Research Article

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Asymptotic approximations of Apostol-Frobenius-Euler polynomials of order α in terms of hyperbolic functions

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Abstract: The study of special functions has become an enthralling area in mathematics because of its properties and wide range of applications that are relevant into other fields of knowledge. Developing topics in special functions involves the investigation of Apostol-type polynomials encompassing the combinations, extensions, and generalizations of some classical polynomials such as Bernoulli, Euler, Genocchi, and tangent polynomials. One particular type of these polynomials is the Apostol-Frobenius-Euler polynomials of order α denoted by $H_n^{\alpha}(z;\ u;\ \lambda)$. Using the saddle point method, Corcino et al. obtained approximations for the higher-order tangent polynomials. They also established a new method to derive its approximations with enlarged region of validity. In this article, it is found that these methods are applicable to the higher-order Apostol-Frobenius-Euler polynomials. Consequently, approximations of higher-order Apostol-Frobenius-Euler polynomials in terms of the hyperbolic functions are obtained for large values of the parameter n, and its uniform approximations with enlarged region of validity are also derived. Moreover, approximations of the generalized Apostol-type Frobenius-Euler polynomials of order α with parameters a, b, and c are obtained by applying the same methods. Graphs are provided to show the accuracy of the exact values of these polynomials and their corresponding approximations for some specific values of the parameters.

Keywords: Apostol-Frobenius-Euler polynomials, generalized Apostol-type Froebnius-Euler polynomials, Euler Polynomials, Hermite polynomials, asymptotic approximation

MSC 2020: 11B68, 11B83, 41A60

1 Introduction

Recent developments in the study of special functions have been constructed and investigated by mixing the concepts of well-known special polynomials such as Bernoulli, Euler, Genocchi, tangent polynomials, Gegenbauer polynomials, Hermite polynomials, and Laguerre polynomials [1–3]. An interesting combination of special polynomials that can be established and explored is through merging the concept of Euler polynomials with the Apostol and Frobenius polynomials.

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The Apostol-Frobenius-Euler polynomials of order α denoted by $H_n^\alpha(z;\, u;\, \lambda)$ are defined by the generating function [4,5]

$$\left(\frac{1-u}{\lambda e^w - u}\right)^\alpha e^{zw} = \sum_{n=0}^{\infty} H_n^\alpha(z;\, u;\, \lambda)\frac{w^n}{n!}, \quad |w| < \left|\log\left(\frac{\lambda}{u}\right)\right|, \tag{1}$$

where $\lambda, u \in \mathbb{C}$ with $\lambda \neq 0$, $u \neq 1$, and $\alpha \in \mathbb{Z}$.

When $\lambda = 1$, this gives the generating function of the Frobenius-Euler polynomials of order α given as [6,7]

$$\left(\frac{1-u}{e^w - u}\right)^\alpha e^{zw} = \sum_{n=0}^{\infty} H_n^\alpha(z;\, u)\frac{w^n}{n!}, \quad |w| < |\log(u)|. \tag{2}$$

When $u = -1$, this gives the generating function of the Apostol-Euler polynomials of order α defined as [8]

$$\left(\frac{2}{\lambda e^w + 1}\right)^\alpha e^{zw} = \sum_{n=0}^{\infty} \mathcal{E}_n^\alpha(z;\, \lambda)\frac{w^n}{n!}, \quad |w| < |\pi i - \log(\lambda)|. \tag{3}$$

Observe that setting $z = 0$ in (1) and (2),

$$H_n^\alpha(0;\, u;\, \lambda) = H_n^\alpha(u;\, \lambda) \quad \text{and} \quad H_n^\alpha(0;\, u) = H_n^\alpha(u), \tag{4}$$

where $H_n^\alpha(u;\, \lambda)$ and $H_n^\alpha(u)$ are called the Apostol-Frobenius-Euler numbers of order α and the Frobenius-Euler numbers of order α, respectively [9,10].

In the case when $\alpha = 1$ in (1) and (2),

$$H_n^1(z;\, u;\, \lambda) = H_n(z;\, u;\, \lambda) \quad \text{and} \quad H_n^1(z;\, u) = H_n(z;\, u), \tag{5}$$

where $H_n(z;\, u;\, \lambda)$ and $H_n(z;\, u)$ are called the Apostol-Frobenius-Euler polynomials and the Frobenius-Euler polynomials, respectively (see [11,12]).

