

Research Article

Jin Li* and Yongling Cheng

Spectral collocation method for convection-diffusion equation

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Abstract: Spectral collocation method, named linear barycentric rational interpolation collocation method (LBRICM), for convection-diffusion (C-D) equation with constant coefficient is considered. We change the discrete linear equations into the matrix equation. Different from the classical methods to solve the C-D equation, we solve the C-D equation with the time variable and space variable obtained at the same time. Furthermore, the convergence rate of the C-D equation by LBRICM is proved. Numerical examples are presented to test our analysis.

Keywords: linear rational interpolation, convection-diffusion equation, Barycentric form, error estimate, matrix equation

MSC 2020: 65D05, 65L60, 31A30

1 Introduction

In this study, we consider the convection-diffusion (C-D) equation with constant coefficient:

$$Du = \frac{\partial u(x, t)}{\partial t} + q \frac{\partial u(x, t)}{\partial x} + s \frac{\partial^2 u(x, t)}{\partial x^2} = f(x, t), \quad (1)$$

where $(x, t) \in [a, b] \times (0, T]$ and q and s are the constants related to fluid diffusion coefficient and concentration, respectively.

There are lots of methods to invest the C-D equation, such as finite difference method [1]. Spectral methods [2] have been widely used in lots of scientific engineering area.

In [3], nonlinear C-D equation is studied by the finite-difference lattice Boltzmann model. In reference [4], error estimate of a nonlinear C-D equation by finite element method is presented. In [5], the nonlinear C-D equation was studied and posteriori error estimates were also given. In [6], some spectral and pseudospectral approximations of Jacobi and Legendre type are considered for the C-D equation. In [7], a high-order accurate method for solving the one-dimensional heat and advection-diffusion equations is presented. In reference [8], C-D equation was solved by new formulas of the high-order derivatives of the fifth-kind Chebyshev polynomials. In [9], a transient C-D equation is considered. Time-fractional diffusion equation was solved by Petrov-Galerkin Lucas polynomials in [10]. In reference [11], the time-fractional diffusion equation was dealt by the explicit Chebyshev-Galerkin scheme. In reference [12], fractional diffusion-wave equation was dealt by the shifted fifth-kind Chebyshev polynomials. In reference [13], two-dimensional nonlinear reaction-diffusion equation with Riesz space-fractional was solved by the Legendre-Chebyshev spectral collocation method. In reference [14], fractional calculus approach was used for oxygen diffusion from capillary to tissues.

* **Corresponding author: Jin Li**, School of Science, Shandong Jianzhu University, Jinan, 250100, P. R. China, e-mail: lijn@lsec.cc.ac.cn
Yongling Cheng: School of Science, Shandong Jianzhu University, Jinan, 250100, P. R. China

As for the barycentric interpolation collocation method, there are barycentric Lagrange interpolation collocation method and barycentric rational interpolation collocation method. We can obtain high accuracy by changing the parameter of barycentric rational interpolation polynomial; in the following, we take the barycentric rational interpolation basis function to approximate the unknown function. Similarly, as the finite difference methods, linear barycentric rational method (LBRM) has been studied [15–19] by choosing the parameter in weight function. The LBRM has been used to solve certain problems such as delay Volterra integro-differential equations [20,21], Volterra integral equations [22,23], boundary value problem [24], plane elastic problems [25], incompressible plane elastic problems [26], nonlinear problems [27], and so on [28,29]. Linear barycentric rational collocation method (LBRM) to solve biharmonic equation [30,31], fractional differential equations [32–34], telegraph equation [35], Volterra integro-differential equation [36], and heat conduction equation [37] are studied.

In this study, we use the linear barycentric rational collocation method to solve the C-D equation. With the help of convergence rate of interpolation barycentric rational function, both the convergence rate of space and time of linear barycentric rational collocation method for the CD equation can be obtained at the same time.

This article is organized as follows: in Section 2, the differentiation matrices and collocation scheme for the C-D equation are presented. In Section 3, the convergence rate of space and time is proved. At last, some numerical examples are listed to illustrate our theorem.

2 Collocation scheme for C-D equation

The area Ω is partitioned into $a = x_1 < x_2 < \dots < x_m = b$, $0 = t_1 < t_2 < \dots < t_n = T$ with $h = \frac{b-a}{m}$ and $\tau = \frac{T}{n}$ being the uniform partition. However, for the quasi-uniform partition, the second kind of Chebyshev point $x = \cos(x_i)$, $t = \cos(t_j)$ is chosen.

Let

$$u(x, t) = \sum_{j=1}^m L_j(x) u_j(t), \quad (2)$$

where

$$u(x_i, t) = u_i(t), \quad i = 1, 2, \dots, m$$

and

$$L_j(x) = \frac{\frac{\lambda_j}{x - x_j}}{\sum_{k=1}^n \frac{\lambda_k}{x - x_k}}, \quad (3)$$

where

$$\lambda_k = \sum_{j \in J_k} (-1)^j \prod_{i=j, j \neq k}^{j+d_x} \frac{1}{x_k - x_i}, \quad J_k = \{j \in \{0, 1, \dots, l - d_x\} : k - d_x \leq j \leq k\}$$

is the basis function [36].

Then, we change equation (1) into the following:

$$\sum_{j=1}^m \delta_{ij} \ddot{u}_j(t) + q \sum_{j=1}^m C_{ij}^{(1)} u_j(t) + s \sum_{j=1}^m C_{ij}^{(2)} u_j(t) = f_i(t), \quad (4)$$

where $L_j(x_i) = \delta_{ij}$, $L_j''(x_i) = C_{ij}^{(2)}$ and $f(x_i, t) = f_i(t)$, $i = 1, 2, \dots, m$. Then, we have

$$\begin{bmatrix} \dot{u}_1(t) \\ \vdots \\ \dot{u}_m(t) \end{bmatrix} + \begin{bmatrix} C_{11}^{(1)} & \cdots & C_{1m}^{(1)} \\ \vdots & & \vdots \\ C_{m1}^{(1)} & \cdots & C_{mn}^{(1)} \end{bmatrix} + \begin{bmatrix} C_{11}^{(2)} & \cdots & C_{1m}^{(2)} \\ \vdots & & \vdots \\ C_{m1}^{(2)} & \cdots & C_{mn}^{(2)} \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ \vdots \\ f_m(t) \end{bmatrix}, \quad (5)$$

where

$$\mathbf{u}(\mathbf{t}) = [u_1(t), u_2(t), \dots, u_m(t)]^T, \quad \mathbf{f}(\mathbf{t}) = [f_1(t), f_2(t), \dots, f_m(t)]^T. \quad (6)$$

