, K is a field and $f(P_e^2) = \{0\} \times K$. While maintaining generality, set $R_e = R_1 \times K$ and $P_e = I \times J$ such that I and J are graded ideals of R_1 and K, respectively. For that reason, since $P_e^2 = \{0\} \times K$ and R_1 is reduced, we conclude that $P_e = \{0\} \times K$. In addition, $\text{Gr-rad}(P_e) = \text{Gr-rad}(\{0\}) \times K = \{0\} \times K = P_e$ since R_1 is reduced. □

A proper gr-I X of R is allegedly a graded maximal ideal (gr-m-I) of R if whenever I is a gr-I of R with $X \subseteq I \subseteq R$, then $I = X$ or $I = R$. Assuredly, every gr-m-I is a gr-p-I. A gr-R R is assumed to be a graded local ring (gr-loc-R) if R has a unique gr-m-I.

Proposition 2.7. *Allow for R to be a gr-loc-R with gr-m-I X. Assume that P is a gr-p-I of R such that $P \subseteq X$. Then, PX is a gr-1-ab-py-I of R.*

Proof. Take note of the fact that $\text{Gr-rad}(PX) = P$. Suppose that $xyz \in PX$ for some nonunit elements $x, y, z \in h(R)$. If $x \in P$ or $y \in P$, then without a doubt, $xy \in PX$. Assume that neither $x \in P$ nor $y \in P$. Then, $xy \notin P$. Since $xyz \in PX \subseteq P$ and $xy \notin P$, we ultimately decide that $z \in P = \text{Gr-rad}(PX)$. Thus, PX is a gr-1-ab-py-I of R. □

Proposition 2.8. *Allow for R to be a gr-R such that every nonzero gr-py-I of R is a gr-p-I. If* $\text{Gr-rad}(0)$ *is a gr-m-I of R, then either* $\text{Gr-rad}(0) = 0$ *or* $\text{Gr-rad}(0)$ *is the unique nonzero proper gr-I of R.*

Proof. If R is a gr-D, then $\text{Gr-rad}(0) = 0$. Assume that R is not a gr-D. Allow for J to be a nonzero proper gr-I of R. Then, $\text{Gr-rad}(0) \subseteq \text{Gr-rad}(J)$, and then as $\text{Gr-rad}(0)$ is a gr-m-I of R, $\text{Gr-rad}(0) = \text{Gr-rad}(J)$. So, $\text{Gr-rad}(J)$ is a gr-m-I of R, which implies that J is a gr-py-I of R by Proposition 1.11 of the study by Refai and Al-Zoubi [3], and then J is a gr-p-I of R, and so $J = \text{Gr-rad}(J) = \text{Gr-rad}(0)$. As a consequence, $\text{Gr-rad}(0)$ is the unique nonzero proper gr-I of R. □

A gr-R R is said to be a graded domain (gr-D) if R has no homogeneous zero divisors, and is said to be a graded field (gr-F) if every nonzero homogeneous element of R is unit [1]. Assuredly, if R is a domain (field) and it is graded, then R is a gr-D (gr-F). Nevertheless, Example 2.4 of the study by Abu-Dawwas [14] shows that a gr-D (gr-F) is not necessarily a domain (field). Recall from [15] and [16, Proposition 2.25], if every element of R is either nilpotent or unit, or alternatively if all of its nonunit elements are products of unit and nilpotent elements, then R is said to be a UN-ring. A straightforward UN-ring example is $\mathbb{Z}/9\mathbb{Z}$. In fact, we present the idea of HUN-rings:

Definition 2.9. A gr-R R is presumably a HUN-ring if every homogeneous element of R is either a unit or a nilpotent.

Absolutely, if R is a UN-ring and it is graded, then R is a HUN-ring. A HUN-ring is not always a UN-ring, as the example below demonstrates:

Example 2.10. Let K be a field and $u \notin K$ with $u^2 = 1$. Assume that $R = \{\alpha + u\beta : \alpha, \beta \in K\}$ and $G = \mathbb{Z}_2$. Then R is a gr-R by $R_0 = K$ and $R_1 = uK$. By Example 2.4 of the study by Abu-Dawwas [14], R is a gr-F, and then R is a HUN-ring. But R is not a UN-ring since $1 + u \in R$ is neither a unit nor a nilpotent.