The first few values of the Apostol-Frobenius-Euler polynomials of order α are determined explicitly as follows:

$$\begin{aligned}
H_0^\alpha(z;\, u;\, \lambda) &= \left(\frac{1-u}{\lambda-u}\right)^\alpha, H_1^\alpha(z;\, u;\, \lambda) = -\left(\frac{1-u}{\lambda-u}\right)^\alpha \frac{uz + (\alpha - z)\lambda}{(\lambda - u)},\\
H_2^\alpha(z;\, u;\, \lambda) &= \left(\frac{1-u}{\lambda-u}\right)^\alpha \frac{u^2z^2 + u(\alpha + 2\alpha z - 2z^2)\lambda + (\alpha - z)^2\lambda^2}{(\lambda-u)^2},\\
H_3^\alpha(z;\, u;\, \lambda) &= -\left(\frac{1-u}{\lambda-u}\right)^\alpha \frac{\lambda[u^3z^3 + u^2(\alpha - 3z^3 + 3\alpha z(1+z))]}{(\lambda-u)^3}\\
&\quad - \left(\frac{1-u}{\lambda-u}\right)^\alpha \frac{u[\alpha + 3z^3 + 3\alpha^2(1+z) - 3\alpha z(1+2z)]}{(\lambda-u)^3}, \quad \text{and}\\
H_4^\alpha(z;\, u;\, \lambda) &= \left(\frac{1-u}{\lambda-u}\right)^\alpha \frac{\lambda[u^4z^4 + u^3(\alpha - 4z^4 + 2\alpha z(2 + z(3+2z)))]}{(\lambda-u)^4}\\
&\quad \left(\frac{1-u}{\lambda-u}\right)^\alpha \frac{u^2[6z^4 - 4\alpha(-1 + 3z^2(1+z))]}{(\lambda-u)^4}.
\end{aligned}$$

The Apostol-Frobenius-Euler polynomials of order $\alpha H_n^\alpha(z;\, u;\, \lambda)$ are λ extensions of the Frobenius-Euler polynomials. Frobenius-Euler polynomials and numbers are named after the great German mathematician Ferdinand Georg Frobenius [13], who made essential works on the context of these polynomials in number theory and the relation of their divisibility properties with the Stirling numbers of the second kind [14,15].

Analogues and other extensions of the Apostol-type Frobenius-Euler polynomials play a significant role in the development of some concepts and applications in calculus, differential equations, number theory, and physics [16–22]. In addition, the Fourier expansions and integral representations of these polynomials are given in [11,23,24].

In the study of Corcino et al. [25–27], uniform approximations for the higher-order tangent polynomials were derived using the saddle point method. In addition, they also presented a new method to obtain its asymptotic expansions with enlarge region of validity. However, the asymptotic approximations of higher-order Apostol-Frobenius-Euler polynomials which are parallel to the results in [27] are not mentioned and provided in other related studies.

In this study, the asymptotic approximations of Apostol-Frobenius-Euler polynomials of order α are derived using the methods employed in [27–29]. Moreover, asymptotic expansion of the generalized Apostol-type Frobenius-Euler polynomials of order α with parameters a, b, and c, which is denoted by $\mathcal{H}_n^{\alpha}(z;\, u;\, a, b, c, \lambda)$ is also obtained. Asymptotic formulas for the special cases and different values of the parameters are also given as corollaries.

2 Uniform approximations

In this section, the main results of the study explore the uniform approximation of the Apostol-Frobenius-Euler polynomials of order α. Using the saddle point method, the following theorem contains the derived approximation.

Theorem 2.1. *For* $n, \alpha \in \mathbb{Z}^+, u, \lambda \in \mathbb{C}\backslash\{0, 1\}$, *and* $z \in \mathbb{C}\backslash\{0\}$ *such that* $|\mathrm{Im} z^{-1}| < 2\pi - \mathrm{Arg}\left(\frac{\lambda}{u}\right)$ or $|z^{-1}| < |z^{-1} - (2\pi i - \delta)|$, *the Apostol-Frobenius-Euler polynomials of order* α *satisfy*

$$H_n^{\alpha}\left(nz + \frac{\alpha}{2};\, u;\, \lambda\right) = (nz)^n \left(\frac{1-u}{2\sqrt{\lambda u}}\right)^{\alpha} \mathrm{csch}^{\alpha}\left(\frac{\delta z + 1}{2z}\right)\left[1 - \frac{\alpha\left(\alpha + (\alpha+1)\mathrm{csch}^2\left(\frac{\delta z + 1}{2z}\right)\right)}{8nz^2} + O\left(\frac{1}{n^2}\right)\right], \tag{6}$$

where $\delta = \log\left(\frac{\lambda}{u}\right)$ *and the logarithm is taken to be the principal branch.*