In order to obtain the time discrete, we take the barycentric interpolation function to the t similarly as x :

$$u_{ij} = u_i(t_j) = u(x_i, t_j), \quad i, j = 1, 2, \dots, m, n \quad (7)$$

and

$$u_i(t) = \sum_{k=1}^n L_k(t) u_{ik}, \quad (8)$$

then we have

$$\begin{bmatrix} \sum_{k=1}^n \dot{L}_k(t_j) u_{1k} \\ \vdots \\ \sum_{k=1}^n \dot{L}_k(t_j) u_{mk} \end{bmatrix} + \begin{bmatrix} C_{11}^{(1)} & \cdots & C_{1m}^{(1)} \\ \vdots & & \vdots \\ C_{m1}^{(1)} & \cdots & C_{mn}^{(1)} \end{bmatrix} \begin{bmatrix} \sum_{k=1}^n L_k(t_j) u_{1k} \\ \vdots \\ \sum_{k=1}^n L_k(t_j) u_{nk} \end{bmatrix} + \begin{bmatrix} C_{11}^{(2)} & \cdots & C_{1m}^{(2)} \\ \vdots & & \vdots \\ C_{m1}^{(2)} & \cdots & C_{mn}^{(2)} \end{bmatrix} \begin{bmatrix} \sum_{k=1}^n L_k(t_j) u_{1k} \\ \vdots \\ \sum_{k=1}^n L_k(t_j) u_{nk} \end{bmatrix} = \begin{bmatrix} f_1(t_j) \\ \vdots \\ f_m(t_j) \end{bmatrix}, \quad (9)$$

where $j = 1, 2, \dots, n$.

We transform the linear equations into the matrix form as:

$$\left[\left(\mathbf{I}_n \otimes \mathbf{D}^{(1)} \right) + q \left(\mathbf{C}^{(1)} \otimes \mathbf{I}_n \right) + s \left(\mathbf{C}^{(2)} \otimes \mathbf{I}_n \right) \right] \mathbf{U} = \mathbf{F} \quad (10)$$

and

$$\mathbf{L}\mathbf{U} = \mathbf{F}, \quad (11)$$

where $\mathbf{L} = \left(\mathbf{I}_n \otimes \mathbf{D}^{(1)} \right) + q \left(\mathbf{C}^{(1)} \otimes \mathbf{I}_n \right) + s \left(\mathbf{C}^{(2)} \otimes \mathbf{I}_n \right)$ and $\otimes$ denotes the Kronecker product [32,38].

3 Convergence and error analysis

We define the error of $u(x)$ with $r(x)$ as:

$$e(x) = u(x) - r(x) = (x - x_i) \cdots (x - x_{i+d}) u[x_i, x_{i+1}, \dots, x_{i+d}, x] \quad (12)$$

and

$$|e(x)| = \frac{\sum_{i=0}^{n-d} \lambda_i(x) [u(x) - p_i(x)]}{\sum_{i=0}^{n-d} \lambda_i(x)} = \frac{A(x)}{B(x)} = O(h^{d+1}), \quad (13)$$

where

$$A(x) = \sum_{i=0}^{n-d} (-1)^i u[x_i, \dots, x_{i+d}, x] \quad (14)$$

and

$$B(x) = \sum_{i=0}^{n-d} \lambda_i(x), \quad (15)$$

where

$$\lambda_i(x) = \frac{(-1)^i}{(x - x_i) \cdots (x - x_{i+d})} \quad (16)$$

and

$$|e^k(x)| \leq Ch^{d+1-k}, u \in C^{d+k}[a, b], k = 0, 1, \dots$$

We first define barycentric rational interpolants as:

$$r_{mn}(x, t) = \frac{\sum_{i=-d_1}^{m+d_1} \sum_{j=-d_2}^{n+d_2} \frac{w_{i,j}}{(x - x_i)(t - t_j)} \tilde{u}_{i,j}}{\sum_{i=-d_1}^{m+d_1} \sum_{j=-d_2}^{n+d_2} \frac{w_{i,j}}{(x - x_i)(t - t_j)}}, \quad (17)$$

where

$$w_{i,j} = (-1)^{i-d_1+j-d_2} \sum_{k_1 \in J_i, k_2 \in J_j, h_1=k_1, h_2 \neq i_1} \frac{1}{|x_i - x_{h_1}|} \prod_{h_2=k_2, h_2 \neq j}^{k_2+d_2} \frac{1}{|t_j - t_{h_2}|} \quad (18)$$

and $J_i = \{k_1 \in I_m : i - d_1 \leq k_1 \leq i\}, I_m = \{0, \dots, m - d_1\}$.

The error of $u(x, t)$ with $r_{mn}(x, t)$ is defined as:

$$\begin{aligned} e(x, t) &:= u(x, t) - r_{mn}(x, t) \\ &= (x - x_i) \cdots (x - x_{i+d_1}) u[x_i, x_{i+1}, \dots, x_{i+d_1}, x] \\ &\quad + (t - t_j) \cdots (t - t_{j+d_2}) u[t_j, t_{j+1}, \dots, t_{j+d_2}, t]. \end{aligned} \quad (19)$$

Now, we present the following theorem.

Theorem 1. For the error $e(x, t)$ in (19) and $u(x, t)$, we have

$$|e(x, t)| \leq C(h^{d_1+1} + \tau^{d_2+1}). \quad (20)$$

Proof. For $(x, t) \in \Omega$ and $e(x, t)$, we have

$$e(x, t) = u(x, t) - r_{mn}(x, t) = \frac{\sum_{i=0}^{m-d_1} \sum_{j=0}^{n-d_2} \lambda_i(x) \lambda_j(t) (u(x, t) - r_{mn}(x, t))}{\sum_{i=0}^{m-d_1} \sum_{j=0}^{n-d_2} \lambda_i(x) \lambda_j(t)} \quad (21)$$

and

$$\begin{aligned} u(x, t) - r_{mn}(x, t) &= u(x, t) - u(x_m, t) + u(x_m, t) - r_{mn}(x, t) \\ &= (x - x_i) \cdots (x - x_{i+d_1}) u[x_i, x_{i+1}, \dots, x_{i+d_1}, x_m, t] + (t - t_j) \cdots (t - t_{j+d_2}) u[t_j, t_{j+1}, \dots, t_{j+d_2}, x, t]. \end{aligned} \quad (22)$$

We reach that

$$u(x, t) - r_{mn}(x, t) = \frac{\sum_{i=0}^{m-d_1} (-1)^i u[x_i, x_{i+1}, \dots, x_{i+d_1}, x_m, t]}{\sum_{i=0}^{m-d_1} \lambda_i(x)} + \frac{\sum_{j=0}^{n-d_2} (-1)^j u[t_j, t_{j+1}, \dots, t_{j+d_2}, x, t]}{\sum_{j=0}^{n-d_2} \lambda_j(t)}. \quad (23)$$