For a gr-R R, the set of all homogeneous zero divisors of R, $HZ(R)$, and the set of all zero divisors of R, $Z(R)$, are not the same. Indeed, $HZ(R) \subseteq Z(R)$, but, in Example 2.10, $1 + u \in Z(R)$ as $(1+u)(1-u) = 0$, while $1 + u \notin HZ(R)$ as $1 + u \notin h(R)$. For a gr-R R, $HZ(R)$ is not necessarily a gr-I of R since it is not necessarily an ideal; consider $R = \mathbb{Z}_6[i]$, $G = \mathbb{Z}_2$, $R_0 = \mathbb{Z}_6$, and $R_1 = i\mathbb{Z}_6$, Note that $2, 3 \in HZ(R)$ with $2 + 3 = 5 \notin HZ(R)$. Nevertheless, if $HZ(R)$ is a gr-I in some gr-R R, then $HZ(R)$ should be a gr-p-I of R. To see this, let $x, y \in h(R)$, where $xy \in HZ(R)$. Then, there exists $0 \neq z \in h(R)$ such that $xyz = 0$. If $yz \neq 0$, then $x \in HZ(R)$. If $yz = 0$, then $y \in HZ(R)$. Indeed, the following lemma exists:

Lemma 2.11. *Let R be a gr-R, where $HZ(R)$ is a gr-I of R. Consequently, $\{0\}$ is a gr-py-I of R if and only if $HZ(R) = \text{Gr-rad}(0)$.*

Proof. Suppose that $\{0\}$ is a gr-py-I of R. Let $x \in \text{Gr-rad}(0)$. Then, for all $g \in G$, there exists a positive integer n_g in such a way that $x_g^{n_g} = 0$, and then $x_g \in HZ(R)$, for all $g \in G$, and so $x \in HZ(R)$ as $HZ(R)$ is an ideal. Hence, $\text{Gr-rad}(0) \subseteq Z(R)$. Let $y \in HZ(R)$. Then, there exists $0 \neq z \in h(R)$, where $zy = 0$, and then $y \in \text{Gr-rad}(0)$ as $\{0\}$ is a gr-py-I and $z \neq 0$. Hence, $HZ(R) = \text{Gr-rad}(0)$. Conversely, let $a, b \in h(R)$ in such a way that $ab = 0$. If $a = 0$, then it is done. If $a \neq 0$, then $b \in HZ(R) = \text{Gr-rad}(0)$. Thus, $\{0\}$ is a gr-py-I of R. □

Theorem 2.12. *Let R be a HUN-ring. If $\{0\}$ is a gr-py-I of R, then R is a gr-loc-R with gr-m-I* $\text{Gr-rad}(0)$.

Proof. Since $\{0\}$ is a gr-py-I of R, $HZ(R) = \text{Gr-rad}(0)$ by Lemma 2.11, and so $HZ(R)$ is a gr-I of R. Let J be a gr-I of R in such a way that $HZ(R) \subseteq J \subseteq R$ and $HZ(R) \neq J$. Then, there is the existence of $x \in J$, where $x \notin HZ(R)$; therefore, there exists $g \in G$, where $x_g \notin HZ(R)$. Note that, $x_g \in J$ as J is a gr-I. Since $x_g \notin HZ(R)$, x_g is not a nilpotent, so x_g is a unit as R is a HUN-ring, and hence $J = R$. Thus, $HZ(R)$ is a gr-m-I of R. Allow for K to be a proper gr-I of R, and suppose that $a \in K$. Since $a_g \in K$ for all $g \in G$ and K is a proper, a_g is a nonunit for all $g \in G$, and then a_g is a nilpotent for all $g \in G$, i.e., $a_g \in HZ(R)$ for all $g \in G$, then $a \in HZ(R)$. So, $J \subseteq HZ(R)$, and hence $HZ(R)$ is the only gr-m-I of R. Thus, R is a gr-loc-R with gr-m-I $HZ(R) = \text{Gr-rad}(0)$. □

In the following theorem, we give a stronger and better conclusion than Theorem 2.12. Undeniably, we investigate the notion of graded n-ideals that were appeared in the study by Al-Zoubi et al. [17]. A proper gr-I P of R is presumably a graded n-ideal (gr-n-I) of R whenever $x, y \in h(R)$, where $xy \in P$ and $x \notin \text{Gr-rad}(0)$, then $y \in P$.

