Research Article

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Rejection and symmetric difference of bipolar picture fuzzy graph

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Abstract: Due to the absence of a negative of three membership functions, there are drawbacks to the existing definition of a picture fuzzy graph (PFG). In that definition of bipolar picture fuzzy graph (BPFG), membership function, neutral membership function, nonmembership function, negative of membership function, negative of neutral membership function, and negative of nonmembership function are involved. A BPFG is the extension of PFG. In this manuscript, we present some properties of symmetric difference, and rejection of BPFG.

Keywords: bipolar picture fuzzy graph, symmetric difference, rejection, degree of vertex, total degree of vertex

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1 Introduction

Graph theory, a fascinating branch of applied mathematics, has a rich history characterized by multiple independent discoveries. Notably, Euler, a prominent mathematician from 1707 to 1782, made a significant breakthrough in 1736 by solving the famous Konigsberg bridge problem. Around a century later, in 1847, Kirchhoff introduced the theory of trees, aiming to address the complex set of simultaneous linear equations governing the flow of electric current within intricate networks of circuits and branches. Graphs, as fundamental components of graph theory, have proven to be an exceptionally versatile tool for modeling a wide array of real-world processes. They are instrumental in representing the intricate relationships between various entities within complex systems, ranging from ecosystems to electrical and computer networks, and beyond. Euler's groundbreaking solution to the Konigsberg bridges puzzle in 1735 laid the foundation for the development of graph theory and led to some fundamental insights about graphs as mathematical structures. Notably, the pursuit of solving diagram tracing puzzles also contributed to the discovery of concepts that would later be recognized as defining features of specific types of graphs. It was not until the late nineteenth century that Rouse Ball made the pivotal connection between diagram tracing puzzles and the Konigsberg bridges problem, as documented in his work "Mathematical Recreations and Problems." This historical narrative underscores the iterative and collaborative nature of mathematical discovery, where ideas and concepts are gradually linked, refined, and consolidated over time.

Graph theory has applications in social science, data analysis, topological space, video recognition, cluster analysis, algebra, laser scanners, and communication. The majority of classical set for formal modelling and reasoning are straightforward and predictable. Crisp is a yes or no person who cannot accept less or more. A statement in classical dual logic is either true or false, with no in-between. In set theory, an element is either in

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or out of the set. Zadeh [1] introduced fuzzy set (FS) as an effective and acceptable extension of crisp set for dealing with ambiguous and obscure data in unpredictable circumstances. It is defined as a true membership function with ranges in [0,1]. These concepts play a key role in understanding situations involving approximation reasoning as well as illustrating complex occurrences that conventional mathematics finds difficult to completely explain. Traditional FSs are expanded upon or extended by image FSs and bipolar picture FSs. Atanassov [2] established intuitionistic FS involving two degrees.

Fuzzy graphs (FGs) have a wide range of applications due to their ability to deal with uncertainty and imprecision in a structured mathematical framework. They are critical tools for simulating real-world systems in which relationships are not strictly binary but exhibit degrees of ambiguity. FGs can be used to factor in multiple criteria and preferences in fields such as decision-making, allowing for more informed decisions. They play an important role in pattern recognition by accommodating fuzzy patterns that do not neatly fit into predefined categories, thereby improving applications such as image processing and machine learning. Furthermore, in complex systems modelling, they are critical for capturing the intricate, ever-changing relationships within transportation networks, social networks, and biological systems.

Unipolar intelligence is less fundamental than bipolar intelligence based on truth, and calmness is a constraint for truth-based environments. When reality disappears in a black hole owing to Hawking radiation or specified/anti-particular emissions, the most powerful phenomenon that endures is bipolarity. A bipolar fuzzy graph (FG) is a mathematical model that extends the traditional FG concept by introducing bipolar membership values. Unlike standard FGs, which typically represent the strength of connections between nodes using positive values between 0 and 1, bipolar FGs allow for the inclusion of both positive and negative membership values. This additional dimension of information allows for a more nuanced representation of relationships, with positive values indicating positive influence or affinity and negative values indicating repulsion or opposition. Bipolar FGs are used in decision-making, sentiment analysis, and modelling complex systems where the direction and polarity of connections between elements are critical to understanding the network's dynamics. A bipolar picture FS is more fundamental than picture FS.

On the other hand, Rosenfeld [3] presented a new idea of one degree FG. After that, some researchers studied about FG [4–6], bipolar FG [7–9], and fuzzy analysis or fuzzy algebra [10–18]. The concept Bondage number in intuitionistic FG is discussed by Shao et al. [19]. Zuo et al. [20] initiated the concept of picture fuzzy graph (PFG). Coung [21] worked on PFG. Shoaib et al. [22–25] worked on different classes of FGs. Shoaib et al. [26] discussed the concept of upper and lower truncation of PFGs. Khan et al. [27] introduced a BPFG. We present some properties, namely symmetric difference and rejection of bipolar picture fuzzy graph (BPFG).

Research objectives

(1) Develop a generalized model called the bipolar picture fuzzy model, which extends the existing picture fuzzy model.
(2) Define the membership grades within the bipolar picture fuzzy model, which consist of six functions: the membership function, the neutral function, the non-membership function, negative of the membership function, negative of the neutral function, and negative of the non-membership function.
(3) Investigate the operations, namely rejection and symmetric difference, of BPFG with the help of examples and theorems.

Some abbreviations in word form are shown in Table 1. PFG and BPFG are both important extensions of FG that allow for the representation of uncertainty. PFGs represent relationships in three dimensions, whereas

Table 1: Abbreviations

Abbreviation	Word
BPFG	Bipolar picture fuzzy graph
PFG	Picture fuzzy graph
FS	Fuzzy set
FG	Fuzzy graph

BPFGs add another dimension by including positive and negative influences, resulting in a richer model for certain applications.

2 Preliminaries

In this section, we lay the foundation with fundamental definitions, revisiting key concepts to pave the way for a thorough exploration of vertex degree. Our objective is to foster a keen interest in understanding this notion more deeply. We use the symbols ∨ to denote maximum values and ∧ to represent minimum values.

Definition 2.1. [27] The functions $\chi_W, \phi_W, \psi_W, \alpha_W, \beta_W$, and γ_W are defined on the set E, which is a subset of V×V, and they map to particular intervals: [0, 1] for χ_W, ϕ_W, and ψ_W, and [−1, 0] for α_W, β_W, and γ_W. These operations meet the following conditions:

$$\chi_W(pq) \leq \wedge\{\chi_U(p), \chi_U(q)\},$$

$$\phi_W(pq) \leq \wedge\{\phi_U(p), \phi_U(q)\},$$

$$\psi_W(pq) \geq \vee\{\psi_U(p), \psi_U(q)\},$$

$$\alpha_W(pq) \geq \vee\{\alpha_U(p), \alpha_U(q)\},$$

$$\beta_W(pq) \geq \vee\{\beta_U(p), \beta_U(q)\},$$

$$\gamma_W(pq) \leq \wedge\{\gamma_U(p), \gamma_U(q)\}.$$

The sum of $\chi_U(p), \phi_U(p)$, and $\psi_U(p)$ lies between 0 and 1, expressed as $0 \leq \chi_U(p) + \phi_U(p) + \psi_U(p) \leq 1$, for all p in V. The sum of $\alpha_U(p), \beta_U(p)$, and $\gamma_U(p)$ is within the range of −1 to 0, denoted as $-1 \leq \alpha_U(p) + \beta_U(p) + \gamma_U(p) \leq 0$ for all p in V. Similarly, the sum of $\chi_W(pq)$, $\phi_W(pq)$, and $\psi_W(pq)$ is bounded between 0 and 1, expressed as $0 \leq \chi_W(pq) + \phi_W(pq) + \psi_W(pq) \leq 1$ for all pq in E. The sum of $\alpha_W(pq)$, $\beta_W(pq)$, and $\gamma_W(pq)$ falls within the interval [−1,0], represented as $-1 \leq \alpha_W(pq) + \beta_W(pq) + \gamma_W(pq) \leq 0$, for all pq in E.

Example 2.2. Consider a graph comprising three vertices, denoted as p, q, and r, along with three edges: pq, qr, and rp, configured in the following manner (Figure 1):

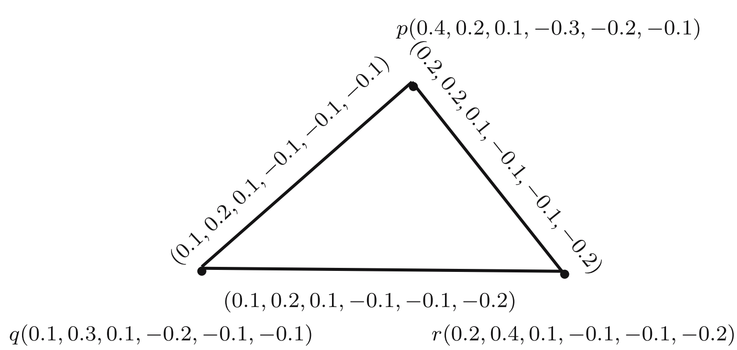

Figure 1: BPFG.

3 BPFG

Definition 3.1. [22] Consider a crisp graph denoted as $G^* = (V, E)$. When we have a non-empty universal set X, we introduce the concept of a picture FS, which can be denoted as $U = <p : \chi_U(p), \phi_U(p), \psi_U(p)>, p \in X$. The constraint applies within this context: $0 \leq \chi_U(p) + \phi_U(p) + \psi_U(p) \leq 1$. Specifically, $\chi_U : V \to [0, 1]$ signifies the degree of the true membership function, $\phi_U : V \to [0,1]$ represents the degree of the neutral membership function, and $\psi_U : V \to [0,1]$ characterizes the degree of the falsity membership function. In addition, we establish the concept of the refusal membership degree as $\pi_U(p) = 1 - \chi_U(p) + \phi_U(p) + \psi_U(p)$.

Definition 3.2. A BPFG $\mathbb{G} = (U, W)$ on a graph $G^* = (V, E)$ is said to be strong if

$$\chi_W(pq) = \wedge\{\chi_U(p), \chi_U(q)\},$$
$$\phi_W(pq) = \wedge\{\phi_U(p), \phi_U(q)\},$$
$$\psi_W(pq) = \vee\{\psi_U(p), \psi_U(q)\},$$
$$\alpha_W(pq) = \vee\{\alpha_U(p), \alpha_U(q)\},$$
$$\beta_W(pq) = \vee\{\beta_U(p), \beta_U(q)\},$$
$$\gamma_W(pq) = \wedge\{\gamma_U(p), \gamma_U(q)\}$$

$\forall pq$ in E.

Definition 3.3. A BPFG $\mathbb{G}$ is said to be complete if

$$\chi_W(pq) = \wedge\{\chi_U(p), \chi_U(q)\},$$
$$\phi_W(pq) = \wedge\{\phi_U(p), \phi_U(q)\},$$
$$\psi_W(pq) = \vee\{\psi_U(p), \psi_U(q)\},$$
$$\alpha_W(pq) = \vee\{\alpha_U(p), \alpha_U(q)\},$$
$$\beta_W(pq) = \vee\{\beta_U(p), \beta_U(q)\},$$
$$\gamma_W(pq) = \wedge\{\gamma_U(p), \gamma_U(q)\}$$

$\forall p, q$ in V.

4 Operations on BPFG

Definition 4.1. The rejection $\mathbb{G}_1|\mathbb{G}_2 = (U_1|U_2, W_1|W_2)$ of two BPFGs $\mathbb{G}_1 = (U_1, W_1)$ and $\mathbb{G}_2 = (U_2, W_2)$ on crisp graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is defined as follows:

(i)

$$(\chi_{U_1}|\chi_{U_2})((k_1, k_2)) = \wedge\{\chi_{U_1}(k_1), \chi_{U_2}(k_2)\}$$
$$(\phi_{U_1}|\phi_{U_2})((k_1, k_2)) = \wedge\{\phi_{U_1}(k_1), \phi_{U_2}(k_2)\}$$
$$(\psi_{U_1}|\psi_{U_2})((k_1, k_2)) = \vee\{\psi_{U_1}(k_1), \psi_{U_2}(k_2)\}$$
$$(\alpha_{U_1}|\alpha_{U_2})((k_1, k_2)) = \vee\{\alpha_{U_1}(k_1), \alpha_{U_2}(k_2)\}$$
$$(\beta_{U_1}|\beta_{U_2})((k_1, k_2)) = \vee\{\beta_{U_1}(k_1), \beta_{U_2}(k_2)\}$$
$$(\gamma_{U_1}|\gamma_{U_2})((k_1, k_2)) = \wedge\{\gamma_{U_1}(k_1), \gamma_{U_2}(k_2)\}$$

$\forall (k_1, k_2) \in (V_1 \times V_2)$,

(ii)

$$(\chi_{W_1}|\chi_{W_2})((m,k_2)(m,n_2)) = \wedge\{\chi_{U_1}(m), \chi_{U_2}(k_2), \chi_{U_2}(n_2)\}$$
$$(\phi_{W_1}|\phi_{W_2})((m,k_2)(m,n_2)) = \wedge\{\phi_{U_1}(m), \phi_{U_2}(k_2), \phi_{U_2}(n_2)\}$$
$$(\psi_{W_1}|\psi_{W_2})((m,k_2)(m,n_2)) = \vee\{\psi_{U_1}(m), \psi_{U_2}(k_2), \psi_{U_2}(n_2)\}$$
$$(\alpha_{W_1}|\alpha_{W_2})((m,k_2)(m,n_2)) = \vee\{\alpha_{U_1}(m), \alpha_{U_2}(k_2), \alpha_{U_2}(n_2)\}$$
$$(\beta_{W_1}|\beta_{W_2})((m,k_2)(m,n_2)) = \vee\{\beta_{U_1}(m), \beta_{U_2}(k_2), \beta_{U_2}(n_2)\}$$
$$(\gamma_{W_1}|\gamma_{W_2})((m,k_2)(m,n_2)) = \wedge\{\gamma_{U_1}(m), \gamma_{U_2}(k_2), \gamma_{U_2}(n_2)\}$$