By the similarly analysis in Floater and Kai [29], we have

$$\left| \sum_{i=0}^{m-d_1} \lambda_i(x) \right| \geq \frac{1}{d_1! h^{d_1+1}} \quad (24)$$

and

$$\left| \sum_{j=0}^{n-d_2} \lambda_j(t) \right| \geq \frac{1}{d_2! \tau^{d_2+1}}. \quad (25)$$

Combining (23), (24), and (25) together, the proof of Theorem 1 is completed. $\square$

Corollary 1. For the $e(x, t)$ defined in (19), we have

$$\begin{cases} |e_x(x, t)| \leq C(h^{d_1} + \tau^{d_2+1}), & u(x, t) \in \Omega, \\ |e_t(x, t)| \leq C(h^{d_1+1} + \tau^{d_2}), & u(x, t) \in \Omega, \\ |e_{xx}(x, t)| \leq C(h^{d_1-1} + \tau^{d_2+1}), & u(x, t) \in \Omega. \end{cases} \quad (26)$$

Let $u(x, t)$ be the analysis solution of (1), $u(x_m, t_n)$ is its approximate value, there holds

$$\lim_{m,n \rightarrow \infty} u(x_m, t_n) = u(x, t), \quad (27)$$

$$Du(x_m, t_n) = f(x, t), \quad (28)$$

and

$$\lim_{m,n \rightarrow \infty} Du(x_m, t_n) = f(x, t). \quad (29)$$

Based on the aforementioned lemma, we obtain the following theorem.

Theorem 2. Let operator D be defined as (28) and $f(x, t) \in \Omega$, and we have

$$|u(x, t) - u(x_m, t_n)| \leq C(h^{d_1-1} + \tau^{d_2}).$$

Proof. As

$$\begin{aligned} Du(x, t) - Du(x_m, t_n) &= u_t(x, t) - u_t(x_m, t_n) + q[u_x(x, t) - u_x(x_m, t_n)] + s[u_{xx}(x, t) - u_{xx}(x_m, t_n)] \\ &=: R_1(x, t) + R_2(x, t) + R_3(x, t), \end{aligned} \quad (30)$$

where

$$\begin{aligned} R_1(x, t) &= u_t(x, t) - u_t(x_m, t_n), \\ R_2(x, t) &= q[u_x(x, t) - u_x(x_m, t_n)], \\ R_3(x, t) &= s[u_{xx}(x, t) - u_{xx}(x_m, t_n)]. \end{aligned}$$

As for the R_1 , we have

$$\begin{aligned} R_1(x, t) &= u_t(x, t) - u_t(x_m, t_n) \\ &= u_t(x, t) - u_t(x_m, t) + u_t(x_m, t) - u_t(x_m, t_n) \\ &= \frac{\sum_{i=0}^{m-d_1} (-1)^i u_t[X_i, X_{i+1}, \dots, X_{i+d_1}, x, t]}{\sum_{i=0}^{m-d_1} \lambda_i(x)} + \frac{\sum_{j=0}^{n-d_2} (-1)^j u_t[t_j, t_{j+1}, \dots, t_{j+d_2}, x_m, t]}{\sum_{j=0}^{n-d_2} \lambda_j(t)} \\ &= e_t(x, t) + e_t(x_m, t). \end{aligned} \quad (31)$$

By the corollary, we obtain

$$|R_1(x, t)| \leq |e_t(x, t) + e_t(x_m, t)| \leq C(h^{d_1+1} + \tau^{d_2}). \quad (32)$$

Similarly, for $R_2(x, t)$ and $R_3(x, t)$, we have

$$R_2(x, t) = u_x(x, t) - u_x(x_m, t_n) = e_x(x, t) + e_x(x_m, t), \quad (33)$$

$$|R_2(x, t)| \leq |e_x(x, t) + e_x(x_m, t)| \leq C(h^{d_1} + \tau^{d_2+1}), \quad (34)$$

and

$$R_3(x, t) = u_{xx}(x, t) - u_{xx}(x_m, t_n) = e_{xx}(x, t_n) + e_{xx}(x_m, t_n), \quad (35)$$

$$|R_3(x, t)| \leq |e_{xx}(x, t) + e_{xx}(x_m, t)| \leq C(h^{d_1-1} + \tau^{d_2+1}). \quad (36)$$

Table 1: Convergence rate of equidistant nodes with $d_2 = 9$ and $t = 1$

m, n	$d_1 = 1$	h^a	$d_1 = 2$	h^a	$d_1 = 3$	h^a	$d_1 = 4$	h^a
10×10	3.5548×10^{-3}		3.9350×10^{-4}		1.1116×10^{-4}		1.0400×10^{-5}	
20×20	1.4005×10^{-3}	1.3438	8.0420×10^{-5}	2.2907	1.0650×10^{-5}	3.3837	5.8204×10^{-7}	4.1593
40×40	5.3926×10^{-4}	1.3769	1.5825×10^{-5}	2.3453	1.0249×10^{-6}	3.3774	2.9676×10^{-8}	4.2937
80×80	2.0200×10^{-4}	1.4166	3.0026×10^{-6}	2.3979	9.6236×10^{-8}	3.4127	1.4282×10^{-9}	4.3770

Combining the identity (30), (32), (34), and (36), the proof of theorem is complete. $\square$

4 Numerical examples

Three examples are presented to illustrate our theorem. All examples were performed on personal computer by Matlab 2013a with a (configuration: Intel(R) Core(TM) i5-8265U CPU @ 1.60GHz 1.80 GHz).

Example 1. Consider the C-D equation:

$$u_t(x, t) = u_{xx}(x, t) - u_x(x, t) + f_{b_0}(x, t),$$

with the condition

$$u(x, 0) = 0, u_x(1, t) + b_0 u(1, t) = 0, u_x(0, t) - u(0, t) = 0.$$

Its analysis solutions is

$$u(x, t) = [3e^x - e(1 + x + x^2)]e^{-\delta(b_0)t}(1 - e^{-t}).$$

In this example, we test the linear barycentric rational with the equidistant nodes. Table 1 shows that the convergence rate is $O(h^{d_1})$ with $d_2 = 9$ first given for the space area for $t = 1$. In Table 2, for the space area partition $d_1 = 9$ first given, the convergence rate of times is $O(\tau^{d_2})$, which agrees with our theorem analysis.