Theorem 2.13. *For any gr-R R, the following are interchangeable*:
- *R is a HUN-ring.*
- *$\langle x \rangle$ is a gr-n-I of R, for every $x \in h(R)$ with $\langle x \rangle \neq R$.*
- *Every proper gr-I is a gr-n-I.*
- *R has a unique gr-p-I, which is* $\text{Gr-rad}(0)$.
- *R is a gr-loc-R with gr-m-I* $\text{Gr-rad}(0)$.
- *$R/(\text{Gr-rad}(0))$ is a gr-F.*

Proof.

(1) ⇒ (2): Let $x \in h(R)$ with $\langle x \rangle \neq R$. Consider $a, b \in h(R)$ in such a way that $ab \in \langle x \rangle$ and $a \notin \text{Gr-rad}(0)$. So, a is a unit, and then $b \in \langle x \rangle$, also $\langle x \rangle$ is a gr-n-I of R.

(2) ⇒ (3): Let P be a proper gr-I of R. Presume that $x, y \in h(R)$, where $xy \in P$ and $x \notin \text{Gr-rad}(0)$. Considering $xy \in \langle xy \rangle$ and $\langle xy \rangle$ is a gr-n-I, $y \in \langle xy \rangle \subseteq P$. Therefore, P is a gr-n-I of R.

(3) ⇒ (4): Let P be a gr-p-I of R. By equation (3) and [17, Theorem 1], $P = \text{Gr-rad}(0)$.

(4) ⇒ (5): Since R has one gr-p-I, which is $\text{Gr-rad}(0)$, we conclude that R is a gr-loc-R with gr-m-I $\text{Gr-rad}(0)$.

(5) ⇒ (6): It is obvious to see.

(6) ⇒ (1): Let $x \in h(R)$ such that x is not a nilpotent. Then, $x \notin \text{Gr-rad}(0)$ and $x + \text{Gr-rad}(0)$ is a nonzero homogeneous element in $R/\text{Gr-rad}(0)$, which implies that $x + \text{Gr-rad}(0)$ is a unit, i.e., $(x + \text{Gr-rad}(0))(y + \text{Gr-rad}(0)) = 1 + \text{Gr-rad}(0)$, for some $y \in h(R)$. So, $xy - 1$ is a nilpotent, and then $(xy - 1) + 1 = xy$ is a unit, which gives that x is a unit. Thus, R is a HUN-ring. □

Theorem 2.14. *Let R be a finitely generated gr-loc-R with gr-m-I X. If R is a gr-D and every gr-1-ab-py-I of R is a gr-w-py-I, then either R is a HUN-ring or X is the unique nonzero gr-p-I of R.*

Proof. Assume that R is not a HUN-ring. Assume that R is a gr-D. Let $0 \neq P$ be gr-p-I of R that is not a gr-m-I. Then, PX is a gr-1-ab-py-I of R and $\text{Gr-rad}(PX) = P$ by Proposition 2.7. Then, PX is a gr-w-py-I of R. Let $0 \neq p \in P$ and $x \in X - P$. Then, $p = \sum_{g \in G} p_g$, where $p_g \in P$ as P is a gr-I, and also, there exists $h \in G$ such that $x_h \notin P$. Note that, $x_h \in X$ as X is a gr-I. We have $0 \neq p_g x_h \in PX$ for all $g \in G$ with $p_g \neq 0$, and $x_h \notin P = \text{Gr-rad}(PX)$. Thus, $p_g \in PX$ for all $g \in G$ with $p_g \neq 0$, so $p \in PX$ as $p_g = 0 \in PX$ too. Hence, $P = PX$. We obtain $P = 0$ from the Nakayama's lemma, a contradiction. Thus, X is the unique nonzero gr-p-I of R. □

Recall from the study by Bataineh and Abu-Dawwas [18] that a proper gr-I P of R is presumably a graded semi-primary ideal (gr-s-py-I) of R if whenever $x, y \in h(R)$, where $xy \in P$, consequently $x \in \text{Gr-rad}(P)$ or $y \in \text{Gr-rad}(P)$, or equivalently, $\text{Gr-rad}(P)$ is a gr-p-I of R [19, Proposition 4]. It has been proved in Lemma 2.7 of the study by Abu-Dawwas [14] that every gr-1-ab-py-I of R is a gr-s-py-I. We establish the concept of g-semi-primary ideals (g-s-py-I's), and then we present a case where every e-w-1-ab-py-I of R is an e-s-py-I.