$\forall m \in V_1$ and $k_2n_2 \notin E_2$,

(iii)

$$(\chi_{W_1}|\chi_{W_2})((k_1,z)(n_1,z)) = \wedge\{\chi_{U_1}(k_1), \chi_{U_1}(n_1), \chi_{U_2}(z)\}$$
$$(\phi_{W_1}|\phi_{W_2})((k_1,z)(n_1,z)) = \wedge\{\phi_{U_1}(k_1), \phi_{U_1}(n_1), \phi_{U_2}(z)\}$$
$$(\psi_{W_1}|\psi_{W_2})((k_1,z)(n_1,z)) = \vee\{\psi_{U_1}(k_1), \psi_{U_1}(n_1), \psi_{U_2}(z)\}$$
$$(\alpha_{W_1}|\alpha_{W_2})((k_1,z)(n_1,z)) = \vee\{\alpha_{U_1}(k_1), \alpha_{U_1}(n_1), \alpha_{U_2}(z)\}$$
$$(\beta_{W_1}|\beta_{W_2})((k_1,z)(n_1,z)) = \vee\{\beta_{U_1}(k_1), \beta_{U_1}(n_1), \beta_{U_2}(z)\}$$
$$(\gamma_{W_1}|\gamma_{W_2})((k_1,z)(n_1,z)) = \wedge\{\gamma_{U_1}(k_1), \gamma_{U_1}(n_1), \gamma_{U_2}(z)\}$$

$\forall z \in V_2$ and $k_1n_1 \notin E_1$,

(iv)

$$(\chi_{W_1}|\chi_{W_2})((k_1,k_2)(n_1,n_2)) = \wedge\{\chi_{U_1}(k_1), \chi_{U_1}(n_1), \chi_{U_2}(k_2), \chi_{U_2}(n_2)\}$$
$$(\phi_{W_1}|\phi_{W_2})((k_1,k_2)(n_1,n_2)) = \wedge\{\phi_{U_1}(k_1), \phi_{U_1}(n_1), \phi_{U_2}(k_2), \phi_{W_2}(n_2)\}$$
$$(\psi_{W_1}|\psi_{W_2})((k_1,k_2)(n_1,n_2)) = \vee\{\psi_{U_1}(k_1), \psi_{U_1}(n_1), \psi_{U_2}(k_2), \psi_{U_2}(n_2)\}$$
$$(\alpha_{W_1}|\alpha_{W_2})((k_1,k_2)(n_1,n_2)) = \vee\{\alpha_{U_1}(k_1), \alpha_{U_1}(n_1), \alpha_{U_2}(k_2), \alpha_{U_2}(n_2)\}$$
$$(\beta_{W_1}|\beta_{W_2})((k_1,k_2)(n_1,n_2)) = \vee\{\beta_{U_1}(k_1), \beta_{U_1}(n_1), \beta_{U_2}(k_2), \beta_{U_2}(n_2)\}$$
$$(\gamma_{W_1}|\gamma_{W_2})((k_1,k_2)(n_1,n_2)) = \wedge\{\gamma_{U_1}(k_1), \gamma_{U_1}(n_1), \gamma_{U_2}(k_2), \gamma_{U_2}(n_2)\}$$

$\forall k_1n_1 \notin E_1$ and $k_2n_2 \notin E_2$.

Example 4.2. Suppose $\mathbb{G}_1$ and $\mathbb{G}_2$ are two BPFGs as in Figures 2 and 3. Figure 4 presents the rejection of two BPFGs $\mathbb{G}_1$ and $\mathbb{G}_2$, that is $\mathbb{G}_1|\mathbb{G}_2$ in Figure 4.

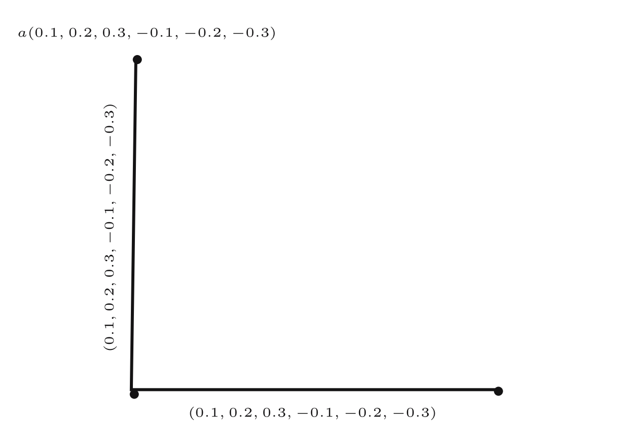

Figure 2: $\mathbb{G}_1$.

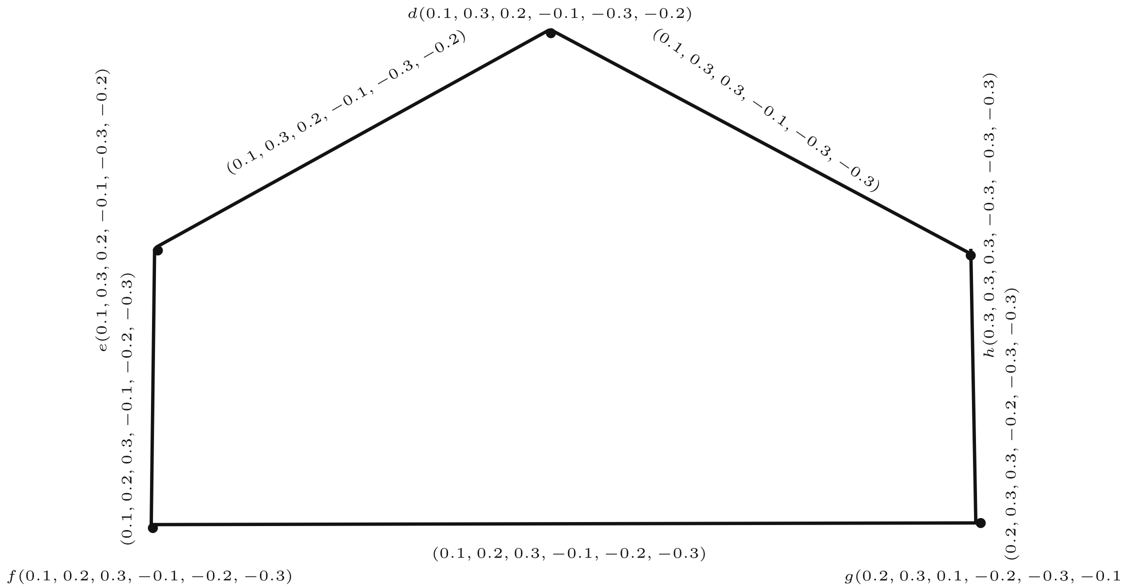

Figure 3: $\mathbb{G}_2$.

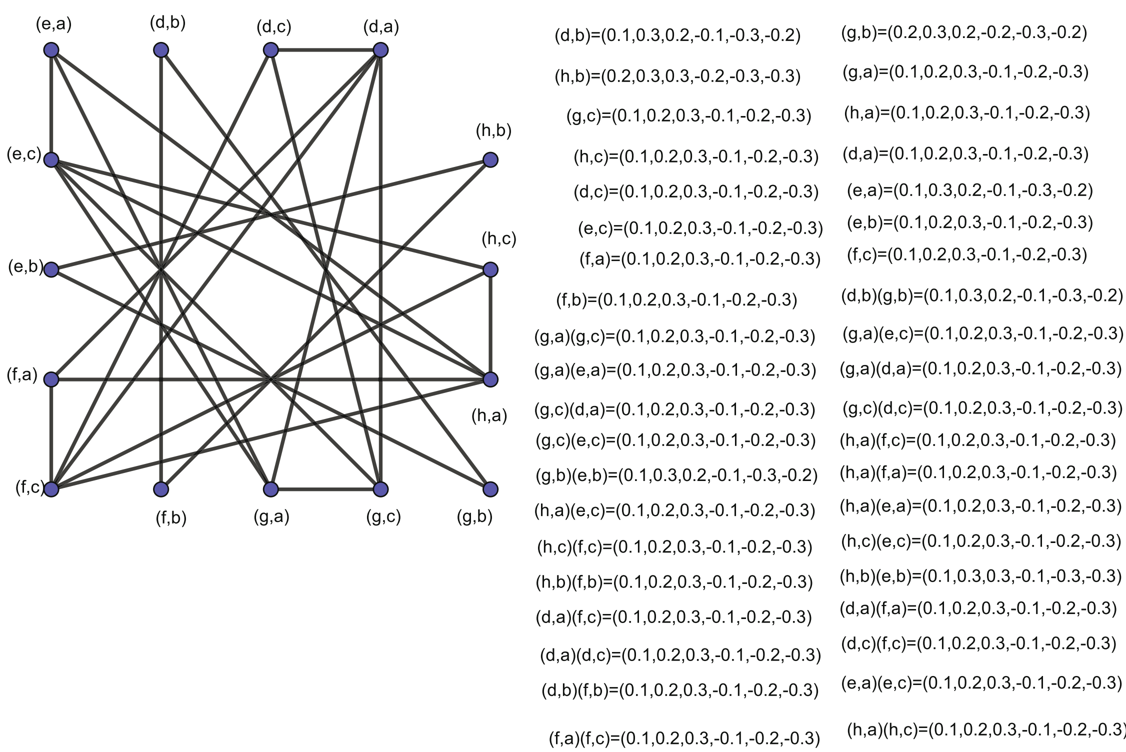

Figure 4: $\mathbb{G}_1|\mathbb{G}_2$.

For node (e, a),

$$\begin{aligned}
(\chi_{U_1}|\chi_{U_2})((a, e)) &= \wedge\{\chi_{U_1}(a), \chi_{U_2}(e)\} = \wedge\{0.1, 0.1\} = 0.1,\\
(\phi_{U_1}|\phi_{U_2})((a, e)) &= \wedge\{\phi_{U_1}(a), \phi_{U_2}(e)\} = \wedge\{0.3, 0.2\} = 0.2,\\
(\psi_{U_1}|\psi_{U_2})((a, e)) &= \vee\{\psi_{U_1}(a), \psi_{U_2}(e)\} = \vee\{0.2, 0.3\} = 0.3,\\
(\alpha_{U_1}|\alpha_{U_2})((a, e)) &= \vee\{\alpha_{U_1}(a), \alpha_{U_2}(e)\} = \vee\{-0.1, -0.1\} = -0.1,\\
(\beta_{U_1}|\beta_{U_2})((a, e)) &= \vee\{\beta_{U_1}(a), \beta_{U_2}(e)\} = \vee\{-0.2, -0.3\} = -0.2,\\
(\gamma_{U_1}|\gamma_{U_2})((a, e)) &= \wedge\{\gamma_{U_1}(a), \gamma_{U_2}(e)\} = \wedge\{-0.3, -0.2\} = -0.3
\end{aligned}$$

for $a \in V_1$ and $e \in V_2$.

Proposition 4.3. The rejection of two BPFGs $\mathbb{G}_1$ and $\mathbb{G}_2$ is a BPFG.

Proof. Suppose $\mathbb{G}_1 = (U_1, W_1)$ and $\mathbb{G}_2 = (U_2, W_2)$ are two BPFGs on crisp graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, and $((k_1, k_2)(n_1, n_2)) \in E_1 \times E_2$.

(i) If $k_1 = n_1$, $k_2n_2 \notin E_2$,

$$\begin{aligned}
(\chi_{W_1}|\chi_{W_2})((k_1, k_2)(n_1, n_2)) &= \wedge\{\chi_{U_1}(k_1), \chi_{U_2}(k_2), \chi_{U_2}(n_2)\}\\
&\leq \wedge\{\wedge\{\chi_{U_1}(k_1), \chi_{U_2}(k_2)\}, \wedge\{\chi_{U_1}(n_1), \chi_{U_2}(n_2)\}\}\\
&= \wedge\{(\chi_{U_1}|\chi_{U_2})(k_1, k_2), (\chi_{U_1}|\chi_{U_2})(n_1, n_2)\},
\end{aligned}$$

$$\begin{aligned}
(\phi_{W_1}|\phi_{W_2})((k_1, k_2)(n_1, n_2)) &= \wedge\{\phi_{U_1}(k_1), \phi_{U_2}(k_2), \phi_{U_2}(n_2)\}\\
&\leq \wedge\{\wedge\{\phi_{U_1}(k_1), \phi_{U_2}(k_2)\}, \wedge\{\phi_{U_1}(n_1), \phi_{U_2}(n_2)\}\}\\
&= \wedge\{(\phi_{U_1}|\phi_{U_2})(k_1, k_2), (\phi_{U_1}|\phi_{U_2})(n_1, n_2)\},
\end{aligned}$$

$$\begin{aligned}
(\psi_{W_1}|\psi_{W_2})((k_1, k_2)(n_1, n_2)) &= \vee\{\psi_{U_1}(k_1), \psi_{U_2}(k_2), \psi_{U_2}(n_2)\}\\
&\geq \vee\{\vee\{\psi_{U_1}(k_1), \psi_{U_2}(k_2)\}, \vee\{\psi_{U_1}(n_1), \psi_{U_2}(n_2)\}\}\\
&= \vee\{(\psi_{U_1}|\psi_{U_2})(k_1, k_2), (\psi_{U_1}|\psi_{U_2})(n_1, n_2)\},
\end{aligned}$$

$$\begin{aligned}
(\alpha_{W_1}|\alpha_{W_2})((k_1, k_2)(n_1, n_2)) &= \vee\{\alpha_{U_1}(k_1), \alpha_{U_2}(k_2), \alpha_{U_2}(n_2)\}\\
&\geq \vee\{\vee\{\alpha_{U_1}(k_1), \alpha_{U_2}(k_2)\}, \vee\{\alpha_{U_1}(n_1), \alpha_{U_2}(n_2)\}\}\\
&= \vee\{(\alpha_{U_1}|\alpha_{U_2})(k_1, k_2), (\alpha_{U_1}|\alpha_{U_2})(n_1, n_2)\},
\end{aligned}$$

$$\begin{aligned}
(\beta_{W_1}|\beta_{W_2})((k_1, k_2)(n_1, n_2)) &= \vee\{\beta_{U_1}(k_1), \beta_{U_2}(k_2), \beta_{U_2}(n_2)\}\\
&\geq \vee\{\vee\{\beta_{U_1}(k_1), \beta_{U_2}(k_2)\}, \vee\{\beta_{U_1}(n_1), \beta_{U_2}(n_2)\}\}\\
&= \vee\{(\beta_{U_1}|\beta_{U_2})(k_1, k_2), (\beta_{U_1}|\beta_{U_2})(n_1, n_2)\},
\end{aligned}$$

$$\begin{aligned}
(\gamma_{W_1}|\gamma_{W_2})((k_1, k_2)(n_1, n_2)) &= \wedge\{\gamma_{U_1}(k_1), \gamma_{U_2}(k_2), \gamma_{U_2}(n_2)\}\\
&\leq \wedge\{\wedge\{\gamma_{U_1}(k_1), \gamma_{U_2}(k_2)\}, \wedge\{\gamma_{U_1}(n_1), \gamma_{U_2}(n_2)\}\}\\
&= \wedge\{(\gamma_{U_1}|\gamma_{U_2})(k_1, k_2), (\gamma_{U_1}|\gamma_{U_2})(n_1, n_2)\}.
\end{aligned}$$