We choose $m = 40, n = 40, d_1 = 9, d_2 = 9$, and $\beta = 1$ to test our algorithm. Figure 1 shows the error estimate of equidistant nodes, and Figure 2 shows the error estimate of Chebyshev nodes. From Figures 1 and 2, we know that the barycentric rational interpolation collocation method has higher accuracy under the condition of Chebyshev nodes, and the barycentric rational interpolation collocation method also has higher accuracy under the condition of equidistant nodes.

In Table 3, we test the linear barycentric rational with the Chebyshev nodes, which that shows the convergence rate is $O(h^{d_2+1})$ with $d_1 = 9$ first given for the space area. In Table 4, for $d_2 = 9$ first given, the convergence rate of times is $O(\tau^{d_1+1})$, which agrees with our theorem analysis.

Table 2: Convergence rate of equidistant nodes with $d_1 = 9$ and $t = 1$

m, n	$d_2 = 1$	h^a	$d_2 = 2$	h^a	$d_2 = 3$	h^a	$d_2 = 4$	h^a
10×10	1.0353×10^{-2}		2.1124×10^{-3}		6.3532×10^{-4}		1.7118×10^{-5}	
20×20	4.7875×10^{-3}	1.1127	5.1680×10^{-4}	2.0312	7.0607×10^{-5}	3.1696	1.1376×10^{-6}	3.9115
40×40	2.3078×10^{-3}	1.0528	1.2764×10^{-4}	2.0176	8.3487×10^{-6}	3.0802	7.1789×10^{-8}	3.9860
80×80	1.1346×10^{-3}	1.0244	3.1707×10^{-5}	2.0092	1.0157×10^{-6}	3.0391	4.4931×10^{-9}	3.9980

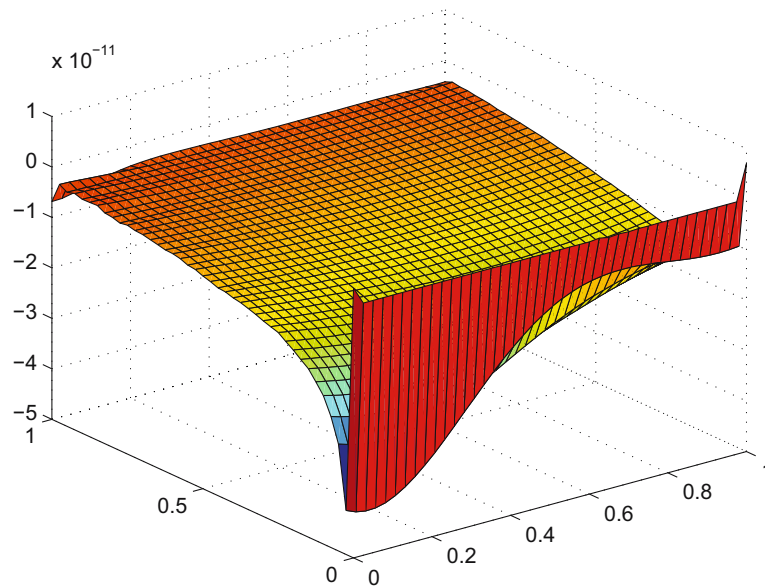


Figure 1: Error estimate of equidistant nodes with $m = 40$, $n = 40$, $d_1 = 9$, $d_2 = 9$, and $\beta = 1$.

In Table 5, we show that the error estimate of time equals $t = 1, 10, 40, 80$, and 100 with $m = 40$, $n = 40$, $d_1 = 9$, $d_2 = 9$, and $\beta = 1$, and the error estimate is given as blow for the equidistant nodes and Chebyshev nodes.

Example 2. Consider the C-D equation:

$$\begin{aligned} u_t(x, t) &= u_{xx}(x, t) - u_x(x, t) + f(x, t), \quad 0 < x < 1; t > 0 \\ u(x, 0) &= 0, \quad 0 < x < 1 \\ u_x(1, t) + 2u(1, t) &= 0, \quad u_x(0, t) - \frac{3}{2}u(0, t) = 0, \quad t > 0. \end{aligned}$$

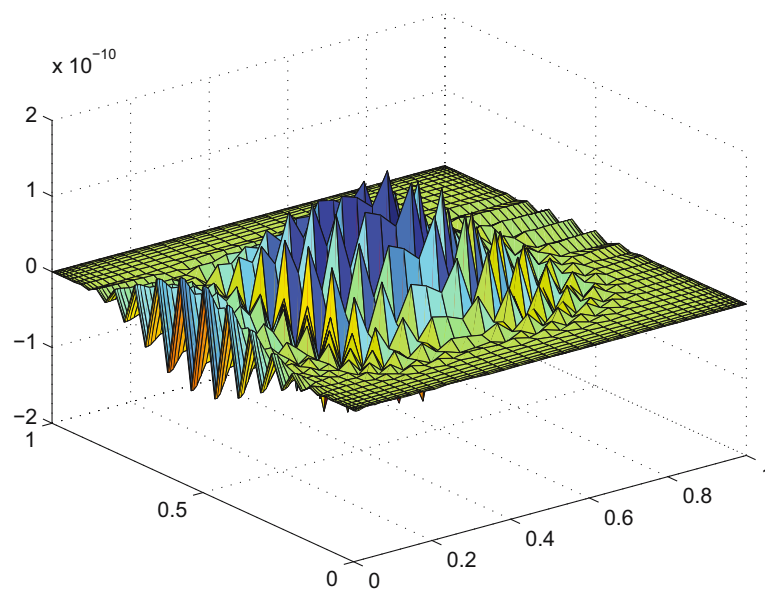


Figure 2: Error estimate of Chebyshev nodes with $m = 40$, $n = 40$, $d_1 = 9$, $d_2 = 9$, and $\beta = 1$.