Proof. Applying the Cauchy integral formula to (1)

$$H_n^{\alpha}(z;\, u;\, \lambda) = \frac{n!}{2\pi i}\int_C \frac{(1-u)^{\alpha} e^{zw}}{(\lambda e^w - u)^{\alpha}} \frac{\mathrm{d}w}{w^{n+1}}, \tag{7}$$

where C is a circle about 0 with radius lesser than $\left|2\pi i - \ln\left(\frac{\lambda}{u}\right)\right|$. Writing $\delta = \log\left(\frac{\lambda}{u}\right)$, (7) can be expressed as

$$H_n^{\alpha}(z;\, u;\, \lambda) = \frac{n!}{2\pi i}\int_C \left(\frac{1-u}{u}\right)^{\alpha} \frac{e^{zw}}{(e^{\delta+w} - 1)^{\alpha}} \frac{\mathrm{d}w}{w^{n+1}}.$$

With $\left(2e^{\frac{(\delta+w)}{2}}\sinh\left(\frac{\delta+w}{2}\right)\right)^{\alpha} = (e^{\delta+w} - 1)^{\alpha}$, it follows that

$$H_n^{\alpha}(z;\, u;\, \lambda) = \frac{n!}{2\pi i}\left(\frac{1-u}{2\sqrt{\lambda u}}\right)^{\alpha} \int_C f(w)\frac{e^{zw}}{e^{\frac{\alpha w}{2}}}\frac{\mathrm{d}w}{w^{n+1}},$$

where $f(w) = 1/\sinh^{\alpha}\left(\frac{\delta+w}{2}\right) = \mathrm{csch}^{\alpha}\left(\frac{\delta+w}{2}\right)$. It can be observed that the function $f(w)$ is a meromorphic function with simple poles of order α at the zeros of $\sinh\left(\frac{\delta+w}{2}\right)$, which are given by $w_j = 2j\pi i - \delta, j = \pm1, \pm2, \ldots$.

It follows that by taking $z \to nz$ and letting $nz \to \infty$ with fixed z,

$$H_n^{\alpha}\left(nz + \frac{\alpha}{2};\, u;\, \lambda\right) = \frac{n!}{2\pi i}\left(\frac{1-u}{2\sqrt{\lambda u}}\right)^{\alpha} \int_C f(w) e^{n(zw - \log w)}\frac{\mathrm{d}w}{w}. \tag{8}$$

The main contribution of the integrand to the integral in (8) originates at the saddle point of the argument of the exponential [30]. Thus, if z^{-1} is not a pole, then the approximations of $H_n^{\alpha}\left(nz + \frac{\alpha}{2};\, u;\, \lambda\right)$ can be derived by expanding $f(w)$ around the saddle point $w = z^{-1}$. It follows from Lemmas 1 and 2 and Theorem 1 of [31] that

$$H_n^{\alpha}\left(nz + \frac{\alpha}{2};\, u;\, \lambda\right) = (nz)^n \left(\frac{1-u}{2\sqrt{\lambda u}}\right)^{\alpha}\sum_{k=0}^{\infty}\frac{f^{(k)}(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k}, \tag{9}$$

where $p_k(n)$ are the polynomials

$$p_0(n) = 1, \quad p_1(n) = 0, \quad p_2(n) = -n, \quad p_3(n) = 2n, \tag{10}$$

$$p_k(n) = (1-k)p_{k-1}(n) + np_{k-2}(n), \quad k \geq 3. \tag{11}$$

Expanding the sum in (9) and keeping only the first three terms yield (6) (Figure 1).

The following corollaries give the uniform approximations of the Frobenius-Euler polynomials of order α, Apostol-Euler polynomials of order α, and Euler polynomials of order α.