(ii) If $k_2 = n_2$, $k_1n_1 \notin E_1$,

$$\begin{aligned}
(\chi_{W_1}|\chi_{W_2})((k_1, k_2)(n_1, n_2)) &= \wedge\{\chi_{U_1}(k_1), \chi_{U_1}(n_1), \chi_{U_2}(k_2)\}\\
&\leq \wedge\{\wedge\{\chi_{U_1}(k_1), \chi_{U_2}(k_2)\}, \wedge\{\chi_{U_1}(n_1), \chi_{U_2}(n_2)\}\}\\
&= \wedge\{(\chi_{U_1}|\chi_{U_2})(k_1, k_2), (\chi_{U_1}|\chi_{U_2})(n_1, n_2)\},
\end{aligned}$$

$$\begin{aligned}
(\phi_{W_1}|\phi_{W_2})((k_1, k_2)(n_1, n_2)) &= \wedge\{\phi_{U_1}(k_1), \phi_{U_1}(n_1), \phi_{U_2}(k_2)\}\\
&\leq \wedge\{\wedge\{\phi_{U_1}(k_1), \phi_{U_2}(k_2)\}, \wedge\{\phi_{U_1}(n_1), \phi_{U_2}(n_2)\}\}\\
&= \wedge\{(\phi_{U_1}|\phi_{U_2})(k_1, k_2), (\phi_{U_1}|\phi_{U_2})(n_1, n_2)\},
\end{aligned}$$

$$\begin{aligned}(\psi_{W_1}|\psi_{W_2})((k_1,k_2)(n_1,n_2)) &= \vee\{\psi_{U_1}(k_1),\psi_{U_1}(n_1),\psi_{U_2}(k_2)\}\\ &\geq \vee\{\vee\{\psi_{U_1}(k_1),\psi_{U_2}(k_2)\},\vee\{\psi_{U_1}(n_1),\psi_{U_2}(n_2)\}\}\\ &= \vee\{(\psi_{U_1}|\psi_{U_2})(k_1,k_2),(\psi_{U_1}|\psi_{U_2})(n_1,n_2)\},\end{aligned}$$

$$\begin{aligned}(\alpha_{W_1}|\alpha_{W_2})((k_1,k_2)(n_1,n_2)) &= \vee\{\alpha_{U_1}(k_1),\alpha_{U_1}(n_1),\alpha_{U_2}(k_2)\}\\ &\geq \vee\{\vee\{\alpha_{U_1}(k_1),\alpha_{U_2}(k_2)\},\vee\{\alpha_{U_1}(n_1),\alpha_{U_2}(n_2)\}\}\\ &= \vee\{(\alpha_{U_1}|\alpha_{U_2})(k_1,k_2),(\alpha_{U_1}|\alpha_{U_2})(n_1,n_2)\},\end{aligned}$$

$$\begin{aligned}(\beta_{W_1}|\beta_{W_2})((k_1,k_2)(n_1,n_2)) &= \vee\{\beta_{U_1}(k_1),\beta_{U_1}(n_1),\beta_{U_2}(k_2)\}\\ &\geq \vee\{\vee\{\beta_{U_1}(k_1),\beta_{U_2}(k_2)\},\vee\{\beta_{U_1}(n_1),\beta_{U_2}(n_2)\}\}\\ &= \vee\{(\beta_{U_1}|\beta_{U_2})(k_1,k_2),(\beta_{U_1}|\beta_{U_2})(n_1,n_2)\},\end{aligned}$$

$$\begin{aligned}(\gamma_{W_1}|\gamma_{W_2})((k_1,k_2)(n_1,n_2)) &= \wedge\{\gamma_{U_1}(k_1),\gamma_{U_1}(n_1),\gamma_{U_2}(k_2)\}\\ &\leq \wedge\{\wedge\{\gamma_{U_1}(k_1),\gamma_{U_2}(k_2)\},\wedge\{\gamma_{U_1}(n_1),\gamma_{U_2}(n_2)\}\}\\ &= \wedge\{(\gamma_{U_1}|\gamma_{U_2})(k_1,k_2),(\gamma_{U_1}|\gamma_{U_2})(n_1,n_2)\}.\end{aligned}$$

(iii) If $k_1n_1 \notin E_1$ and $k_2n_2 \notin E_2$,

$$\begin{aligned}(\chi_{W_1}|\chi_{W_2})((k_1,k_2)(n_1,n_2)) &= \wedge\{\chi_{U_1}(k_1),\chi_{U_1}(n_1),\chi_{U_2}(k_2),\chi_{U_2}(n_2)\}\\ &\leq \wedge\{\wedge\{\chi_{U_1}(k_1),\chi_{U_2}(k_2)\},\wedge\{\chi_{U_1}(n_1),\chi_{U_2}(n_2)\}\}\\ &= \wedge\{(\chi_{U_1}|\chi_{U_2})(k_1,k_2),(\chi_{U_1}|\chi_{U_2})(n_1,n_2)\},\end{aligned}$$

$$\begin{aligned}(\phi_{W_1}|\phi_{W_2})((k_1,k_2)(n_1,n_2)) &= \wedge\{\phi_{U_1}(k_1),\phi_{U_1}(n_1),\phi_{U_2}(k_2),\chi_{U_2}(n_2)\}\\ &\leq \wedge\{\wedge\{\phi_{U_1}(k_1),\phi_{U_2}(k_2)\},\wedge\{\phi_{U_1}(n_1),\phi_{U_2}(n_2)\}\}\\ &= \wedge\{(\phi_{U_1}|\phi_{U_2})(k_1,k_2),(\phi_{U_1}|\phi_{U_2})(n_1,n_2)\},\end{aligned}$$

$$\begin{aligned}(\psi_{W_1}|\psi_{W_2})((k_1,k_2)(n_1,n_2)) &= \vee\{\psi_{U_1}(k_1),\psi_{U_1}(n_1),\psi_{U_2}(k_2),\psi_{U_2}(n_2)\}\\ &\geq \vee\{\vee\{\psi_{U_1}(k_1),\psi_{U_2}(k_2)\},\vee\{\psi_{U_1}(n_1),\psi_{U_2}(n_2)\}\}\\ &= \vee\{(\psi_{U_1}|\psi_{U_2})(k_1,k_2),(\psi_{U_1}|\psi_{U_2})(n_1,n_2)\},\end{aligned}$$

$$\begin{aligned}(\alpha_{W_1}|\alpha_{W_2})((k_1,k_2)(n_1,n_2)) &= \vee\{\alpha_{U_1}(k_1),\alpha_{U_1}(n_1),\alpha_{U_2}(k_2),\alpha_{U_2}(n_2)\}\\ &\geq \vee\{\vee\{\alpha_{U_1}(k_1),\alpha_{U_2}(k_2)\},\vee\{\alpha_{U_1}(n_1),\alpha_{U_2}(n_2)\}\}\\ &= \vee\{(\alpha_{U_1}|\alpha_{U_2})(k_1,k_2),(\alpha_{U_1}|\alpha_{U_2})(n_1,n_2)\},\end{aligned}$$

$$\begin{aligned}(\beta_{W_1}|\beta_{W_2})((k_1,k_2)(n_1,n_2)) &= \vee\{\beta_{U_1}(k_1),\beta_{U_1}(n_1),\beta_{U_2}(k_2),\beta_{U_2}(n_2)\}\\ &\geq \vee\{\vee\{\beta_{U_1}(k_1),\beta_{U_2}(k_2)\},\vee\{\beta_{U_1}(n_1),\beta_{U_2}(n_2)\}\}\\ &= \vee\{(\beta_{U_1}|\beta_{U_2})(k_1,k_2),(\beta_{U_1}|\beta_{U_2})(n_1,n_2)\},\end{aligned}$$

$$\begin{aligned}(\gamma_{W_1}|\gamma_{W_2})((k_1,k_2)(n_1,n_2)) &= \wedge\{\gamma_{U_1}(k_1),\gamma_{U_1}(n_1),\gamma_{U_2}(k_2),\gamma_{U_2}(n_2)\}\\ &\leq \wedge\{\wedge\{\gamma_{U_1}(k_1),\gamma_{U_2}(k_2)\},\wedge\{\gamma_{U_1}(n_1),\gamma_{U_2}(n_2)\}\}\\ &= \wedge\{(\gamma_{U_1}|\gamma_{U_2})(k_1,k_2),(\gamma_{U_1}|\gamma_{U_2})(n_1,n_2)\}.\end{aligned}$$

Therefore, $\mathbf{G}_1|\mathbf{G}_2 = (U_1|U_2, W_1|W_2)$ is a BPFG. □

Remark 4.4. The rejection of two complete BPFGs $\mathbf{G}_1 = (U_1, W_1)$ and $\mathbf{G}_2 = (U_2, W_2)$ is a complete BPFG.

Definition 4.5. Suppose $\mathbf{G}_1 = (U_1, W_1)$ and $\mathbf{G}_2 = (U_2, W_2)$ are two BPFGs on crisp graphs. For any node $(k_1, k_2) \in V_1 \times V_2$, we have

$$\begin{aligned}(d_\chi)_{\mathbb{G}_1|\mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\chi_{W_1}|\chi_{W_2})((k_1, k_2)(n_1, n_2))\\ &= \sum_{k_1=n_1, k_2n_2\notin E_2} \wedge \{\chi_{U_1}(k_1), \chi_{U_2}(k_2), \chi_{U_2}(n_2)\}\\ &+ \sum_{k_2=n_2, k_1n_1\notin E_1} \wedge \{\chi_{U_1}(k_1), \chi_{U_1}(n_1), \chi_{U_2}(k_2)\}\\ &+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\notin E_2} \wedge \{\chi_{U_1}(k_1), \chi_{U_1}(n_1), \chi_{U_2}(k_2), \chi_{U_2}(n_2)\},\end{aligned}$$

$$\begin{aligned}(d_\phi)_{\mathbb{G}_1|\mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\phi_{W_1}|\phi_{W_2})((k_1, k_2)(n_1, n_2))\\ &= \sum_{k_1=n_1, k_2n_2\notin E_2} \wedge \{\phi_{U_1}(k_1), \phi_{U_2}(k_2), \phi_{U_2}(n_2)\}\\ &+ \sum_{k_2=n_2, k_1n_1\notin E_1} \wedge \{\phi_{U_1}(k_1), \phi_{U_1}(n_1), \phi_{U_2}(k_2)\}\\ &+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\notin E_2} \wedge \{\phi_{U_1}(k_1), \phi_{U_1}(n_1), \phi_{U_2}(k_2), \phi_{U_2}(n_2)\},\end{aligned}$$

$$\begin{aligned}(d_\psi)_{\mathbb{G}_1|\mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\psi_{W_1}|\psi_{W_2})((k_1, k_2)(n_1, n_2))\\ &= \sum_{k_1=n_1, k_2n_2\notin E_2} \vee \{\psi_{U_1}(k_1), \psi_{U_2}(k_2), \psi_{U_2}(n_2)\}\\ &+ \sum_{k_2=n_2, k_1n_1\notin E_1} \vee \{\psi_{U_1}(k_1), \psi_{U_1}(n_1), \psi_{U_2}(k_2)\}\\ &+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\notin E_2} \vee \{\psi_{U_1}(k_1), \psi_{U_1}(n_1), \psi_{U_2}(k_2), \psi_{U_2}(n_2)\},\end{aligned}$$

$$\begin{aligned}(d_\alpha)_{\mathbb{G}_1|\mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\alpha_{W_1}|\alpha_{W_2})((k_1, k_2)(n_1, n_2))\\ &= \sum_{k_1=n_1, k_2n_2\notin E_2} \vee \{\alpha_{U_1}(k_1), \alpha_{U_2}(k_2), \alpha_{U_2}(n_2)\}\\ &+ \sum_{k_2=n_2, k_1n_1\notin E_1} \vee \{\alpha_{U_1}(k_1), \alpha_{U_1}(n_1), \alpha_{U_2}(k_2)\}\\ &+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\notin E_2} \vee \{\alpha_{U_1}(k_1), \alpha_{U_1}(n_1), \alpha_{U_2}(k_2), \alpha_{U_2}(n_2)\},\end{aligned}$$

$$\begin{aligned}(d_\beta)_{\mathbb{G}_1|\mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\beta_{W_1}|\beta_{W_2})((k_1, k_2)(n_1, n_2))\\ &= \sum_{k_1=n_1, k_2n_2\notin E_2} \vee \{\beta_{U_1}(k_1), \beta_{U_2}(k_2), \beta_{U_2}(n_2)\}\\ &+ \sum_{k_2=n_2, k_1n_1\notin E_1} \vee \{\beta_{U_1}(k_1), \beta_{U_1}(n_1), \beta_{U_2}(k_2)\}\\ &+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\notin E_2} \vee \{\beta_{U_1}(k_1), \beta_{U_1}(n_1), \beta_{U_2}(k_2), \beta_{U_2}(n_2)\},\end{aligned}$$

$$\begin{aligned}(d_\gamma)_{\mathbb{G}_1|\mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\gamma_{W_1}|\gamma_{W_2})((k_1, k_2)(n_1, n_2))\\ &= \sum_{k_1=n_1, k_2n_2\notin E_2} \wedge \{\gamma_{U_1}(k_1), \gamma_{U_2}(k_2), \gamma_{U_2}(n_2)\}\\ &+ \sum_{k_2=n_2, k_1n_1\notin E_1} \wedge \{\gamma_{U_1}(k_1), \gamma_{U_1}(n_1), \gamma_{U_2}(k_2)\}\\ &+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\notin E_2} \wedge \{\gamma_{U_1}(k_1), \gamma_{U_1}(n_1), \gamma_{U_2}(k_2), \gamma_{U_2}(n_2)\}.\end{aligned}$$

Definition 4.6. Suppose $\mathbb{G}_1 = (U_1, W_1)$ and $\mathbb{G}_2 = (U_2, W_2)$ are two BPFGs on crisp graphs. Then $\forall (k_1, k_2) \in V_1 \times V_2$.