Table 3: Convergence rate of Chebyshev nodes with $d_1 = 9$ and $t = 1$

m, n	$d_2 = 1$	h^a	$d_2 = 2$	h^a	$d_2 = 3$	h^a	$d_2 = 4$	h^a
10×10	1.1867×10^{-3}		9.1568×10^{-5}		1.7898×10^{-5}		1.1576×10^{-6}	
20×20	2.6617×10^{-4}	2.1565	1.0834×10^{-5}	3.0792	1.0571×10^{-6}	4.0816	4.9817×10^{-8}	4.5383
40×40	6.0449×10^{-5}	2.1386	1.2475×10^{-6}	3.1185	5.9525×10^{-8}	4.1505	1.4384×10^{-9}	5.1141
80×80	1.4033×10^{-5}	2.1069	1.4806×10^{-7}	3.0748	3.5266×10^{-9}	4.0771	8.8800×10^{-9}	—

Table 4: Convergence rate of Chebyshev nodes with $d_2 = 9$ and $t = 1$

n	$d_1 = 1$	h^a	$d_1 = 2$	h^a	$d_1 = 3$	h^a	$d_1 = 4$	h^a
10×10	2.4674×10^{-3}		7.7254×10^{-5}		8.2887×10^{-6}		1.0909×10^{-6}	
20×20	1.0512×10^{-3}	1.2309	1.8978×10^{-5}	2.0253	6.8668×10^{-7}	3.5934	1.7571×10^{-9}	9.2781
40×40	3.7121×10^{-4}	1.5017	1.8696×10^{-6}	3.3435	1.8229×10^{-8}	5.2353	3.7351×10^{-11}	5.5559
80×80	1.2165×10^{-4}	1.6095	6.5999×10^{-6}	—	7.6684×10^{-6}	—	4.4403×10^{-6}	—

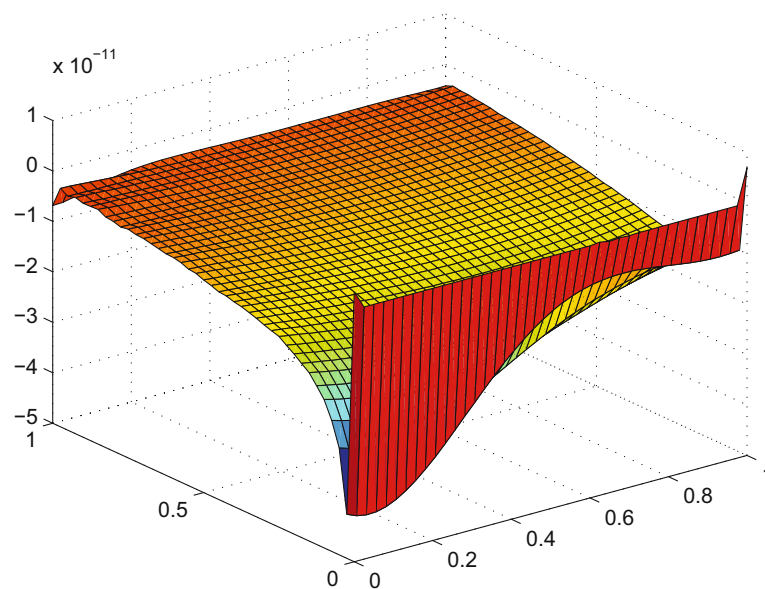
Table 5: Convergence rate of long time with $m = 40$, $n = 40$, $d_1 = 9$, $d_2 = 9$, and $\beta = 1$

n	$t = 1$	$t = 10$	$t = 40$	$t = 80$	$t = 100$
Equidistant nodes	4.2413×10^{-11}	3.4452×10^{-6}	3.8497×10^{-3}	6.4714×10^{-3}	5.0873×10^{-3}
Chebyshev nodes	1.5197×10^{-10}	1.2543×10^{-7}	3.6741×10^{-3}	1.3245×10^{-01}	3.1104×10^{-01}

Its analysis solutions is

$$u(x, t) = \left(1 + \frac{1}{2}x - x^2\right)e^x te^{-\beta t}$$

and

**Figure 3:** Convergence rate of equidistant nodes with $m = 40$, $n = 40$, $d_1 = 9$, $d_2 = 9$, and $\beta = 0.6$.

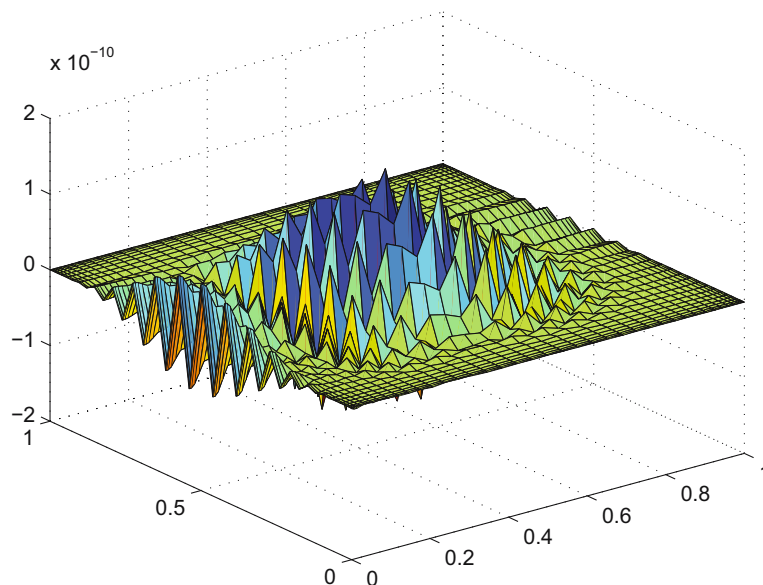


Figure 4: Error estimate of Chebyshev nodes with $m = 40$, $n = 40$, $d_1 = 9$, $d_2 = 9$, and $\beta = 0.6$.

Table 6: Convergence rate of equidistant nodes with $d_1 = 9$ and $t = 1$

m, n	$d_2 = 1$	h^a	$d_2 = 2$	h^a	$d_2 = 3$	h^a	$d_2 = 4$	h^a
10×10	2.0117×10^{-2}		9.9996×10^{-4}		2.6462×10^{-4}		9.8527×10^{-6}	
20×20	7.6706×10^{-3}	1.3910	1.9840×10^{-4}	2.3334	2.4016×10^{-5}	3.4619	5.3162×10^{-7}	4.2120
40×40	2.8333×10^{-3}	1.4368	3.7231×10^{-5}	2.4139	2.1654×10^{-6}	3.4713	2.5515×10^{-8}	4.3810
80×80	1.0279×10^{-3}	1.4628	6.7990×10^{-6}	2.4531	1.9398×10^{-7}	3.4807	2.0626×10^{-9}	3.6288

Table 7: Convergence rate of equidistant nodes with $d_2 = 9$ and $t = 1$

m, n	$d_1 = 1$	h^a	$d_1 = 2$	h^a	$d_1 = 3$	h^a	$d_1 = 4$	h^a
10×10	2.6369×10^{-2}		3.4104×10^{-3}		1.0652×10^{-3}		6.8651×10^{-5}	
20×20	6.0571×10^{-3}	1.0630	3.9190×10^{-4}	2.0623	5.6503×10^{-5}	3.1775	2.1560×10^{-6}	3.9337
40×40	1.4660×10^{-3}	1.0205	4.7384×10^{-5}	2.0218	3.2950×10^{-6}	3.0737	6.6715×10^{-8}	3.9880
80×80	3.5956×10^{-4}	1.0149	5.8076×10^{-6}	2.0157	1.9852×10^{-7}	3.0402	2.0590×10^{-9}	4.0053

$$f(x, t) = \frac{1}{2} e^x e^{-\beta} (2 + x - 2x^2 - 2\beta t - \beta x t + 2\beta x^2 t + 3t + 4xt).$$