Corollary 2.2. *For $n, \alpha \in \mathbb{Z}^+$, $u \in \mathbb{C}\backslash\{0,1\}$, and $z \in \mathbb{C}\backslash\{0\}$ such that $|\mathrm{Im}\, z^{-1}| < 2\pi - \mathrm{Arg}\left(\frac{1}{u}\right)$ or $|z^{-1}| < |z^{-1} - (2\pi i + v)|$, the Frobenius-Euler polynomials of order α satisfy*

$$H_n^\alpha\left(nz + \frac{\alpha}{2}; u\right) = (nz)^n \frac{(1-u)^\alpha}{(2\sqrt{u})^\alpha} \mathrm{csch}^\alpha\left(\frac{1-zv}{2z}\right)\left[1 - \frac{\alpha\left(\alpha + (\alpha+1)\mathrm{csch}^2\left(\frac{1-zv}{2z}\right)\right)}{8nz^2} + O\left(\frac{1}{n^2}\right)\right], \tag{12}$$

where $v = \log(u)$ and the logarithm is taken to be the principal branch.

Proof. This follows from Theorem 2.1 by taking $\lambda = 1$. □

Corollary 2.3. *For $n, \alpha \in \mathbb{Z}^+$, $\lambda \in \mathbb{C}\backslash\{0,1\}$, and $z \in \mathbb{C}\backslash\{0\}$ such that $|\mathrm{Im}\, z^{-1}| < \pi - \mathrm{Arg}(\lambda)$ or $|z^{-1}| < |z^{-1} - (\pi i - \tau)|$, the Apostol-Euler polynomials of order α satisfy*

$$\mathcal{E}_n^\alpha\left(nz + \frac{\alpha}{2}; \lambda\right) = \frac{(nz)^n}{(\sqrt{\lambda})^\alpha} \mathrm{sech}^\alpha\left(\frac{1+z\tau}{2z}\right)\left[1 - \frac{\alpha\left(\alpha - (\alpha+1)\,\mathrm{sech}^2\left(\frac{1-z\tau}{2z}\right)\right)}{8nz^2} + O\left(\frac{1}{n^2}\right)\right], \tag{13}$$

where $\tau = \log(\lambda)$ and the logarithm is taken to be the principal branch.

Proof. This follows from Theorem 2.1 by taking $u = -1$. □

Corollary 2.4. *For $n, \alpha \in \mathbb{Z}^+$ and $z \in C\backslash 0$ such that $|\mathrm{Im}\, z^{-1}| < \pi$ or $|z^{-1}| < |z^{-1} - \pi|$*

$$E_n^\alpha\left(nz + \frac{\alpha}{2}\right) = (nz)^n \mathrm{sech}^\alpha\left(\frac{1}{2z}\right)\left[1 - \frac{\alpha\left(\alpha - (\alpha+1)\,\mathrm{sech}^2\left(\frac{1}{2z}\right)\right)}{8nz^2} + O\left(\frac{1}{n^2}\right)\right]. \tag{14}$$

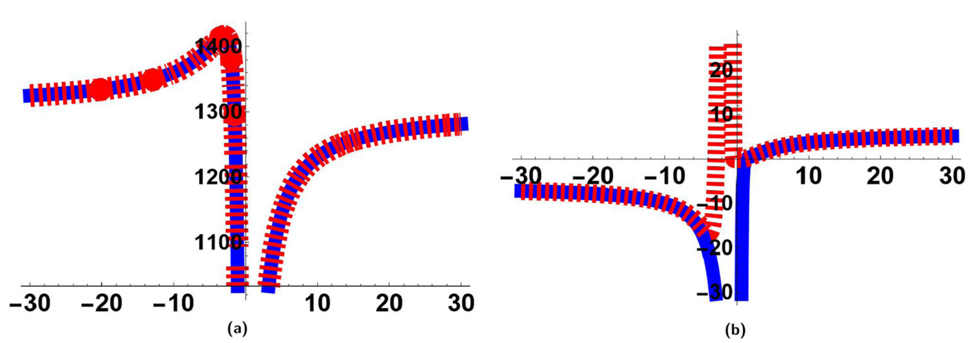

Figure 1: Solid lines represent the Apostol-Frobenius-Euler polynomials of order $\alpha H_n^\alpha(nz + \frac{\alpha}{2}; u; \lambda)$ for several values of n, whereas dashed lines represent the right-hand side of (6) with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^1$ where we choose $\sigma = 0.5$. (a) $n = 8, \alpha = 7, u = 3, \lambda = 4, b = 0.5$ and (b) $n = 5, \alpha = 4, u = 3, \lambda = 7, b = 0.5$.