Definition 2.15. Allow R to be a gr-R, $g \in G$ and P be a gr-I of R with $P_g \neq R_g$. Then, P is said to be a g-semi-primary ideal (g-s-py-I) of R if $\text{Gr-rad}(P)$ is a g-p-I of R.

Proposition 2.16. *Assume R is a first strongly gr-R. Then, every e-w-1-ab-py-I of R is an e-s-py-I supposing that $\text{Gr-rad}(0)$ is an e-p-I of R.*

Proof. Presume that $\text{Gr-rad}(0)$ is an e-p-I of R. Let P be an e-w-1-ab-py-I of R. Assume that $x, y \in R_e$ with $xy \in \text{Gr-rad}(P)$ and $x \notin \text{Gr-rad}(P)$. We can suppose that x is not a unit. Now, there is the existence of a positive integer n, where $x^n y^n \in P$. Accordingly, $x^{n+1}y^n \in P$ and $n + 1 \geq 2$. If $x^{n+1}y^n \neq 0$, then $x^{n+1} \in P$ or $y^n \in P$. Thus, $y \in \text{Gr-rad}(P)$ since $x \notin \text{Gr-rad}(P)$. Consider $x^{n+1}y^n = 0$. If $x^{n+1}P_e = \{0\} \subseteq \text{Gr-rad}(0)$, then $P_e \subseteq \text{Gr-rad}(0)$ since $\text{Gr-rad}(0)$ is an e-p-I and $x \notin \text{Gr-rad}(0)$. If $g \in G$ with $P_g \neq 0$, then $P_g = R_g P_e \subseteq R_g(\text{Gr-rad}(0)) \subseteq \text{Gr-rad}(0)$. So, $P_g \subseteq \text{Gr-rad}(0)$ for all $g \in G$. Thus, $P \subseteq \text{Gr-rad}(0)$, and then $\text{Gr-rad}(P) = \text{Gr-rad}(0)$ is an e-p-I. If $x^{n+1}P_e \neq 0$, then there exists $a \in P_e$ such that $x^{n+1}a \neq 0$, and so $0 \neq x^{n+1}(a + y^n) \in P$. If $a + y^n$ is a unit, then $x^{n+1} \in P$, a contradiction. Thus, $a + y^n$ is a nonunit. Since $x^{n+1} \notin P$, we obtain $a + y^n \in \text{Gr-rad}(P)$. Thus, $y \in \text{Gr-rad}(P)$. Consequently, $\text{Gr-rad}(P)$ is an e-p-I of R. Conversely, since $\{0\}$ is an e-w-1-ab-py-I of R, $\{0\}$ is an e-s-py-I, and hence $\text{Gr-rad}(0)$ is an e-p-I of R. □

Proposition 2.17. *Let R be a gr-R. Then, every gr-w-1-ab-py-I of R is a gr-1-ab-py-I if and only if $\{0\}$ is a gr-1-ab-py-I ideal of R.*

Proof. Speculate that $\{0\}$ is a gr-1-ab-py-I ideal of R. Let P be a gr-w-1-ab-py-I of R. Assume that $x, y, z \in h(R)$ are nonunits such that $xyz \in P$ and $z \notin \text{Gr-rad}(P)$. If $xyz \neq 0$, then $xy \in P$. Now, consider $xyz = 0$. Hence, $xy = 0$ or $z \in \text{Gr-rad}(0)$ since $\{0\}$ is a gr-1-ab-py-I. The second case, however, cannot happen since $z \notin \text{Gr-rad}(P)$. Hence, $xy = 0 \in P$, in the desired manner. Conversely, since $\{0\}$ is a gr-w-1-ab-py-I of R, $\{0\}$ is a gr-1-ab-py-I of R. □