Table 8: Convergence rate of Chebyshev nodes with $d_2 = 9$ and $t = 10$

m, n	$d_1 = 1$	h^a	$d_1 = 2$	h^a	$d_1 = 3$	h^a	$d_1 = 4$	h^a
10×10	3.5254×10^{-2}		1.4203×10^{-3}		1.5504×10^{-3}		1.5309×10^{-3}	
20×20	1.2567×10^{-2}	1.4882	1.0407×10^{-4}	3.7706	6.3260×10^{-6}	7.9371	5.5968×10^{-6}	8.0956
40×40	3.9582×10^{-3}	1.6667	9.8234×10^{-6}	3.4051	1.0552×10^{-7}	5.9057	2.0826×10^{-8}	8.0701
80×80	1.7563×10^{-3}	1.1723	1.0063×10^{-6}	3.2871	3.5781×10^{-7}	—	1.2045×10^{-6}	—

Table 9: Convergence rate of Chebyshev nodes with $d_1 = 9$ and $t = 10$

m, n	$d_2 = 1$	h^a	$d_2 = 2$	h^a	$d_2 = 3$	h^a	$d_2 = 4$	h^a
10×10	5.2543×10^{-2}		3.7959×10^{-2}		2.4572×10^{-2}		1.2281×10^{-2}	
20×20	1.8911×10^{-2}	1.4743	7.4472×10^{-3}	2.3497	3.4116×10^{-3}	2.8485	1.4325×10^{-3}	3.0998
40×40	4.7752×10^{-3}	1.9856	1.0058×10^{-3}	2.8884	2.4478×10^{-4}	3.8009	5.6804×10^{-5}	4.6564
80×80	1.1944×10^{-3}	1.9993	1.2625×10^{-4}	2.9940	1.5335×10^{-5}	3.9966	2.3166×10^{-6}	4.6159

Table 10: Convergence rate of long time with $m = 40$, $n = 40$, $d_1 = 9$, $d_2 = 9$, and $\beta = 1$

n	$t = 1$	$t = 10$	$t = 40$	$t = 80$	$t = 100$
Equidistant nodes	1.1725×10^{-9}	8.0311×10^{-9}	1.2664×10^{-4}	3.2401×10^{-3}	7.0226×10^{-3}
Chebyshev nodes	3.8538×10^{-8}	1.4320×10^{-9}	3.7099×10^{-5}	5.5803×10^{-3}	2.3399×10^{-2}

We still choose $m = 40$, $n = 40$, $d_1 = 9$, $d_2 = 9$, and $\beta = 1$ to test our theorem. Figure 3 shows the error estimate of equidistant nodes, and Figure 4 shows the error figure of Chebyshev nodes. From Figures 3 and 4, we know that the barycentric rational interpolation collocation method has higher accuracy under the condition of Chebyshev nodes, and the barycentric rational interpolation collocation method also has higher accuracy under the condition of equidistant nodes.

In this example, in order to test our theorem analysis, we have first given $d_1 = 9$ and $t = 1$ to be the exact solution. Table 6 shows that the convergence rate of equidistant nodes is $O(\tau^{d_2})$, which agrees with our theorem analysis. In Table 7, in order to test the convergence rate of space, $d_2 = 9$ is first given to be the exact solution, and the convergence rate of times is $O(h^{d_1})$, which is one order higher than our theorem analysis.

In Table 8, we test the linear barycentric rational with the Chebyshev nodes, and shows that the convergence rate is $O(h^{d_1+1})$ with $d_2 = 9$ and $t = 10$ first given for the space area. In Table 9, for the space area partition $d_1 = 9$ first given, the convergence rate of times is $O(\tau^{d_2+1})$, which is one order higher than our theorem analysis.

Table 11: Convergence rate of equidistant nodes with $d_1 = 9$ and $t = 1$

m, n	$d_2 = 1$	h^a	$d_2 = 2$	h^a	$d_2 = 3$	h^a	$d_2 = 4$	h^a
10×10	8.0557×10^{-4}		1.2053×10^{-5}		3.0259×10^{-6}		1.6513×10^{-7}	
20×20	4.1120×10^{-4}	0.9702	3.4240×10^{-6}	1.8156	4.1782×10^{-7}	2.8564	1.4356×10^{-8}	3.5239
40×40	1.7850×10^{-4}	1.2039	8.0203×10^{-7}	2.0940	4.8292×10^{-8}	3.1130	9.0948×10^{-10}	3.9805
80×80	7.0234×10^{-5}	1.3457	1.6862×10^{-7}	2.2499	4.8670×10^{-9}	3.3107	1.4183×10^{-9}	—

Table 12: Convergence rate of equidistant nodes with $d_2 = 9$ and $t = 1$

m, n	$d_1 = 1$	h^a	$d_1 = 2$	h^a	$d_1 = 3$	h^a	$d_1 = 4$	h^a
10×10	3.9671×10^{-4}		7.6555×10^{-6}		2.4860×10^{-6}		1.9775×10^{-7}	
20×20	1.2339×10^{-4}	1.6848	9.9756×10^{-7}	2.9400	1.5130×10^{-7}	4.0383	7.5609×10^{-9}	4.7090
40×40	3.7144×10^{-5}	1.7320	1.2982×10^{-7}	2.9419	9.5794×10^{-9}	3.9813	2.8567×10^{-10}	4.7261
80×80	1.0934×10^{-5}	1.7643	1.6644×10^{-8}	2.9634	7.0405×10^{-10}	3.7662	8.2156×10^{-10}	—