$$
\begin{aligned}
(td_\chi)_{\mathbb{G}_1|\mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\chi_{W_1}|\chi_{W_2})((k_1, k_2)(n_1, n_2)) + (\chi_{U_1}|\chi_{U_2})(k_1, k_2)\\
&= \sum_{k_1=n_1, k_2n_2\notin E_2} \wedge \{\chi_{U_1}(k_1), \chi_{U_2}(k_2), \chi_{U_2}(n_2)\}\\
&+ \sum_{k_2=n_2, k_1n_1\notin E_1} \wedge \{\chi_{U_1}(k_1), \chi_{U_1}(n_1), \chi_{U_2}(k_2)\}\\
&+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\notin E_2} \wedge \{\chi_{U_1}(k_1), \chi_{U_1}(n_1), \chi_{U_2}(k_2), \chi_{U_2}(n_2)\}\\
&+ \wedge \{\chi_{U1}(k_1), \chi_{U2}(k_2)\},
\end{aligned}
$$

$$
\begin{aligned}
(td_\phi)_{\mathbb{G}_1|\mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\phi_{W_1}|\phi_{W_2})((k_1, k_2)(n_1, n_2)) + (\phi_{U_1}|\phi_{U_2})(k_1, k_2)\\
&= \sum_{k_1=n_1, k_2n_2\notin E_2} \wedge \{\phi_{U_1}(k_1), \phi_{U_2}(k_2), \phi_{U_2}(n_2)\}\\
&+ \sum_{k_2=n_2, k_1n_1\notin E_1} \wedge \{\phi_{U_1}(k_1), \phi_{U_1}(n_1), \phi_{U_2}(k_2)\}\\
&+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\notin E_2} \wedge \{\phi_{U_1}(k_1), \phi_{U_1}(n_1), \phi_{U_2}(k_2), \phi_{U_2}(n_2)\}\\
&+ \wedge \{\phi_{U1}(k_1), \phi_{U2}(k_2)\},
\end{aligned}
$$

$$
\begin{aligned}
(td_\psi)_{\mathbb{G}_1|\mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\psi_{W_1}|\psi_{W_2})((k_1, k_2)(n_1, n_2)) + (\psi_{U_1}|\psi_{U_2})(k_1, k_2)\\
&= \sum_{k_1=n_1, k_2n_2\notin E_2} \vee \{\psi_{U_1}(k_1), \psi_{U_2}(k_2), \psi_{U_2}(n_2)\}\\
&+ \sum_{k_2=n_2, k_1n_1\notin E_1} \vee \{\psi_{U_1}(k_1), \psi_{U_1}(n_1), \psi_{U_2}(k_2)\}\\
&+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\notin E_2} \vee \{\psi_{U_1}(k_1), \psi_{U_1}(n_1), \psi_{U_2}(k_2), \psi_{U_2}(n_2)\}\\
&+ \vee \{\psi_{U1}(k_1), \psi_{U2}(k_2)\},
\end{aligned}
$$

$$
\begin{aligned}
(td_\alpha)_{\mathbb{G}_1|\mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\alpha_{W_1}|\alpha_{W_2})((k_1, k_2)(n_1, n_2)) + (\alpha_{U_1}|\alpha_{U_2})(k_1, k_2)\\
&= \sum_{k_1=n_1, k_2n_2\notin E_2} \vee \{\alpha_{U_1}(k_1), \alpha_{U_2}(k_2), \alpha_{U_2}(n_2)\}\\
&+ \sum_{k_2=n_2, k_1n_1\notin E_1} \vee \{\alpha_{U_1}(k_1), \alpha_{U_1}(n_1), \alpha_{U_2}(k_2)\}\\
&+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\notin E_2} \vee \{\alpha_{U_1}(k_1), \alpha_{U_1}(n_1), \alpha_{U_2}(k_2), \alpha_{U_2}(n_2)\}\\
&+ \vee \{\alpha_{U1}(k_1), \alpha_{U2}(k_2)\},
\end{aligned}
$$

$$
\begin{aligned}
(td_\beta)_{\mathbb{G}_1|\mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\beta_{W_1}|\beta_{W_2})((k_1, k_2)(n_1, n_2)) + (\beta_{U_1}|\beta_{U_2})(k_1, k_2)\\
&= \sum_{k_1=n_1, k_2n_2\notin E_2} \vee \{\beta_{U_1}(k_1), \beta_{U_2}(k_2), \beta_{U_2}(n_2)\}\\
&+ \sum_{k_2=n_2, k_1n_1\notin E_1} \vee \{\beta_{U_1}(k_1), \beta_{U_1}(n_1), \beta_{U_2}(k_2)\}\\
&+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\notin E_2} \vee \{\beta_{U_1}(k_1), \beta_{U_1}(n_1), \beta_{U_2}(k_2), \beta_{U_2}(n_2)\}\\
&+ \vee \{\beta_{U1}(k_1), \beta_{U2}(k_2)\},
\end{aligned}
$$

$$
\begin{aligned}
(td_{\gamma})_{\mathbb{G}_1|\mathbb{G}_2}(k_1,k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\gamma_{W_1}|\gamma_{W_2})((k_1,k_2)(n_1,n_2)) + (\gamma_{U_1}|\gamma_{U_2})(k_1,k_2)\\
&= \sum_{k_1=n_1,k_2n_2\notin E_2} \wedge\{\gamma_{U_1}(k_1),\gamma_{U_2}(k_2),\gamma_{U_2}(n_2)\} + \sum_{k_2=n_2,k_1n_1\notin E_1} \wedge\{\gamma_{U_1}(k_1),\gamma_{U_1}(n_1),\gamma_{U_2}(k_2)\}\\
&\quad + \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\notin E_2} \wedge\{\gamma_{U_1}(k_1),\gamma_{U_1}(n_1),\gamma_{U_2}(k_2),\gamma_{U_2}(n_2)\} + \wedge\{\gamma_{U1}(k_1),\gamma_{U2}(k_2)\}.
\end{aligned}
$$

Example 4.7. We calculate the degree and the total degree of node (a, d), for two functions.

$$
\begin{aligned}
(d_{\chi})_{\mathbb{G}_1|\mathbb{G}_2}(a,d) &= \wedge\{\chi_{U_2}(d),\chi_{U_1}(a),\chi_{U_1}(c)\} + \wedge\{\chi_{U_2}(d),\chi_{U_1}(a),\chi_{U_2}(f),\chi_{U_1}(c)\}\\
&\quad + \wedge\{\chi_{U_2}(d),\chi_{U_1}(a),\chi_{U_2}(g),\chi_{U_1}(c)\}\\
&= \wedge\{0.1,0.1,0.1\} + \wedge\{0.1,0.1,0.2,0.1\} + \wedge\{0.1,0.1,0.1,0.1\}\\
&= 0.1 + 0.1 + 0.1\\
&= 0.3,
\end{aligned}
$$

$$
\begin{aligned}
(d_{\alpha})_{\mathbb{G}_1|\mathbb{G}_2}(a,d) &= \wedge\{\alpha_{U_2}(d),\alpha_{U_1}(a),\alpha_{U_1}(c)\} + \wedge\{\alpha_{U_2}(d),\alpha_{U_1}(a),\alpha_{U_2}(f),\alpha_{U_1}(c)\}\\
&\quad + \wedge\{\alpha_{U_2}(d),\alpha_{U_1}(a),\alpha_{U_2}(g),\alpha_{U_1}(c)\}\\
&= \wedge\{-0.1,-0.1,-0.1\} + \wedge\{-0.1,-0.1,-0.1,-0.1\} + \wedge\{-0.1,-0.1,-0.2,-0.1\}\\
&= -0.1 - 0.1 - 0.1\\
&= -0.3.
\end{aligned}
$$

For total vertex degree,

$$
\begin{aligned}
(td_{\chi})_{\mathbb{G}_1|\mathbb{G}_2}(a,d) &= \wedge\{\chi_{U_2}(d),\chi_{U_1}(a),\chi_{U_1}(c)\} + \wedge\{\chi_{U_2}(d),\chi_{U_1}(a),\chi_{U_2}(f),\chi_{U_1}(c)\}\\
&\quad + \wedge\{\chi_{U_2}(d),\chi_{U_1}(a),\chi_{U_2}(g),\chi_{U_1}(c)\} + \wedge\{\chi_{U_2}(d),\chi_{U_1}(a)\}\\
&= \wedge\{0.1,0.1,0.1\} + \wedge\{0.1,0.1,0.2,0.1\} + \wedge\{0.1,0.1,0.1,0.1\} + \wedge\{0.1,0.1\}\\
&= 0.1 + 0.1 + 0.1 + 0.1\\
&= 0.4,
\end{aligned}
$$

$$
\begin{aligned}
(td_{\alpha})_{\mathbb{G}_1|\mathbb{G}_2}(a,d) &= \wedge\{\alpha_{U_2}(d),\alpha_{U_1}(a),\alpha_{U_1}(c)\} + \wedge\{\gamma_{U_2}(d),\gamma_{U_1}(a),\alpha_{U_2}(f),\alpha_{U_1}(c)\}\\
&\quad + \wedge\{\alpha_{U_2}(d),\alpha_{U_1}(a),\alpha_{U_2}(g),\alpha_{U_1}(c)\} + \wedge\{\alpha_{U_2}(d),\alpha_{U_1}(a)\}\\
&= \wedge\{-0.1,-0.1,-0.1\} + \wedge\{-0.1,-0.1,-0.2,-0.1\} + \wedge\{-0.1,-0.1,-0.1,-0.1\}\\
&\quad + \wedge\{-0.1,-0.1\}\\
&= -0.1 - 0.2 - 0.1 - 0.1\\
&= -0.5.
\end{aligned}
$$

Definition 4.8. The symmetric difference $\mathbb{G}_1 \oplus \mathbb{G}_2 = (U_1 \oplus U_2, W_1 \oplus W_2)$ of two BPFGs $\mathbb{G}_1 = (U_1, W_1)$ and $\mathbb{G}_2 = (U_2, W_2)$ on crisp graphs is defined as follows:

(i)

$$
\begin{aligned}
(\chi_{U_1} \oplus \chi_{U_2})((k_1,k_2)) &= \wedge\{\chi_{U_1}(k_1),\chi_{U_2}(k_2)\}\\
(\phi_{U_1} \oplus \phi_{U_2})((k_1,k_2)) &= \wedge\{\phi_{U_1}(k_1),\phi_{U_2}(k_2)\}\\
(\psi_{U_1} \oplus \psi_{U_2})((k_1,k_2)) &= \vee\{\psi_{U_1}(k_1),\psi_{U_2}(k_2)\}\\
(\alpha_{U_1} \oplus \alpha_{U_2})((k_1,k_2)) &= \vee\{\alpha_{U_1}(k_1),\alpha_{U_2}(k_2)\}\\
(\beta_{U_1} \oplus \beta_{U_2})((k_1,k_2)) &= \vee\{\beta_{U_1}(k_1),\beta_{U_2}(k_2)\}\\
(\gamma_{U_1} \oplus \gamma_{U_2})((k_1,k_2)) &= \wedge\{\gamma_{U_1}(k_1),\gamma_{U_2}(k_2)\}
\end{aligned}
$$

$\forall (k_1,k_2) \in (V_1 \times V_2)$.

(ii)

$$(\chi_{W_1} \oplus \chi_{W_2})((m, k_2)(m, n_2)) = \wedge\{\chi_{U_1}(m), \chi_{W_2}(k_2n_2)\}$$
$$(\phi_{W_1} \oplus \phi_{W_2})((m, k_2)(m, n_2)) = \wedge\{\phi_{U_1}(m), \phi_{W_2}(k_2n_2)\}$$
$$(\psi_{W_1} \oplus \psi_{W_2})((m, k_2)(m, n_2)) = \vee\{\psi_{U_1}(m), \psi_{W_2}(k_2n_2)\}$$
$$(\alpha_{W_1} \oplus \alpha_{W_2})((m, k_2)(m, n_2)) = \vee\{\alpha_{U_1}(m), \alpha_{W_2}(k_2n_2)\}$$
$$(\beta_{W_1} \oplus \beta_{W_2})((m, k_2)(m, n_2)) = \vee\{\beta_{U_1}(m), \beta_{W_2}(k_2n_2)\}$$
$$(\gamma_{W_1} \oplus \gamma_{W_2})((m, k_2)(m, n_2)) = \wedge\{\gamma_{U_1}(m), \gamma_{W_2}(k_2n_2)\}$$

$\forall m \in V_1$ and $k_2n_2 \in E_2$.

(iii)

$$(\chi_{W_1} \oplus \chi_{W_2})((k_1, z)(n_1, z)) = \wedge\{\chi_{W_1}(k_1n_1), \chi_{U_2}(z)\}$$
$$(\phi_{W_1} \oplus \phi_{W_2})((k_1, z)(n_1, z)) = \wedge\{\phi_{W_1}(k_1n_1), \phi_{U_2}(z)\}$$
$$(\psi_{W_1} \oplus \psi_{W_2})((k_1, z)(n_1, z)) = \vee\{\psi_{W_1}(k_1n_1), \psi_{U_2}(z)\}$$
$$(\alpha_{W_1} \oplus \alpha_{W_2})((k_1, z)(n_1, z)) = \vee\{\alpha_{W_1}(k_1n_1), \alpha_{U_2}(z)\}$$
$$(\beta_{W_1} \oplus \beta_{W_2})((k_1, z)(n_1, z)) = \vee\{\beta_{W_1}(k_1n_1), \beta_{U_2}(z)\}$$
$$(\gamma_{W_1} \oplus \gamma_{W_2})((k_1, z)(n_1, z)) = \wedge\{\gamma_{W_1}(k_1n_1), \gamma_{U_2}(z)\}$$

$\forall z \in V_2$ and $k_1n_1 \in E_1$.