Proposition 2.18. *Allow R to be a gr-R in such a way that $HZ(R)$ is a gr-I of R. If $\{0\}$ is a gr-1-ab-py-I ideal of R, then either $HZ(R) = Grad(0)$ or R is a HUN-ring with $HZ(R) = \text{Ann}_R(x) = \{a \in R : ax = 0\}$ for some $x \in h(R)$.*

Proof. Suppose that $HZ(R) \neq \text{Gr-rad}(0)$. Let $a \in HZ(R) - (\text{Gr-rad}(0))$. There is the existence of $0 \neq x \in h(R)$, where $ax = 0$. Suppose that R has a homogeneous nonunit regular element named s. We have $sxa = 0$ and $a \notin \text{Gr-rad}(0)$. Then, $sx = 0$, and so $x = 0$, wholly unattainable. Thus, homogeneous nonunit elements of R are in $HZ(R)$. So, R is a HUN-ring. Let $y \in HZ(R)$. We have $yxa = 0$ and $a \notin \text{Gr-rad}(0)$, and so $yx = 0$. Thus, $HZ(R) \subseteq \text{Ann}_R(x)$. Let $r \in \text{Ann}_R(x)$. Then, $r_g \in \text{Ann}_R(x)$ for all $g \in G$ as $\text{Ann}_R(x)$ is a gr-I by ([20], page 3, line 11), which implies that $r_g \in HZ(R)$ for all $g \in G$, and so $r \in HZ(R)$ as $HZ(R)$ is a gr-I. Thus, $HZ(R) = \text{Ann}_R(x)$. □

Proposition 2.19. *Let R be a reduced first strongly gr-R. Then, every e-w-1-ab-py-I of R is an e-1-ab-py-I on the condition that R_e is a domain.*

Proof. Presume that every e-w-1-ab-py-I of R is an e-1-ab-py-I. Similarly as in Lemma 2.7 of the study by Abu-Dawwas [14], one can prove that every e-1-ab-py-I of R is an e-s-py-I. So, every e-w-1-ab-py-I of R is an e-s-py-I. Hence, by Proposition 2.16, $\{0\} = \text{Gr-rad}(0)$ is an e-p-I of R. Therefore, R_e is a domain. Conversely, let P be an e-w-1-ab-py-I of R. Assume that $x, y, z \in R_e$ are nonunits such that $xyz \in P$. If $xyz = 0$, then $x = 0$ or $y = 0$ or $z = 0$ as R_e is a domain, and so it is done. Consider $xyz \neq 0$. Then, as P is an e-w-1-ab-py-I, either $xy \in P$ or $z \in \text{Gr-rad}(0)$. Therefore, P is an e-1-ab-py-I of R. □

3 Conclusion

In this article, we looked at and explored more outcomes to graded weakly 1-absorbing primary ideals. We proved that if R_e is a nonlocal ring and P is an e-w-1-ab-py-I of R that is not an e-w-py-I, then either $P_e^3 = 0$ or $P_e^2 = \langle s \rangle$ with s as an idempotent such that $\langle 1 - s \rangle$ is a maximal ideal of R_e (Theorem 2.4). In addition, we showed that if every nonzero gr-py-I of R is a gr-p-I and $\text{Gr-rad}(0)$ is a gr-m-I of R, then either $\text{Gr-rad}(0) = 0$ or $\text{Gr-rad}(0)$ is the unique nonzero proper gr-I of R (Proposition 2.8). In addition, we proved that if R is a HUN-ring and $\{0\}$ is a gr-py-I of R, then R is a gr-loc-R with gr-m-I $\text{Gr-rad}(0)$ (Theorem 2.12). Moreover, a nice characterization was introduced in Theorem 2.13. We also showed that if R is a finitely generated gr-loc-R with gr-m-I X, R is a gr-D, and every gr-1-ab-py-I of R is a gr-w-py-I, then R is either HUN-ring or X is the unique nonzero gr-p-I of R (Theorem 2.14). Furthermore, we proved that if R is the first strongly gr-R, then every e-w-1-ab-py-I of R is an e-s-py-I if and only if $\text{Gr-rad}(0)$ is an e-p-I of R (Proposition 2.16). Finally, we showed that if R is a reduced first strongly gr-R, then every e-w-1-ab-py-I of R is an e-1-ab-py-I if and only if R_e is a domain (Proposition 2.19). As a proposal for further work on the topic, we are going to introduce a deep study on the concept of graded 1-absorbing prime ideals that have been established in the study by Abu-Dawwas et al. [9].