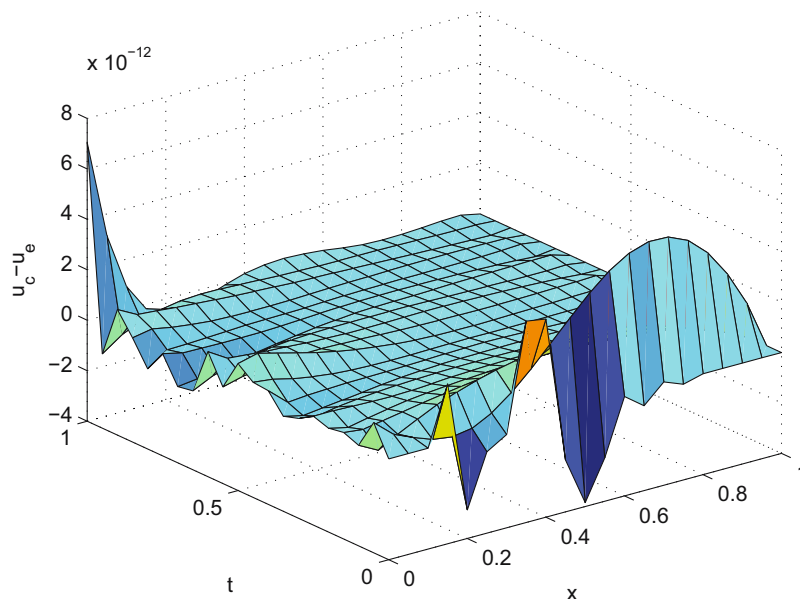


Figure 5: Convergence rate of equidistant nodes with $m = 20$, $n = 20$, $d_1 = 9$, $d_2 = 9$, and $t = 1$.

In Table 10, we show that the error estimate of time equals $t = 1, 10, 40, 80$, and 100 with $m = 40$, $n = 40$, $d_1 = 9$, $d_2 = 9$, and $\beta = 1$, the error estimate is given for the equidistant nodes and Chebyshev nodes. It is shown that linear barycentric rational collocation method is still effective for the long time.

Example 3. Consider the nonlinear C-D equation:

$$u_t(x, t) = (u + u^2)u_{xx}(x, t) + (2u + 1)u_x^2(x, t) - 2uu_x(x, t) + 2u(1 - 2u - 6u^2),$$

$$0 < x < 1; \quad t > 0 \quad u(x, 0) = e^{2x}, \quad u(x, 1) = e^{2x+2}.$$

Its analysis solutions is

$$u(x, t) = e^{2x+2t},$$

and the boundary condition and initial condition are also given by the analysis solution.

In this example, we take direct linearization scheme with u_0 :

$$u_t(x, t) = (u_0 + u_0^2)u_{xx}(x, t) + (2u_0 + 1)u_x^2(x, t) - 2u_0u_x(x, t) + 2u_0(1 - 2u_0 - 6u_0^2)$$

and we test our algorithm to solve the nonlinear C-D equation. In order to test our theorem analysis, we have first given $d_1 = 9$ and $t = 1$ to be the exact solution. Table 11 shows that the convergence rate of equidistant nodes is $O(\tau^{d_2})$, which agrees with our theorem analysis. In Table 12, in order to test the convergence rate of space, $d_2 = 9$ is first given to be the exact solution and the convergence rate of times is $O(h^{d_1+1})$, which is one order higher than our theorem analysis. From this example, the same convergence rate is the same as the linear C-D equation, while the theorem analysis is beyond our goal and will be given in another study.

Figure 5 shows the error estimate of equidistant nodes. From Figure 5, we know that the barycentric rational interpolation collocation method has higher accuracy under the condition of equidistant nodes.

5 Conclusion

In this study, (1+1)-dimensional C-D equation has been solved by LBRICM, and the discrete C-D equation can be written to matrix equation using the Kronecker product. For the linear C-D equation, the convergence rate of space and time at the same time is proved with the constant coefficient. However, for the nonlinear C-D equation, the convergence rate is the same. As for (2+1)-dimensional linear and nonlinear C-D equation, magneto-micropolar equations, and fractional C-D equation, we will study in further article to overcome the difficulty of the full coefficient matrix.

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References

- [1] N. A. Kudryashov and D. I. Sineshchikov, *Analytical solution of a nonlinear convection-diffusion equation with polynomial sources*, Model. Anal. Inform. Sist. **23** (2016), 309–316.
- [2] J. Shen, T. Tang, and L. Wang, *Spectral Methods Algorithms, Analysis and Applications*, Springer, London, 2011.
- [3] H. Wang, B. Shi, H. Liang, and Z. Chai, *Finite difference lattice Boltzmann modal for nonlinear convection-diffusion equations*, Appl. Math. Comput. **309** (2017), 334–349, DOI: <https://doi.org/10.1016/j.amc.2017.04.015>.
- [4] R. Eymard, T. Gallonet, and R. Herbin, *Error estimate for approximate solutions of a nonlinear convection-diffusion problem*, Adv. Differential Equations **7** (2002), 419–440, DOI: <https://doi.org/10.1093/imanum/18.4.563>.
- [5] J. D. Frutos, B. G. Archilla, and J. Novo, *Nonlinear convection-diffusion problems: fully discrete approximations and a posteriori error estimates*, IMA J. Numer. Anal. **31** (2011), 1402–1430, DOI: <https://doi.org/10.1093/imanum/drq017>.
- [6] C. Canuto and A. Quarteroni, *Error estimates for spectral and pseudospectral approximations of hyperbolic equations*, SIAM J. Numer. Anal. **19** (1982), no. 3, 629–642.
- [7] A. Mohebbi and M. Dehghan, *High-order compact solution of the one-dimensional heat and advection-diffusion equations*, Appl. Math. Model. **34** (2010), 3071–3084, DOI: <https://doi.org/10.1016/j.apm.2010.01.013>.
- [8] W. M. Abd-Elhameed and Y. H. Youssri, *New formulas of the high-order derivatives of fifth-kind Chebyshev polynomials: Spectral solution of the convection-diffusion equation*, Numer. Methods Partial Differential Equations (2023), 1–17, DOI: <https://doi.org/10.1002/num.22756>.
- [9] N. R. Bayramov and J. K. Kraus, *On the stable solution of transient convection-diffusion equations*, J. Comput. Appl. Math. **280** (2015), 275–293, DOI: <https://doi.org/10.1016/j.cam.2014.12.001>.
- [10] Y. H. Youssri and A. G. Atta, *Petrov-Galerkin Lucas polynomials procedure for the time-fractional diffusion equation*, Contemp. Math. [Internet], **4**(2023 Apr. 8), no. 2, 230–248, [cited 2023 Jun. 22], <https://ojs.wiserpub.com/index.php/CM/article/view/2420>.
- [11] M. Moustafa, Y. H. Youssri, and A. G. Atta, *Explicit Chebyshev-Galerkin scheme for the time-fractional diffusion equation*, J. Modern Phys. C. **35** (2024), 2450002, DOI: <https://doi.org/10.1142/S0129183124500025>.
- [12] A. G. Atta, W. M. Abd-Elhameed, and Y. H. Youssri, *Shifted fifth-kind Chebyshev polynomials Galerkin-based procedure for treating fractional diffusion-wave equation*, Int. J. Modern Phys. C **33** (2022), no. 8, 2250102, DOI: <https://doi.org/10.1142/S0129183122501029>.
- [13] M. A. Abdelkawy and S. A. Alyami, *Legendre-Chebyshev spectral collocation method for two-dimensional nonlinear reaction-diffusion equation with Riesz space-fractional*, Chaos Solitons Fractals **151** (2021), 111279, DOI: <https://doi.org/10.1016/j.chaos.2021.111279>.
- [14] V. F. Morales-Delgado, J. F. Gómez-Aguilar, K. M. Saad, M. A. Khan, and P. Agarwal, *Analytic solution for oxygen diffusion from capillary to tissues involving external force effects: A fractional calculus approach*, Physica A Stat. Mech. Appl. **523** (2019), 48–65, DOI: <https://doi.org/10.1016/j.physa.2019.02.018>.
- [15] P. Berrut, M. S. Floater, and G. Klein, *Convergence rates of derivatives of a family of barycentric rational interpolants*, Appl. Numer. Math. **61** (2011), no. 9, 989–1000, DOI: <https://doi.org/10.1016/j.apnum.2011.05.001>.
- [16] P. Berrut and G. Klein, *Recent advances in linear barycentric rational interpolation*, J. Comput. Appl. Math. **259** (2014), Part A, 95–107, DOI: <https://doi.org/10.1016/j.cam.2013.03.044>.
- [17] G. Klein and J. P. Berrut, *Linear rational finite differences from derivatives of barycentric rational interpolants*, Siam J. Numer. Anal. **50** (2012), no. 2, 643–656, DOI: <https://doi.org/10.1137/110827156>.