(iv)

$$\begin{aligned}
(\chi_{W_1} \oplus \chi_{W_2})((k_1, k_2)(n_1, n_2)) &= \wedge\{\chi_{U_1}(k_1), \chi_{U_1}(n_1), \chi_{W_2}(k_2n_2)\} \quad \text{for all } k_1n_1 \notin E_1 \quad \text{and} \quad k_2n_2 \in E_2\\
&\text{or}\\
&= \wedge\{\chi_{U_2}(k_2), \chi_{U_2}(n_2), \chi_{W_1}(k_1n_1)\} \quad \text{for all } k_1n_1 \in E_1 \quad \text{and} \quad k_2n_2 \notin E_2\\
(\phi_{W_1} \oplus \phi_{W_2})((k_1, k_2)(n_1, n_2)) &= \wedge\{\phi_{U_1}(k_1), \phi_{U_1}(n_1), \phi_{W_2}(k_2n_2)\} \quad \text{for all } k_1n_1 \notin E_1 \quad \text{and} \quad k_2n_2 \in E_2\\
&\text{or}\\
&= \wedge\{\phi_{U_2}(k_2), \phi_{U_2}(n_2), \phi_{W_1}(k_1n_1)\} \quad \text{for all } k_1n_1 \in E_1 \quad \text{and} \quad k_2n_2 \notin E_2\\
(\psi_{W_1} \oplus \psi_{W_2})((k_1, k_2)(n_1, n_2)) &= \vee\{\psi_{U_1}(k_1), \psi_{U_1}(n_1), \psi_{W_2}(k_2n_2)\} \quad \text{for all } k_1n_1 \notin E_1 \quad \text{and} \quad k_2n_2 \in E_2\\
&\text{or}\\
&= \vee\{\psi_{U_2}(k_2), \psi_{U_2}(n_2), \psi_{W_2}(k_1n_1)\} \quad \text{for all } k_1n_1 \in E_1 \quad \text{and} \quad k_2n_2 \notin E_2.\\
(\alpha_{W_1} \oplus \alpha_{W_2})((k_1, k_2)(n_1, n_2)) &= \vee\{\alpha_{U_1}(k_1), \alpha_{U_1}(n_1), \alpha_{W_2}(k_2n_2)\} \quad \text{for all } k_1n_1 \notin E_1 \quad \text{and} \quad k_2n_2 \in E_2\\
&\text{or}\\
&= \vee\{\alpha_{U_2}(k_2), \alpha_{U_2}(n_2), \alpha_{W_2}(k_1n_1)\} \quad \text{for all } k_1n_1 \in E_1 \quad \text{and} \quad k_2n_2 \notin E_2.\\
(\beta_{W_1} \oplus \beta_{W_2})((k_1, k_2)(n_1, n_2)) &= \vee\{\beta_{U_1}(k_1), \beta_{U_1}(n_1), \beta_{W_2}(k_2n_2)\} \quad \text{for all } k_1n_1 \notin E_1 \quad \text{and} \quad k_2n_2 \in E_2\\
&\text{or}\\
&= \vee\{\beta_{U_2}(k_2), \beta_{U_2}(n_2), \beta_{W_2}(k_1n_1)\} \quad \text{for all } k_1n_1 \in E_1 \quad \text{and} \quad k_2n_2 \notin E_2.\\
(\gamma_{W_1} \oplus \gamma_{W_2})((k_1, k_2)(n_1, n_2)) &= \wedge\{\gamma_{U_1}(k_1), \gamma_{U_1}(n_1), \gamma_{W_2}(k_2n_2)\} \quad \text{for all } k_1n_1 \notin E_1 \quad \text{and} \quad k_2n_2 \in E_2\\
&\text{or}\\
&= \wedge\{\gamma_{U_2}(k_2), \gamma_{U_2}(n_2), \gamma_{W_1}(k_1n_1)\} \quad \text{for all } k_1n_1 \in E_1 \quad \text{and} \quad k_2n_2 \notin E_2.
\end{aligned}$$

Example 4.9. Suppose $\mathbb{G}_1$ and $\mathbb{G}_2$ are two BPFGs as in Figures 5 and 6. Figure 7 shows the symmetric difference of two BPFGs $\mathbb{G}_1$ and $\mathbb{G}_2$, that is $\mathbb{G}_1 \oplus \mathbb{G}_2$ in Figure 7.

Proposition 4.10. *The symmetric difference of two BPFGs $\mathbb{G}_1$ and $\mathbb{G}_2$, is a BPFG.*

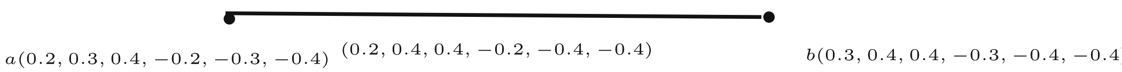

Figure 5: $\mathbb{G}_1$.

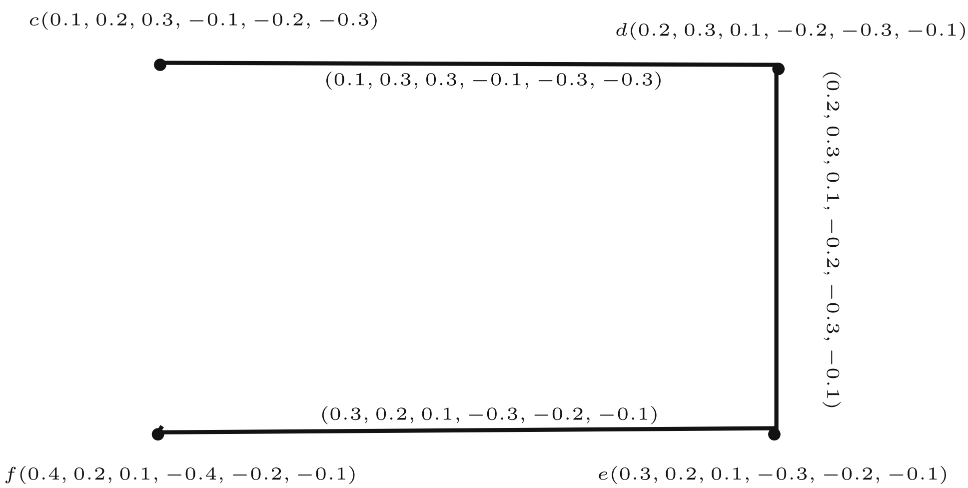

Figure 6: $\mathbb{G}_2$.

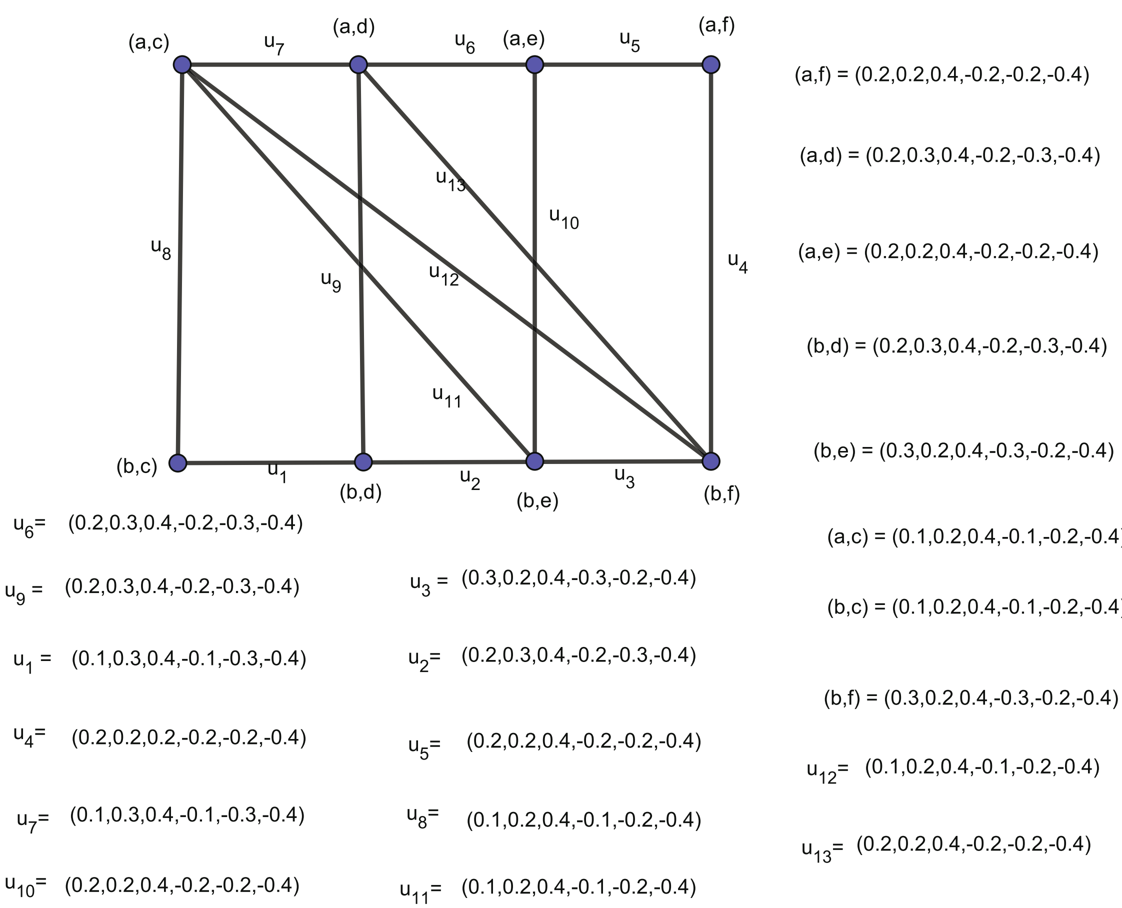

Figure 7: $\mathbb{G}_1 \oplus \mathbb{G}_2$.

Proof. Suppose $\mathbb{G}_1 = (U_1, W_1)$ and $\mathbb{G}_2 = (U_2, W_2)$ are two BPFGs on two crisp graphs and $((k_1, k_2)(n_1, n_2)) \in E_1 \times E_2$.

(i)

If $k_1 = n_1 = m$

$$\begin{aligned}(\chi_{W_1} \oplus \chi_{W_2})((m, k_2)(m, n_2)) &= \wedge\{\chi_{U_1}(m), \chi_{W_2}(k_2n_2)\}\\ &\leq \wedge\{\chi_{U_1}(m), \wedge \{\chi_{U_2}(k_2), \chi_{U_2}(n_2)\}\}\\ &= \wedge\{\wedge\{\{\chi_{U_1}(m), \chi_{U_2}(k_2)\}, \wedge \{\{\chi_{U_1}(m), \chi_{U_2}(n_2)\}\}\}\}\\ &= \wedge\{(\chi_{U_1} \oplus \chi_{U_2})(m, k_2), (\chi_{U_1} \oplus \chi_{U_2})(m, n_2)\},\end{aligned}$$

$$\begin{aligned}(\phi_{W_1} \oplus \phi_{W_2})((m, k_2)(m, n_2)) &= \wedge\{\phi_{U_1}(m), \phi_{W_2}(k_2n_2)\}\\ &\leq \wedge\{\phi_{U_1}(m), \wedge \{\phi_{U_2}(k_2), \phi_{U_2}(n_2)\}\}\\ &= \wedge\{\wedge\{\{\phi_{U_1}(m), \phi_{U_2}(k_2)\}, \wedge \{\{\phi_{U_1}(m), \phi_{U_2}(n_2)\}\}\\ &= \wedge\{(\phi_{U_1} \oplus \phi_{U_2})(m, k_2), (\phi_{U_1} \oplus \phi_{U_2})(m, n_2)\},\end{aligned}$$

$$\begin{aligned}(\psi_{W_1} \oplus \psi_{W_2})((m, k_2)(m, n_2)) &= \vee\{\psi_{U_1}(m), \psi_{W_2}(k_2n_2)\}\\ &\geq \vee\{\psi_{U_1}(m), \vee \{\psi_{U_2}(k_2), \psi_{U_2}(n_2)\}\}\\ &= \vee\{\vee\{\{\psi_{U_1}(m), \psi_{U_2}(k_2)\}, \vee \{\{\psi_{U_1}(m), \psi_{U_2}(n_2)\}\}\\ &= \vee\{(\psi_{U_1} \oplus \psi_{U_2})(m, k_2), (\psi_{U_1} \oplus \psi_{U_2})(m, n_2)\},\end{aligned}$$

$$\begin{aligned}(\alpha_{W_1} \oplus \alpha_{W_2})((m, k_2)(m, n_2)) &= \vee\{\alpha_{U_1}(m), \alpha_{W_2}(k_2n_2)\}\\ &\geq \vee\{\alpha_{U_1}(m), \vee \{\alpha_{U_2}(k_2), \alpha_{U_2}(n_2)\}\}\\ &= \vee\{\vee\{\{\alpha_{U_1}(m), \alpha_{U_2}(k_2)\}, \vee \{\{\alpha_{U_1}(m), \alpha_{U_2}(n_2)\}\}\\ &= \vee\{(\alpha_{U_1} \oplus \alpha_{U_2})(m, k_2), (\alpha_{U_1} \oplus \alpha_{U_2})(m, n_2)\},\end{aligned}$$

$$\begin{aligned}(\beta_{W_1} \oplus \beta_{W_2})((m, k_2)(m, n_2)) &= \vee\{\beta_{U_1}(m), \beta_{W_2}(k_2n_2)\}\\ &\geq \vee\{\beta_{U_1}(m), \vee \{\beta_{U_2}(k_2), \beta_{U_2}(n_2)\}\}\\ &= \vee\{\vee\{\{\beta_{U_1}(m), \beta_{U_2}(k_2)\}, \vee \{\{\beta_{U_1}(m), \beta_{U_2}(n_2)\}\}\\ &= \vee\{(\beta_{U_1} \oplus \beta_{U_2})(m, k_2), (\beta_{U_1} \oplus \beta_{U_2})(m, n_2)\},\end{aligned}$$

$$\begin{aligned}(\gamma_{W_1} \oplus \gamma_{W_2})((m, k_2)(m, n_2)) &= \wedge\{\gamma_{U_1}(m), \gamma_{W_2}(k_2n_2)\}\\ &\leq \wedge\{\gamma_{U_1}(m), \wedge \{\gamma_{U_2}(k_2), \gamma_{U_2}(n_2)\}\}\\ &= \wedge\{\wedge\{\{\gamma_{U_1}(m), \gamma_{U_2}(k_2)\}, \wedge \{\{\gamma_{U_1}(m), \gamma_{U_2}(n_2)\}\}\\ &= \wedge\{(\gamma_{U_1} \oplus \gamma_{U_2})(m, k_2), (\gamma_{U_1} \oplus \gamma_{U_2})(m, n_2)\}.\end{aligned}$$