Acknowledgments: This article was supported by Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R183), Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia. We are very grateful for the reviewer comments that helped to improve our article.

Funding information: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R183), Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia.

Conflict of interest: The authors declare that there is no conflict of interest.

References

[1] C. Nastasescu and F. Oystaeyen, *Methods of Graded Rings, 1836*, Springer-Verlag, Berlin, 2004.
[2] M. Refai, M. Hailat, and S. Obiedat, *Graded radicals and graded prime spectra*, Far East J. Math. Sci. (2000), 59–73.
[3] M. Refai and K. Al-Zoubi, *On graded primary ideals*, Turkish J. Math. **28** (2004), no. 3, 217–229.
[4] S. E. Atani, *On graded weakly prime ideals*, Turkish J. Math. **30** (2006), 351–358.
[5] S. E. Atani, *On graded weakly primary ideals*, Quasigroups Related Syst. **13** (2005), 185–191.
[6] R. Abu-Dawwas and M. Bataineh, *On graded 1-absorbing primary ideals*, Turkish J. Math-Studies Scientific Developments in Geometry, Algebra, and Applied Mathematics, Turkish Journal of Mathematics, Turkey, February 1–3, 2022.
[7] M. Bataineh and R. Abu-Dawwas, *Graded weakly 1-absorbing primary ideals*, Demonstr. Math. **56** (2023), 20220214.
[8] F. Soheilnia and A. Y. Darani, *On graded 2-absorbing and graded weakly 2-absorbing primary ideals*, Kyungpook Math. J. **57** (2017), no. 4, 559–580.

[9] R. Abu-Dawwas, E. Yıldız, Ü. Tekir, and S. Koç, *On graded 1-absorbing prime ideals*, Sao Paulo J. Math. Sci. **15** (2021), 450–462.
[10] F. A. A. Almahdi, M. Tamekkante, and A. N. A. Koam, *Note on weakly 1-absorbing primary ideals*, Filomat **36** (2022), no. 1, 165–173.
[11] A. Badawi and E. Y. Celikel, *On 1-absorbing primary ideals of commutative rings*, J. Algebra Appl. **19** (2020), no. 6, 2050111.
[12] A. Badawi and E. Yetkin, *On weakly 1-absorbing primary ideals of commutative rings*, Algebra Colloquium **29** (2022), no. 2, 189–202.
[13] M. Refai, *Various types of strongly graded rings*, Abhath Al-Yarmouk J. (Pure Sciences and Engineering Series) **4** (1995), no. 2, 9–19.
[14] R. Abu-Dawwas, *On graded strongly 1-absorbing primary ideals*, Khayyam J. Math. **8** (2022), no. 1, 42–52.
[15] G. Calugareanu, *UN-rings*, J. Algebra Appl. **15** (2016), 165–182.
[16] Ü Tekir, S. Koç, and K. H. Oral, *n-Ideals of commutative rings*, Filomat **31** (2017), 2933–2941.
[17] K. Al-Zoubi, F. Turman and E. Y. Çelikel, *Gr-n-ideals in graded commutative rings*, Acta Universitatis Sapientiae: Mathematica **11** (2019), no. 1, 18–28.
[18] M. Bataineh and R. Abu-Dawwas, *On graded 2-prime ideals*, Mathematics **9** (2021), no. 5, 493, doi: https://doi.org/10.3390/math9050493.
[19] A. Melhem, M. Bataineh, and R. Abu-Dawwas, *Characterizations of graded rings over which every graded semi-primary ideal is graded 1-absorbing primary*, Wseas Trans. Math. **20** (2021), 547–553.
[20] R. Abu-Dawwas, H. Saber, T. Alraqad, and R. Jaradat, *On generalizations of graded multiplication modules*, Boletim da Sociedade Paranaense de Matematica **40** (2022), 1–10.