- [18] G. Klein and J.-P. Berrut, *Linear barycentric rational quadrature*, BIT Numer. Math. **52** (2012), 407–424, DOI: <https://doi.org/10.1007/s10543-011-0357-x>.
- [19] M. S. Floater and K. Hormann, *Barycentric rational interpolation with no poles and high rates of approximation*, Numer. Math. **107** (2007), no. 2, 315–331, DOI: <https://doi.org/10.1007/s00211-007-0093-y>.
- [20] A. Abdi, J.-P. Berrut, and S. A. Hosseini, *The linear barycentric rational method for a class of delay Volterra integro-differential equations*, J. Scientific Comput. **75** (2001), no. 3, 1195–1210, DOI: <https://doi.org/10.1007/s10915-017-0608-3>.
- [21] S. Li and Z. Wang, *High Precision Meshless Barycentric Interpolation Collocation Method-Algorithmic Program and Engineering Application*, Science Publishing, 2012.
- [22] J. P. Berrut, S. A. Hosseini, and G. Klein, *The linear barycentric rational quadrature method for Volterra integral equations*, SIAM J. Sci. Comput. **36** (2014), no. 1, 105–123, DOI: <https://doi.org/10.1137/120904020>.
- [23] M. Li and C. Huang, *The linear barycentric rational quadrature method for auto-convolution Volterra integral equations*, J. Sci. Comput. **78** (2019), no. 1, 549–564, DOI: <https://doi.org/10.1007/s10915-018-0779-6>.
- [24] J. Leey and L. Greengardz, *A fast adaptive numerical method for stiff two-point boundary value problems*, Siam. J. Sci. Comput. **18** (1997), no. 2, 403–429, DOI: <https://doi.org/10.1137/S1064827594272797>.
- [25] Z. Wang, L. Zhang, Z. Xu, and J. Li, *Barycentric interpolation collocation method based on mixed displacement-stress formulation for solving plane elastic problems*, Chinese J. Appl. Mechanics **35** (2018), no. 2, 304–309.
- [26] Z. Wang, Z. Xu, and J. Li, *Mixed barycentric interpolation collocation method of displacement-pressure for incompressible plane elastic problems*, Chinese J. Appl. Mechanics. **35** (2018), no. 3, 195–201.
- [27] Z. Wang and S. Li, *Barycentric Interpolation Collocation Method for Nonlinear Problems*, National Defense Industry Press, Beijing, 2015.
- [28] E. Cirillo and K. Hormann, *On the Lebesgue constant of barycentric rational Hermite interpolants at equidistant nodes*, J. Comput. Appl. Math. **349** (2019), 292–301, DOI: <https://doi.org/10.1016/j.cam.2018.06.011>.
- [29] M. S. Floater and H. Kai, *Barycentric rational interpolation with no poles and high rates of approximation*, Numer. Math. **107** (2007), no. 2, 315–331, DOI: <https://doi.org/10.1007/s00211-007-0093-y>.
- [30] J. Li, *Linear barycentric rational collocation method for solving biharmonic equation*, Demonstr. Math. **55** (2022), 587–603, DOI: <https://doi.org/10.1515/dema-2022-0151>.
- [31] J. Li, Y. Cheng, Z. C. Li, and Z. K. Tian, *Linear barycentric rational collocation method for solving generalized Poisson equations*, Math. Biosci. Eng. **20** (2023), no. 3, 4782–4797, DOI: <https://doi.org/10.3934/mbe.2023221>.
- [32] J. Li, X. Su, and K. Zhao, *Barycentric interpolation collocation algorithm to solve fractional differential equations*, Math. Comput. Simulat. **205** (2023), 340–367, DOI: <https://doi.org/10.1016/j.matcom.2022.10.005>.
- [33] J. Li and Y. Cheng, *Barycentric rational interpolation method for solving fractional cable equation*, Electr. Res. Archive **31** (2023), no. 6, 3649–3665, DOI: <https://doi.org/10.3934/era.2023185>.
- [34] J. Li, *Barycentric rational collocation method for fractional reaction-diffusion equation*, AIMS Math. **8** (2023), no. 4, 9009–9026, DOI: <https://doi.org/10.3934/math.2023451>.
- [35] J. Li, X. Su, and J. Qu, *Linear barycentric rational collocation method for solving telegraph equation*, Math. Meth. Appl. Sci. **44** (2021), 11720–11737, DOI: <https://doi.org/10.1002/mma.7548>.
- [36] J. Li and Y. Cheng, *Linear barycentric rational collocation method for solving second-order Volterra integro-differential equation*, Comput. Appl. Math. **39** (2020), 1–9, DOI: <https://doi.org/10.1007/s40314-020-1114-z>.
- [37] J. Li and Y. Cheng, *Linear barycentric rational collocation method for solving heat conduction equation*, Numer. Methods Partial Differential Equations **37** (2021), no. 1, 533–545, DOI: <https://doi.org/10.1002/num.22539>.
- [38] G. W. Stewart, *Matrix Algorithms*, Vol. II, Eigensystems Paperback-August 1, Philadelphia, 2001.