(ii) If $k_2 = n_2 = z$,

$$\begin{aligned}(\chi_{W_1} \oplus \chi_{W_2})((k_1, z)(n_1, z)) &= \wedge\{\chi_{W_1}(k_1n_1), \chi_{U_2}(z)\}\\ &\leq \wedge\{\wedge\{\chi_{U_1}(k_1), \chi_{U_1}(n_1)\}, \chi_{U_2}(z)\}\\ &= \wedge\{\wedge\{\{\chi_{U_1}(k_1), \chi_{U_2}(z)\}, \wedge \{\{\chi_{U_1}(n_1), \chi_{U_2}(z)\}\}\\ &= \wedge\{(\chi_{U_1} \oplus \chi_{U_2})(k_1, z), (\chi_{U_1} \oplus \chi_{U_2})(n_1, z)\},\end{aligned}$$

$$\begin{aligned}(\phi_{W_1} \oplus \phi_{W_2})((k_1, z)(n_1, z)) &= \wedge\{\phi_{W_1}(k_1n_1), \phi_{U_2}(z)\}\\ &\geq \wedge\{\wedge\{\phi_{U_1}(k_1), \phi_{U_1}(n_1)\}, \phi_{U_2}(z)\}\\ &= \wedge\{\wedge\{\{\phi_{U_1}(k_1), \phi_{U_2}(z)\}, \wedge \{\{\phi_{U_1}(n_1), \phi_{U_2}(z)\}\}\\ &= \wedge\{(\phi_{U_1} \oplus \phi_{U_2})(k_1, z), (\phi_{U_1} \oplus \phi_{U_2})(n_1, z)\},\end{aligned}$$

$$\begin{aligned}(\psi_{W_1} \oplus \psi_{W_2})((k_1, z)(n_1, z)) &= \vee\{\psi_{W_1}(k_1n_1), \psi_{U_2}(z)\}\\ &\geq \vee\{\vee\{\psi_{U_1}(k_1), \psi_{U_1}(n_1)\}, \chi_{U_2}(z)\}\\ &= \vee\{\vee\{\{\psi_{U_1}(k_1), \psi_{U_2}(z)\}, \vee \{\{\psi_{U_1}(n_1), \psi_{U_2}(z)\}\}\\ &= \vee\{(\psi_{U_1} \oplus \psi_{U_2})(k_1, z), (\psi_{U_1} \oplus \psi_{U_2})(n_1, z)\},\end{aligned}$$

$$\begin{aligned}(\alpha_{W_1} \oplus \alpha_{W_2})((k_1, z)(n_1, z)) &= \vee\{\alpha_{W_1}(k_1n_1), \alpha_{U_2}(z)\}\\ &\geq \vee\{\vee\{\alpha_{U_1}(k_1), \alpha_{U_1}(n_1)\}, \chi_{U_2}(z)\}\\ &= \vee\{\vee\{\{\alpha_{U_1}(k_1), \alpha_{U_2}(z)\}, \vee \{\{\alpha_{U_1}(n_1), \alpha_{U_2}(z)\}\}\\ &= \vee\{(\alpha_{U_1} \oplus \alpha_{U_2})(k_1, z), (\alpha_{U_1} \oplus \alpha_{U_2})(n_1, z)\},\end{aligned}$$

$$\begin{aligned}(\beta_{W_1} \oplus \beta_{W_2})((k_1, z)(n_1, z)) &= \vee\{\beta_{W_1}(k_1n_1), \beta_{U_2}(z)\}\\ &\geq \vee\{\vee\{\beta_{U_1}(k_1), \beta_{U_1}(n_1)\}, \chi_{U_2}(z)\}\\ &= \vee\{\vee\{\{\beta_{U_1}(k_1), \beta_{U_2}(z)\}, \vee \{\{\beta_{U_1}(n_1), \beta_{U_2}(z)\}\}\\ &= \vee\{(\beta_{U_1} \oplus \beta_{U_2})(k_1, z), (\beta_{U_1} \oplus \beta_{U_2})(n_1, z)\},\end{aligned}$$

$$\begin{aligned}(\gamma_{W_1} \oplus \gamma_{W_2})((k_1, z)(n_1, z)) &= \wedge\{\wedge_{U_1}(k_1), \wedge_{Ue_1}(n_1)\}, \gamma_{U_2}(z)\}\\ &\leq \wedge\{\wedge\{\gamma_{W_1}(k_1), \gamma_{U_1}(n_1)\}, \gamma_{U_2}(z)\}\\ &= \wedge\{\wedge\{\{\gamma_{U_1}(k_1), \gamma_{U_2}(z)\}, \wedge \{\{\gamma_{U_1}(n_1), \gamma_{U_2}(z)\}\}\\ &= \wedge\{(\gamma_{U_1} \oplus \gamma_{U_2})(k_1, z), (\gamma_{U_1} \oplus \gamma_{U_2})(n_1, z)\},\end{aligned}$$

(iii) If $k_1n_1 \notin E_1$ and $k_2n_2 \in E_2$,

$$\begin{aligned}(\chi_{W_1} \oplus \chi_{W_2})((k_1, k_2)(n_1, n_2)) &= \wedge\{\chi_{U_1}(k_1), \chi_{U_1}(n_1), \chi_{W_2}(k_2n_2)\}\\ &\leq \wedge\{\chi_{U_1}(k_1), \chi_{U_1}(n_1), \wedge \{\chi_{U_2}(k_2)\chi_{U_2}(n_2)\}\}\\ &= \wedge\{\wedge\{\chi_{U_1}(k_1), \chi_{U_2}(k_2)\}, \wedge \{\chi_{U_1}(n_1), \chi_{U_2}(n_2)\}\\ &= \wedge\{(\chi_{U_1} \oplus \chi_{U_2})(k_1, k_2), (\chi_{U_1} \oplus \chi_{U_2})(n_1, n_2)\},\end{aligned}$$

$$\begin{aligned}(\phi_{W_1} \oplus \phi_{W_2})((k_1, k_2)(n_1, n_2)) &= \wedge\{\phi_{U_1}(k_1), \phi_{U_1}(n_1), \phi_{W_2}(k_2n_2)\}\\ &\leq \wedge\{\phi_{U_1}(k_1), \phi_{U_1}(n_1), \wedge \{\phi_{U_2}(k_2)\phi_{U_2}(n_2)\}\}\\ &= \wedge\{\wedge\{\phi_{U_1}(k_1), \phi_{U_2}(k_2)\}, \wedge \{\phi_{U_1}(n_1), \phi_{U_2}(n_2)\}\\ &= \wedge\{(\phi_{U_1} \oplus \phi_{U_2})(k_1, k_2), (\phi_{U_1} \oplus \phi_{U_2})(n_1, n_2)\},\end{aligned}$$

$$\begin{aligned}(\psi_{W_1} \oplus \psi_{W_2})((k_1, k_2)(n_1, n_2)) &= \vee\{\psi_{U_1}(k_1), \psi_{U_1}(n_1), \psi_{W_2}(k_2n_2)\}\\ &\geq \vee\{\psi_{U_1}(k_1), \psi_{U_1}(n_1), \vee \{\psi_{U_2}(k_2)\psi_{U_2}(n_2)\}\}\\ &= \vee\{\vee\{\psi_{U_1}(k_1), \psi_{U_2}(k_2)\}, \vee \{\psi_{U_1}(n_1), \psi_{U_2}(n_2)\}\\ &= \vee\{(\psi_{U_1} \oplus \psi_{U_2})(k_1, k_2), (\psi_{U_1} \oplus \psi_{U_2})(n_1, n_2)\},\end{aligned}$$

$$\begin{aligned}(\alpha_{W_1} \oplus \alpha_{W_2})((k_1, k_2)(n_1, n_2)) &= \vee\{\alpha_{U_1}(k_1), \alpha_{U_1}(n_1), \alpha_{W_2}(k_2n_2)\}\\ &\geq \vee\{\alpha_{U_1}(k_1), \alpha_{U_1}(n_1), \vee \{\alpha_{U_2}(k_2)\alpha_{U_2}(n_2)\}\}\\ &= \vee\{\vee\{\alpha_{U_1}(k_1), \alpha_{U_2}(k_2)\}, \vee \{\alpha_{U_1}(n_1), \alpha_{U_2}(n_2)\}\\ &= \vee\{(\alpha_{U_1} \oplus \alpha_{U_2})(k_1, k_2), (\alpha_{U_1} \oplus \alpha_{U_2})(n_1, n_2)\},\end{aligned}$$

$$\begin{aligned}(\beta_{W_1} \oplus \beta_{W_2})((k_1, k_2)(n_1, n_2)) &= \vee\{\beta_{U_1}(k_1), \beta_{U_1}(n_1), \beta_{W_2}(k_2n_2)\}\\ &\geq \vee\{\beta_{U_1}(k_1), \beta_{U_1}(n_1), \vee \{\beta_{U_2}(k_2)\beta_{U_2}(n_2)\}\}\\ &= \vee\{\vee\{\beta_{U_1}(k_1), \beta_{U_2}(k_2)\}, \vee \{\beta_{U_1}(n_1), \beta_{U_2}(n_2)\}\\ &= \vee\{(\beta_{U_1} \oplus \beta_{U_2})(k_1, k_2), (\beta_{U_1} \oplus \beta_{U_2})(n_1, n_2)\},\end{aligned}$$

$$\begin{aligned}(\gamma_{W_1} \oplus \gamma_{W_2})((k_1,k_2)(n_1,n_2)) &= \wedge\{\gamma_{U_1}(k_1), \gamma_{U_1}(n_1), \gamma_{W_2}(k_2n_2)\}\\ &\le \wedge\{\gamma_{U_1}(k_1), \gamma_{U_1}(n_1), \wedge\{\gamma_{U_2}(k_2)\gamma_{U_2}(n_2)\}\}\\ &= \wedge\{\wedge\{\gamma_{U_1}(k_1), \gamma_{U_2}(k_2)\}, \wedge\{\gamma_{U_1}(n_1), \gamma_{U_2}(n_2)\}\\ &= \wedge\{(\gamma_{U_1} \oplus \gamma_{U_2})(k_1,k_2), (\gamma_{U_1} \oplus \gamma_{U_2})(n_1,n_2)\}.\end{aligned}$$

(iv) If $k_1n_1 \in E_1$ and $k_2n_2 \notin E_2$,

$$\begin{aligned}(\chi_{W_1} \oplus \chi_{W_2})((k_1,k_2)(n_1,n_2)) &= \wedge\{\chi_{U_2}(k_2), \chi_{U_2}(n_2), \chi_{W_1}(k_1n_1)\}\\ &\le \wedge\{\chi_{U_2}(k_2), \chi_{U_2}(n_2), \wedge\{\chi_{U_1}(k_1)\chi_{U_1}(n_1)\}\}\\ &= \wedge\{\wedge\{\chi_{U_1}(k_1), \chi_{U_2}(k_2)\}, \wedge\{\chi_{U_1}(n_1), \chi_{U_2}(n_2)\}\\ &= \wedge\{(\chi_{U_1} \oplus \chi_{U_2})(k_1,k_2), (\chi_{U_1} \oplus \chi_{U_2})(n_1,n_2)\},\end{aligned}$$

$$\begin{aligned}(\phi_{W_1} \oplus \phi_{W_2})((k_1,k_2)(n_1,n_2)) &= \wedge\{\phi_{U_2}(k_2), \phi_{U_2}(n_2), \phi_{W_1}(k_1n_1)\}\\ &\le \wedge\{\phi_{U_2}(k_2), \phi_{U_2}(n_2), \wedge\{\phi_{U_1}(k_1)\phi_{U_1}(n_1)\}\}\\ &= \wedge\{\wedge\{\phi_{U_2}(k_2), \phi_{U_1}(k_1)\}, \wedge\{\phi_{U_2}(n_2), \phi_{U_1}(n_1)\}\\ &= \wedge\{(\phi_{U_1} \oplus \phi_{U_2})(k_1,k_2), (\phi_{U_1} \oplus \phi_{U_2})(n_1,n_2)\},\end{aligned}$$

$$\begin{aligned}(\psi_{W_1} \oplus \psi_{W_2})((k_1,k_2)(n_1,n_2)) &= \vee\{\psi_{U_2}(k_2), \psi_{U_2}(n_2), \psi_{W_1}(k_1n_1)\}\\ &\ge \vee\{\psi_{U_2}(k_2), \psi_{U_2}(n_2), \vee\{\psi_{U_1}(k_1)\psi_{U_1}(n_1)\}\}\\ &= \vee\{\vee\{\psi_{U_2}(k_2), \psi_{U_1}(k_1)\}, \vee\{\psi_{U_2}(n_2), \psi_{U_1}(n_1)\}\\ &= \vee\{(\psi_{U_1} \oplus \psi_{U_2})(k_1,k_2), (\psi_{U_1} \oplus \psi_{U_2})(n_1,n_2)\},\end{aligned}$$

$$\begin{aligned}(\alpha_{W_1} \oplus \alpha_{W_2})((k_1,k_2)(n_1,n_2)) &= \vee\{\alpha_{U_2}(k_2), \alpha_{U_2}(n_2), \alpha_{W_1}(k_1n_1)\}\\ &\ge \vee\{\alpha_{U_2}(k_2), \alpha_{U_2}(n_2), \vee\{\alpha_{U_1}(k_1)\alpha_{U_1}(n_1)\}\}\\ &= \vee\{\vee\{\alpha_{U_2}(k_2), \alpha_{U_1}(k_1)\}, \vee\{\alpha_{U_2}(n_2), \alpha_{U_1}(n_1)\}\\ &= \vee\{(\alpha_{U_1} \oplus \alpha_{U_2})(k_1,k_2), (\alpha_{U_1} \oplus \alpha_{U_2})(n_1,n_2)\},\end{aligned}$$

$$\begin{aligned}(\beta_{W_1} \oplus \beta_{W_2})((k_1,k_2)(n_1,n_2)) &= \vee\{\beta_{U_2}(k_2), \beta_{U_2}(n_2), \beta_{W_1}(k_1n_1)\}\\ &\ge \vee\{\beta_{U_2}(k_2), \beta_{U_2}(n_2), \vee\{\beta_{U_1}(k_1)\beta_{U_1}(n_1)\}\}\\ &= \vee\{\vee\{\beta_{U_2}(k_2), \beta_{U_1}(k_1)\}, \vee\{\beta_{U_2}(n_2), \beta_{U_1}(n_1)\}\\ &= \vee\{(\beta_{U_1} \oplus \beta_{U_2})(k_1,k_2), (\beta_{U_1} \oplus \beta_{U_2})(n_1,n_2)\},\end{aligned}$$

$$\begin{aligned}(\gamma_{W_1} \oplus \gamma_{W_2})((k_1,k_2)(n_1,n_2)) &= \wedge\{\gamma_{U_2}(k_2), \gamma_{U_2}(n_2), \gamma_{W_1}(k_1n_1)\}\\ &\le \wedge\{\gamma_{U_2}(k_2), \gamma_{U_2}(n_2), \wedge\{\gamma_{U_1}(k_1)\gamma_{U_1}(n_1)\}\}\\ &= \wedge\{\wedge\{\gamma_{U_1}(k_1), \gamma_{U_2}(k_2)\}, \wedge\{\gamma_{U_1}(n_1), \gamma_{U_2}(n_2)\}\\ &= \wedge\{(\gamma_{U_1} \oplus \gamma_{U_2})(k_1,k_2), (\gamma_{U_1} \oplus \gamma_{U_2})(n_1,n_2)\}.\end{aligned}$$