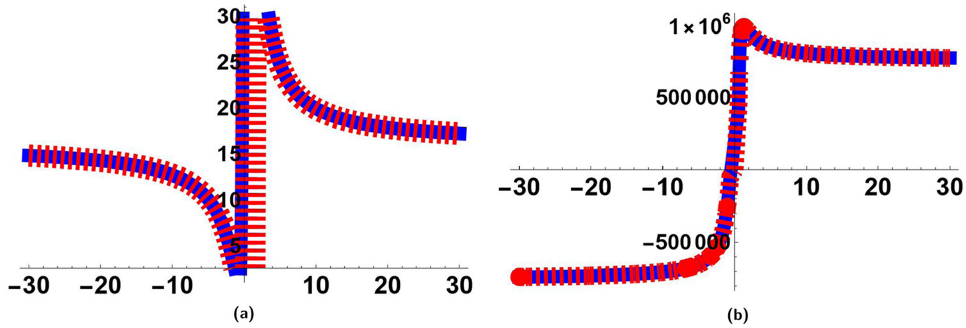

Figure 2: Solid lines represent the Frobenius-Euler polynomials of order $\alpha H_n^{\alpha}(nz + \frac{\alpha}{2}; u)$ for several values of n, whereas dashed lines represent the right-hand side of (12) with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^1$ where we choose $\sigma = 0.5$. (a) $n = 4$, $\alpha = 3$, $u = 5$ and (b) $n = 9$, $\alpha = 4$, $u = 2$.

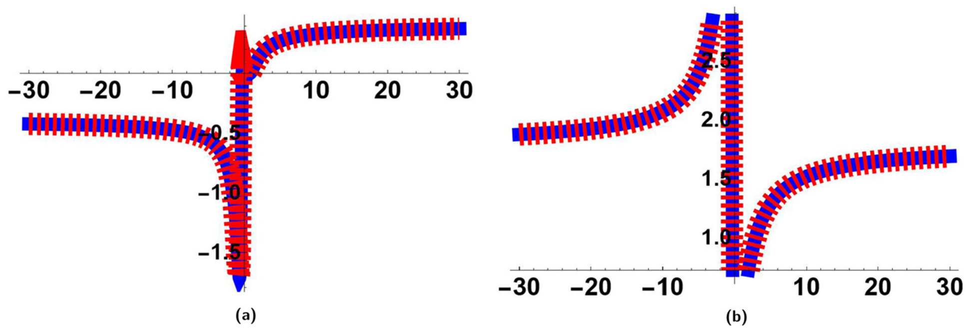

Figure 3: Solid lines represent the Apostol-Euler polynomials of order $\alpha \mathcal{E}_n^{\alpha}(nz + \frac{\alpha}{2}, \lambda)$ for several values of n, whereas dashed lines represent the right-hand side of (13) with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^1$ where we choose $\sigma = 0.5$. (a) $n = 5$, $\alpha = 6$, $\lambda = 4$ and (b) $n = 6$, $\alpha = 4$, $\lambda = 8$.

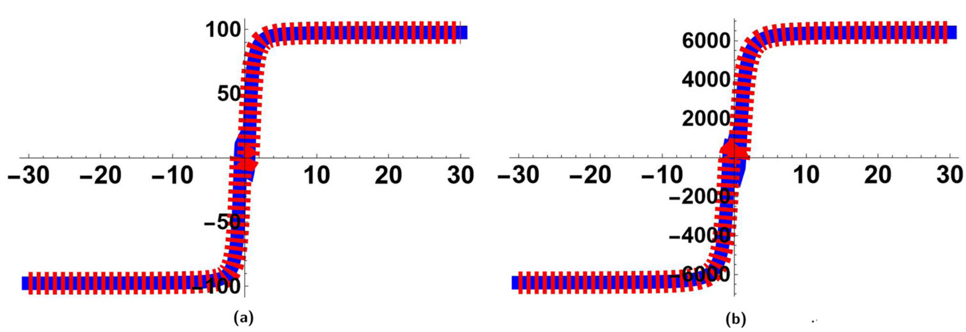

Figure 4: Solid lines represent the Euler polynomials of order $\alpha E_n^{\alpha}(nz + \frac{\alpha}{2})$ for several values of n, whereas dashed lines represent the right-hand side of (14) with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^1$ where we choose $\sigma = 0.5$. (a) $n = 5$, $\alpha = 4$, $b = 0.5$ and (b) $n = 7$, $\alpha = 8$, $b = 0.5$.