Hence, $\mathbb{G}_1 \oplus \mathbb{G}_2$ is a BPFG. □

Definition 4.11. Let $\mathbb{G}_1 = (U_1, W_1)$ and $\mathbb{G}_2 = (U_2, W_2)$ be two BPFGs on crisp graphs. For any vertex $(k_1,k_2) \in V_1 \times V_2$, we have

$$\begin{aligned}(td_\chi)_{\mathbb{G}_1\oplus\mathbb{G}_2}(k_1,k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\chi_{W_1} \oplus \chi_{W_2})((k_1,k_2)(n_1,n_2)) + (\chi_{U_1} \oplus \chi_{U_2})(k_1,k_2)\\ &= \sum_{k_1=n_1, k_2n_2\in E_2} \wedge\{\chi_{U_1}(k_1), \chi_{W_2}(k_2n_2)\} + \sum_{k_1n_1\in E_1, k_2=n_2} \wedge\{\chi_{W_1}(k_1n_1), \chi_{U_2}(k_2)\}\\ &\quad + \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2} \wedge\{\chi_{U_1}(k_1), \chi_{U_1}(n_1), \chi_{W_2}(k_2n_2)\}\\ &\quad + \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2} \wedge\{\chi_{W_1}(k_1n_1), \chi_{U_2}(k_2), \chi_{U_2}(n_2)\} + \wedge\{\chi_{U_1}(k_1), \chi_{U_2}(k_2)\},\end{aligned}$$

$$
\begin{aligned}
(td_{\phi})_{\mathbb{G}_1\oplus\mathbb{G}_2}(k_1,k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\phi_{W_1}\oplus\phi_{W_2})((k_1,k_2)(n_1,n_2)) + (\phi_{U_1}\oplus\phi_{U_2})(k_1,k_2)\\
&= \sum_{k_1=n_1,k_2n_2\in E_2} \wedge\{\phi_{U_1}(k_1),\phi_{W_2}(k_2n_2)\} + \sum_{k_1n_1\in E_1,k_2=n_2} \wedge\{\phi_{W_1}(k_1n_1),\phi_{U_2}(k_2)\}\\
&\quad+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2} \wedge\{\phi_{U_1}(k_1),\phi_{U_1}(n_1),\phi_{W_2}(k_2n_2)\}\\
&\quad+ \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2} \wedge\{\phi_{W_1}(k_1n_1),\phi_{U_2}(k_2),\phi_{U_2}(n_2)\} + \wedge\{\phi_{U_1}(k_1),\phi_{U_2}(k_2)\},
\end{aligned}
$$

$$
\begin{aligned}
(td_{\psi})_{\mathbb{G}_1\oplus\mathbb{G}_2}(k_1,k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\psi_{W_1}\oplus\psi_{W_2})((k_1,k_2)(n_1,n_2)) + (\psi_{U_1}\oplus\psi_{U_2})(k_1,k_2)\\
&= \sum_{k_1=n_1,k_2n_2\in E_2} \vee\{\psi_{U_1}(k_1),\psi_{W_2}(k_2n_2)\} + \sum_{k_1n_1\in E_1,k_2=n_2} \vee\{\psi_{W_1}(k_1n_1),\psi_{U_2}(k_2)\}\\
&\quad+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2} \vee\{\psi_{U_1}(k_1),\psi_{U_1}(n_1),\psi_{W_2}(k_2n_2)\}\\
&\quad+ \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2} \vee\{\psi_{W_1}(k_1n_1),\psi_{U_2}(k_2),\psi_{U_2}(n_2)\}\\
&\quad+ \vee\{\psi_{U_1}(k_1),\psi_{U_2}(k_2)\},
\end{aligned}
$$

$$
\begin{aligned}
(td_{\alpha})_{\mathbb{G}_1\oplus\mathbb{G}_2}(k_1,k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\alpha_{W_1}\oplus\alpha_{W_2})((k_1,k_2)(n_1,n_2)) + (\alpha_{U_1}\oplus\alpha_{U_2})(k_1,k_2)\\
&= \sum_{k_1=n_1,k_2n_2\in E_2} \vee\{\alpha_{U_1}(k_1),\alpha_{W_2}(k_2n_2)\} + \sum_{k_1n_1\in E_1,k_2=n_2} \vee\{\alpha_{W_1}(k_1n_1),\alpha_{U_2}(k_2)\}\\
&\quad+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2} \vee\{\alpha_{U_1}(k_1),\alpha_{U_1}(n_1),\alpha_{W_2}(k_2n_2)\}\\
&\quad+ \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2} \vee\{\alpha_{W_1}(k_1n_1),\alpha_{U_2}(k_2),\alpha_{U_2}(n_2)\} + \vee\{\alpha_{U_1}(k_1),\alpha_{U_2}(k_2)\},
\end{aligned}
$$

$$
\begin{aligned}
(td_{\beta})_{\mathbb{G}_1\oplus\mathbb{G}_2}(k_1,k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\beta_{W_1}\oplus\beta_{W_2})((k_1,k_2)(n_1,n_2)) + (\beta_{U_1}\oplus\beta_{U_2})(k_1,k_2)\\
&= \sum_{k_1=n_1,k_2n_2\in E_2} \vee\{\beta_{U_1}(k_1),\beta_{W_2}(k_2n_2)\} + \sum_{k_1n_1\in E_1,k_2=n_2} \vee\{\beta_{W_1}(k_1n_1),\beta_{U_2}(k_2)\}\\
&\quad+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2} \vee\{\beta_{U_1}(k_1),\beta_{U_1}(n_1),\beta_{W_2}(k_2n_2)\}\\
&\quad+ \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2} \vee\{\beta_{W_1}(k_1n_1),\beta_{U_2}(k_2),\beta_{U_2}(n_2)\} + \vee\{\beta_{U_1}(k_1),\beta_{U_2}(k_2)\},
\end{aligned}
$$

$$
\begin{aligned}
(td_{\gamma})_{\mathbb{G}_1\oplus\mathbb{G}_2}(k_1,k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\gamma_{W_1}\oplus\gamma_{W_2})((k_1,k_2)(n_1,n_2)) + (\gamma_{U_1}\oplus\gamma_{U_2})(k_1,k_2)\\
&= \sum_{k_1=n_1,k_2n_2\in E_2} \wedge\{\gamma_{U_1}(k_1),\gamma_{W_2}(k_2n_2)\} + \sum_{k_1n_1\in E_1,k_2=n_2} \wedge\{\gamma_{W_1}(k_1n_1),\gamma_{U_2}(k_2)\}\\
&\quad+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2} \wedge\{\gamma_{U_1}(k_1),\gamma_{U_1}(n_1),\gamma_{W_2}(k_2n_2)\}\\
&\quad+ \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2} \wedge\{\gamma_{W_1}(k_1n_1),\gamma_{U_2}(k_2),\gamma_{U_2}(n_2)\} + \wedge\{\gamma_{U_1}(k_1),\gamma_{U_2}(k_2)\}.
\end{aligned}
$$

Theorem 4.12. *Suppose $\mathbb{G}_1 = (U_1, W_1)$ and $\mathbb{G}_2 = (U_2, Y_2)$ are two BPFGs on crisp graphs. If $\chi_{U_1} \geq \chi_{W_2}$ and $\chi_{U_2} \geq \chi_{W_1}$, then $\forall (k_1, k_2) \in V_1 \times V_2$,*

$$(td_{\chi})_{\mathbb{G}_1\oplus\mathbb{G}_2}(k_1,k_2) = q(td_{\chi})_{\mathbb{G}_1}(k_1) + s(td_{\chi})_{\mathbb{G}_2}(k_2) - (q-1)\chi_{\mathbb{G}_1}(k_1) - \vee\{\chi_{\mathbb{G}_1}(k_1),\chi_{\mathbb{G}_1}(k_1)\};$$

if $\phi_{U_1} \leq \phi_{W_2}$ and $\phi_{U_2} \leq \phi_{W_1}$, then $\forall (k_1, k_2) \in V_1 \times V_2$,

$$(td_{\phi})_{\mathbb{G}_1\oplus\mathbb{G}_2}(k_1,k_2) = q(td_{\phi})_{\mathbb{G}_1}(k_1) + s(td_{\phi})_{\mathbb{G}_2}(k_2) - (q-1)\phi_{\mathbb{G}_1}(k_1) - \vee\{\phi_{\mathbb{G}_1}(k_1),\phi_{\mathbb{G}_1}(k_1)\};$$

if $\psi_{U_1} \le \psi_{W_2}$ *and* $\psi_{U_2} \ge \psi_{W_1}$, *then* $\forall (k_1, k_2) \in V_1 \times V_2$,

$$(td_\psi)_{\mathbb{G}_1 \oplus \mathbb{G}_2}(k_1, k_2) = q(td_\psi)_{\mathbb{G}_1}(k_1) + s(td_\psi)_{\mathbb{G}_2}(k_2) - (q-1)\psi_{\mathbb{G}_1}(k_1) - \wedge \{\psi_{\mathbb{G}_1}(k_1), \psi_{\mathbb{G}_1}(k_1)\};$$

if $\alpha_{U_1} \le \alpha_{W_2}$ *and* $\alpha_{U_2} \ge \alpha_{W_1}$, *then* $\forall (k_1, k_2) \in V_1 \times V_2$,

$$(td_\alpha)_{\mathbb{G}_1 \oplus \mathbb{G}_2}(k_1, k_2) = q(td_\alpha)_{\mathbb{G}_1}(k_1) + s(td_\alpha)_{\mathbb{G}_2}(k_2) - (q-1)\alpha_{\mathbb{G}_1}(k_1) - \wedge \{\alpha_{\mathbb{G}_1}(k_1), \alpha_{\mathbb{G}_1}(k_1)\};$$

if $\beta_{U_1} \le \beta_{W_2}$ *and* $\beta_{U_2} \ge \beta_{W_1}$, *then* $\forall (k_1, k_2) \in V_1 \times V_2$,

$$(td_\beta)_{\mathbb{G}_1 \oplus \mathbb{G}_2}(k_1, k_2) = q(td_\beta)_{\mathbb{G}_1}(k_1) + s(td_\beta)_{\mathbb{G}_2}(k_2) - (q-1)\beta_{\mathbb{G}_1}(k_1) - \wedge \{\beta_{\mathbb{G}_1}(k_1), \beta_{\mathbb{G}_1}(k_1)\};$$

if $\gamma_{U_1} \ge \gamma_{W_2}$ *and* $\gamma_{U_2} \ge \gamma_{W_1}$, *then* $\forall (k_1, k_2) \in V_1 \times V_2$,

$$(td_\gamma)_{\mathbb{G}_1 \oplus \mathbb{G}_2}(k_1, k_2) = q(td_\gamma)_{\mathbb{G}_1}(k_1) + s(td_\gamma)_{\mathbb{G}_2}(k_2) - (q-1)\gamma_{\mathbb{G}_1}(k_1) - \vee \{\gamma_{\mathbb{G}_1}(k_1), \gamma_{\mathbb{G}_1}(k_1)\};$$

where $s = |V_1| - (d)_{G_1}(k_1)$ *and* $q = |V_2| - (d)_{G_2}(k_2)$.

Proof. $\forall (k_1, k_2) \in V_1 \times V_2$, we have

$$\begin{aligned}
(td_\chi)_{\mathbb{G}_1 \oplus \mathbb{G}_2}(k_1, k_2) = & \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\chi_{W_1} \oplus \chi_{W_2})((k_1,k_2)(n_1,n_2)) + (\chi_{U_1} \oplus \chi_{U_2})(k_1,k_2) \\
= & \sum_{k_1=n_1, k_2n_2\in E_2} \wedge \{\chi_{U_1}(k_1), \chi_{W_2}(k_2n_2)\} + \sum_{k_1n_1\in E_1, k_2=n_2} \wedge \{\chi_{W_1}(k_1n_1), \chi_{U_2}(k_2)\} \\
& + \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2} \wedge \{\chi_{U_1}(k_1), \chi_{U_1}(n_1), \chi_{W_2}(k_2n_2)\} \\
& + \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2} \wedge \{\chi_{W_1}(k_1n_1), \chi_{U_2}(k_2), \chi_{U_2}(n_2)\} + \vee \{\chi_{U_1}(k_1), \chi_{U_2}(k_2)\} \\
= & \sum_{k_2n_2\in E_2} \chi_{W_2}(k_2n_2) + \sum_{k_1n_1\in E_1} \chi_{W_1}(k_1n_1) + \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2} \chi_{W_2}(k_2n_2) \\
& + \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2} \chi_{W_1}(k_1n_1) + \wedge \{\chi_{U_1}(k_1), \chi_{U_2}(k_2)\} \\
= & \sum_{k_2n_2\in E_2} \chi_{W_2}(k_2n_2) + \sum_{k_1n_1\in E_1} \chi_{W_1}(k_1n_1) + \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2} \chi_{W_2}(k_2n_2) \\
& + \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2} \chi_{W_1}(k_1n_1) + \chi_{U_1}(k_1) + \chi_{U_2}(k_2) - \wedge \{\chi_{U_1}(k_1), \chi_{U_2}(k_2)\} \\
= & q(td_\chi)_{\mathbb{G}_1}(k_1) + s(td_\chi)_{\mathbb{G}_2}(k_2) - (q-1)\chi_{\mathbb{G}_1}(k_1) - \vee \{\chi_{\mathbb{G}_1}(k_1), \chi_{\mathbb{G}_1}(k_1)\},
\end{aligned}$$

$$\begin{aligned}
(td_\phi)_{\mathbb{G}_1 \oplus \mathbb{G}_2}(k_1, k_2) = & \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2} (\phi_{W_1} \oplus \phi_{W_2})((k_1,k_2)(n_1,n_2)) + (\phi_{U_1} \oplus \phi_{U_2})(k_1,k_2) \\
= & \sum_{k_1=n_1, k_2n_2\in E_2} \wedge \{\phi_{U_1}(k_1), \phi_{W_2}(k_2n_2)\} + \sum_{k_1n_1\in E_1, k_2=n_2} \wedge \{\phi_{W_1}(k_1n_1), \phi_{U_2}(k_2)\} \\
& + \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2} \wedge \{\phi_{U_1}(k_1), \phi_{U_1}(n_1), \phi_{W_2}(k_2n_2)\} \\
& + \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2} \wedge \{\phi_{W_1}(k_1n_1), \phi_{U_2}(k_2), \phi_{U_2}(n_2)\} + \wedge \{\phi_{U_1}(k_1), \phi_{U_2}(k_2)\} \\
= & \sum_{k_2n_2\in E_2} \phi_{W_2}(k_2n_2) + \sum_{k_1n_1\in E_1} \phi_{W_1}(k_1n_1) + \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2} \phi_{W_2}(k_2n_2)\} \\
& + \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2} \phi_{W_1}(k_1n_1) + \wedge \{\phi_{U_1}(k_1), \phi_{U_2}(k_2)\} \\
= & \sum_{k_2n_2\in E_2} \phi_{W_2}(k_2n_2) + \sum_{k_1n_1\in E_1} \phi_{W_1}(k_1n_1) \\
& + \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2} \phi_{W_2}(k_2n_2)\} \\
& + \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2} \phi_{W_1}(k_1n_1) + \phi_{U_1}(k_1) + \phi_{U_2}(k_2) - \wedge \{\phi_{U_1}(k_1), \phi_{U_2}(k_2)\} \\
= & q(td_\phi)_{\mathbb{G}_1}(k_1) + s(td_\phi)_{\mathbb{G}_2}(k_2) - (q-1)\phi_{\mathbb{G}_1}(k_1) - \vee \{\phi_{\mathbb{G}_1}(k_1), \phi_{\mathbb{G}_1}(k_1)\},
\end{aligned}$$

$$
\begin{aligned}
(td_\psi)_{\mathbb{G}_1 \oplus \mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2) \in E_1 \times E_2.} (\psi_{W_1} \oplus \psi_{W_2})((k_1, k_2)(n_1, n_2)) + (\psi_{U_1} \oplus \psi_{U_2})(k_1, k_2) \\
&= \sum_{k_1 = n_1, k_2n_2 \in E_2} \vee \{\psi_{U_1}(k_1), \psi_{W_2}(k_2n_2)\} + \sum_{k_1n_1 \in E_1, k_2 = n_2} \vee \{\psi_{W_1}(k_1n_1), \psi_{U_2}(k_2)\} \\
&\quad + \sum_{k_1n_1 \notin E_1 \text{ and } k_2n_2 \in E_2} \vee \{\psi_{U_1}(k_1), \psi_{U_1}(n_1), \psi_{W_2}(k_2n_2)\} \\
&\quad + \sum_{k_1n_1 \in E_1 \text{ and } k_2n_2 \notin E_2} \vee \{\psi_{W_1}(k_1n_1), \psi_{U_2}(k_2), \psi_{U_2}(n_2)\} + \wedge \{\psi_{U_1}(k_1), \psi_{U_2}(k_2)\} \\
&= \sum_{k_2n_2 \in E_2} \psi_{W_2}(k_2n_2) + \sum_{k_1n_1 \in E_1} \psi_{W_1}(k_1n_1) + \sum_{k_1n_1 \notin E_1 \text{ and } k_2n_2 \in E_2} \psi_{W_2}(k_2n_2)\} \\
&\quad + \sum_{k_1n_1 \in E_1 \text{ and } k_2n_2 \notin E_2} \psi_{W_1}(k_1n_1) + \vee \{\psi_{U_1}(k_1), \psi_{U_2}(k_2)\} \\
&= \sum_{k_2n_2 \in E_2} \psi_{W_2}(k_2n_2) + \sum_{k_1n_1 \in E_1} \psi_{W_1}(k_1n_1) + \sum_{k_1n_1 \notin E_1 \text{ and } k_2n_2 \in E_2} \psi_{W_2}(k_2n_2)\} \\
&\quad + \sum_{k_1n_1 \in E_1 \text{ and } k_2n_2 \notin E_2} \psi_{W_1}(k_1n_1) + \psi_{U_1}(k_1) + \psi_{U_2}(k_2) - \vee \{\psi_{U_1}(k_1), \psi_{U_2}(k_2)\} \\
&= q(td_\psi)_{\mathbb{G}_1}(k_1) + s(td_\psi)_{\mathbb{G}_2}(k_2) - (q-1)\psi_{\mathbb{G}_1}(k_1) - \wedge \{\psi_{\mathbb{G}_1}(k_1), \psi_{\mathbb{G}_1}(k_1)\},
\end{aligned}
$$

$$
\begin{aligned}
(td_\alpha)_{\mathbb{G}_1 \oplus \mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2) \in E_1 \times E_2} (\alpha_{W_1} \oplus \alpha_{W_2})((k_1, k_2)(n_1, n_2)) + (\alpha_{U_1} \oplus \alpha_{U_2})(k_1, k_2) \\
&= \sum_{k_1 = n_1, k_2n_2 \in E_2} \vee \{\alpha_{U_1}(k_1), \alpha_{W_2}(k_2n_2)\} + \sum_{k_1n_1 \in E_1, k_2 = n_2} \vee \{\alpha_{W_1}(k_1n_1), \alpha_{U_2}(k_2)\} \\
&\quad + \sum_{k_1n_1 \notin E_1 \text{ and } k_2n_2 \in E_2} \vee \{\alpha_{U_1}(k_1), \alpha_{U_1}(n_1), \alpha_{W_2}(k_2n_2)\} \\
&\quad + \sum_{k_1n_1 \in E_1 \text{ and } k_2n_2 \notin E_2} \vee \{\alpha_{W_1}(k_1n_1), \alpha_{U_2}(k_2), \alpha_{U_2}(n_2)\} + \wedge \{\alpha_{U_1}(k_1), \alpha_{U_2}(k_2)\} \\
&= \sum_{k_2n_2 \in E_2} \alpha_{W_2}(k_2n_2) + \sum_{k_1n_1 \in E_1} \alpha_{W_1}(k_1n_1) + \sum_{k_1n_1 \notin E_1 \text{ and } k_2n_2 \in E_2} \alpha_{W_2}(k_2n_2)\} \\
&\quad + \sum_{k_1n_1 \in E_1 \text{ and } k_2n_2 \notin E_2} \alpha_{W_1}(k_1n_1) + \vee \{\alpha_{U_1}(k_1), \alpha_{U_2}(k_2)\} \\
&= \sum_{k_2n_2 \in E_2} \alpha_{W_2}(k_2n_2) + \sum_{k_1n_1 \in E_1} \alpha_{W_1}(k_1n_1) + \sum_{k_1n_1 \notin E_1 \text{ and } k_2n_2 \in E_2} \alpha_{W_2}(k_2n_2)\} \\
&\quad + \sum_{k_1n_1 \in E_1 \text{ and } k_2n_2 \notin E_2} \alpha_{W_1}(k_1n_1) + \alpha_{U_1}(k_1) + \alpha_{U_2}(k_2) - \vee \{\alpha_{U_1}(k_1), \alpha_{U_2}(k_2)\} \\
&= q(td_\alpha)_{\mathbb{G}_1}(k_1) + s(td_\alpha)_{\mathbb{G}_2}(k_2) - (q-1)\alpha_{\mathbb{G}_1}(k_1) - \wedge \{\alpha_{\mathbb{G}_1}(k_1), \alpha_{\mathbb{G}_1}(k_1)\},
\end{aligned}
$$

$$
\begin{aligned}
(td_\beta)_{\mathbb{G}_1 \oplus \mathbb{G}_2}(k_1, k_2) &= \sum_{(k_1,k_2)(n_1,n_2) \in E_1 \times E_2} (\beta_{W_1} \oplus \beta_{W_2})((k_1, k_2)(n_1, n_2)) + (\beta_{U_1} \oplus \beta_{U_2})(k_1, k_2) \\
&= \sum_{k_1 = n_1, k_2n_2 \in E_2} \vee \{\beta_{U_1}(k_1), \beta_{W_2}(k_2n_2)\} + \sum_{k_1n_1 \in E_1, k_2 = n_2} \vee \{\beta_{W_1}(k_1n_1), \beta_{U_2}(k_2)\} \\
&\quad + \sum_{k_1n_1 \notin E_1 \text{ and } k_2n_2 \in E_2} \vee \{\beta_{U_1}(k_1), \beta_{U_1}(n_1), \beta_{W_2}(k_2n_2)\} \\
&\quad + \sum_{k_1n_1 \in E_1 \text{ and } k_2n_2 \notin E_2} \vee \{\beta_{W_1}(k_1n_1), \beta_{U_2}(k_2), \beta_{U_2}(n_2)\} + \wedge \{\beta_{U_1}(k_1), \beta_{U_2}(k_2)\} \\
&= \sum_{k_2n_2 \in E_2} \beta_{W_2}(k_2n_2) + \sum_{k_1n_1 \in E_1} \beta_{W_1}(k_1n_1) + \sum_{k_1n_1 \notin E_1 \text{ and } k_2n_2 \in E_2} \beta_{W_2}(k_2n_2)\} \\
&\quad + \sum_{k_1n_1 \in E_1 \text{ and } k_2n_2 \notin E_2} \beta_{W_1}(k_1n_1) + \vee \{\beta_{U_1}(k_1), \beta_{U_2}(k_2)\} \\
&= \sum_{k_2n_2 \in E_2} \beta_{W_2}(k_2n_2) + \sum_{k_1n_1 \in E_1} \beta_{W_1}(k_1n_1) + \sum_{k_1n_1 \notin E_1 \text{ and } k_2n_2 \in E_2} \beta_{W_2}(k_2n_2)\} \\
&\quad + \sum_{k_1n_1 \in E_1 \text{ and } k_2n_2 \notin E_2} \beta_{W_1}(k_1n_1) + \beta_{U_1}(k_1) + \beta_{U_2}(k_2) - \vee \{\beta_{U_1}(k_1), \beta_{U_2}(k_2)\} \\
&= q(td_\beta)_{\mathbb{G}_1}(k_1) + s(td_\beta)_{\mathbb{G}_2}(k_2) - (q-1)\beta_{\mathbb{G}_1}(k_1) - \wedge \{\beta_{\mathbb{G}_1}(k_1), \beta_{\mathbb{G}_1}(k_1)\},
\end{aligned}
$$

$$
\begin{aligned}
(td_\gamma)_{\mathbb{G}_1\oplus\mathbb{G}_2}(k_1,k_2) &= \sum_{(k_1,k_2)(n_1,n_2)\in E_1\times E_2.} (\gamma_{W_1}\oplus\ \gamma_{W_2})((k_1,k_2)(n_1,n_2)) + (\gamma_{U_1}\oplus\ \gamma_{U_2})(k_1,k_2)\\
&= \sum_{k_1=n_1,k_2n_2\in E_2} \wedge\{\gamma_{U_1}(k_1),\gamma_{W_2}(k_2n_2)\} + \sum_{k_1n_1\in E_1,k_2=n_2} \wedge\{\gamma_{W_1}(k_1n_1),\gamma_{U_2}(k_2)\}\\
&+ \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2} \wedge\{\gamma_{U_1}(k_1),\gamma_{U_1}(n_1),\gamma_{W_2}(k_2n_2)\}\\
&+ \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2} \wedge\{\gamma_{W_1}(k_1n_1),\gamma_{U_2}(k_2),\gamma_{U_2}(n_2)\} + \vee\{\gamma_{U_1}(k_1),\gamma_{U_2}(k_2)\}\\
&= \sum_{k_2n_2\in E_2}\gamma_{W_2}(k_2n_2) + \sum_{k_1n_1\in E_1}\gamma_{W_1}(k_1n_1) + \sum_{k_1n_1\notin E_1 \text{ and } k_2n_2\in E_2}\gamma_{W_2}(k_2n_2)\}\\
&+ \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2}\gamma_{W_1}(k_1n_1) + \wedge\{\gamma_{U_1}(k_1),\gamma_{U_2}(k_2)\}\\
&= \sum_{k_2n_2\in E_2}\gamma_{W_2}(k_2n_2) + \sum_{k_1n_1\in E_1}\gamma_{W_1}(k_1n_1) + \sum_{k_1n_1\notin E_1 \quad and \quad k_2n_2\in E_2}\gamma_{W_2}(k_2n_2)\}\\
&+ \sum_{k_1n_1\in E_1 \text{ and } k_2n_2\notin E_2}\gamma_{W_1}(k_1n_1) + \gamma_{U_1}(k_1) + \gamma_{U_2}(k_2) - \wedge\{\gamma_{U_1}(k_1),\gamma_{U_2}(k_2)\}\\
&= q(td_\gamma)_{\mathbb{G}_1}(k_1) + s(td_\gamma)_{\mathbb{G}_2}(k_2) - (q-1)\gamma_{\mathbb{G}_1}(k_1) - \vee\{\gamma_{\mathbb{G}_1}(k_1),\gamma_{\mathbb{G}_1}(k_1)\}.
\end{aligned}
$$

So, $s = |V_1| - (d)_{G_1}(k_1)$ and $q = |V_2| - (d)_{G_2}(k_2)$. □

Example 4.13. We calculate the total degree of nodes as follows:

$$(d_\chi)_{\mathbb{G}_1\oplus\mathbb{G}_2}(a,e) = q(d_\chi)_{\mathbb{G}_1}(a) + s(d_\chi)_{\mathbb{G}_2}(e)$$

$$s = |V_1| - (d)_{G_1}(a) = 2 - 1 = 1.$$

Now,

$$q = |V_2| - (d)_{G_2}(e) = 4 - 2 = 2$$

$$
\begin{aligned}
(td_\gamma)_{\mathbb{G}_1\oplus\mathbb{G}_2}(a,e) &= q(td_\gamma)_{\mathbb{G}_1}(a) + s(td_\gamma)_{\mathbb{G}_2}(e) - (s-1)\gamma_{\mathbb{G}_2}(e) - (q-1)\gamma_{\mathbb{G}_1}(a)\\
&\quad - \ \vee\{\gamma_{\mathbb{G}_1}(a),\gamma_{\mathbb{G}_2}(e)\}\\
&= -1.1
\end{aligned}
$$

$$
\begin{aligned}
(td_\chi)_{\mathbb{G}_1\oplus\mathbb{G}_2}(a,e) &= q(td_\chi)_{\mathbb{G}_1}(a) + s(td_\chi)_{\mathbb{G}_2}(e) - (s-1)\chi_{\mathbb{G}_2}(e) - (q-1)\chi_{\mathbb{G}_1}(a)\\
&\quad - \ \vee\{\chi_{\mathbb{G}_1}(a),\chi_{\mathbb{G}_2}(e)\}\\
&= 1.1.
\end{aligned}
$$

5 Conclusion

BPFG represents a broader conceptual framework that encompasses both FG and PFG. Within this context, BPFG exhibits significantly greater levels of flexibility and comparability when compared to intuitionistic fuzzy graph and PFG. In this research article, we delve into the intricate aspects of BPFG, particularly focusing on its symmetric differences and the concept of rejection. In addition, we engage in a comprehensive discussion surrounding the degree and total degree of nodes within the BPFG structure. This exploration aims to shed light on the enhanced capabilities and analytical potential by using BPFG as opposed to its counterparts, FG and PFG. In future work, we will apply different results on BPFG.

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