Proof. This follows from 2.1 by taking $\lambda = 1$ and $u = -1$. □

Remark 2.5. Taking $\alpha = 1$ in (14), approximation for the classical Euler polynomials $E_n(nz + \frac{1}{2})$ is obtained similar to that of Lopez and Temme (Corollary 2, [31]).

Figures 2–4 show the accuracy of the asymptotic formulae obtained in Corollaries (12), (13), and (14), respectively.

3 Enlarged region of validity

The previous section contains an approximation valid in the region $|z^{-1}| < |z^{-} - w_j|$ with poles $w_j = \pm 1, \pm 2, \ldots$. This region may be enlarged by isolating the contribution of the poles of $f(w)$. In this section, an asymptotic expansion with an enlarged region of validity is obtained in the following theorem.

Theorem 3.1. *For $\lambda, u \in \mathbb{C}\backslash\{0, 1\}$, $\alpha \in \mathbb{Z}^{+}$, and $z \in \mathbb{C}$ such that $|z^{-1}| < |z^{-1} - w_k|$ for all $k = l+1, l+2, \ldots$, the Apostol-Frobenius-Euler polynomials of order α satisfy*

$$H_n^{\alpha}\left(nz + \frac{\alpha}{2}; u; \lambda\right) = \frac{(1-u)^{\alpha}}{(2\sqrt{\lambda u})^{\alpha}}\left\{\sum_{k=1}^{l}\sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j}\left[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle j-1\rangle_s (w_k)^{-(j-1+s)}\left(\frac{(n-s)!}{w_k^{n-s+1}}\right.\right.\right.$$
$$\left.\left.\left. - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}}\right) + \frac{(-1)^j\langle j\rangle_n}{w_k^{j+n}}\right] + (nz)^n\sum_{k=0}^{\infty}\frac{f^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k}\right\}, \tag{15}$$

where the polynomials $p_k(n)$ are given in (11) *and h_l^k is the kth derivative of the function $h_l(w)$ given by* (28) *and*

$$\sum_{j=1}^{m}\frac{r_{k_j}}{(w-w_k)^j}$$

are the given principal parts of the Laurent series corresponding to the poles $w_k = 2k\pi - \delta$, where $\delta = \log\left(\frac{\lambda}{u}\right)$ and the entire function $h_l(w)$ is determined by $f(w) = \operatorname{csch}^{\alpha}\left(\frac{\delta + w}{2}\right)$.

Proof. Using Mittag-Leffler's theorem [32,33], write $f(w) = \cosh^{\alpha}\left(\frac{\delta+w}{2}\right)$ as

$$f(w) = \sum_{k=1}^{l}\left[\sum_{j=1}^{m}\frac{r_{k_j}}{(w-w_k)^j} + q_k(w)\right] + g(w) = \sum_{k=1}^{l}\sum_{j=1}^{\alpha}\frac{r_{k_j}}{(w-w_k)^j} + f_l(w), \tag{16}$$

where

$$f_l(w) = \sum_{k=1}^{l} q_k(w) + g(w), \tag{17}$$

$q_k(w)$ is a polynomial of w, r_{k_j} are residues at w_k, $k = 1, 2, \ldots, l$. With this, $f_l(w)$ has no poles inside the disk $|w| < |w_{m+1}|$. Recall from (8),

$$H_n^{\alpha}\left(nz + \frac{\alpha}{2}; u; \lambda\right) = \frac{n!}{2\pi i}\frac{(1-u)^{\alpha}}{(2\sqrt{\lambda u})^{\alpha}}\int_C f(w)e^{wnz}\frac{\mathrm{d}w}{w^{n+1}}, \tag{18}$$

where $f(w) = 1/\sinh^{\alpha}\left(\frac{\delta+w}{2}\right) = \operatorname{csch}^{\alpha}\left(\frac{\delta+w}{2}\right)$. Substituting (16) into (18) gives

$$H_n^{\alpha}\left(nz + \frac{\alpha}{2}; u; \lambda\right) = \frac{n!}{2\pi i}\frac{(1-u)^{\alpha}}{(2\sqrt{\lambda u})^{\alpha}}\int_C\left(\sum_{k=1}^{l}\sum_{j=1}^{\alpha}\frac{r_{k_j}}{(w-w_k)^j} + f_l(w)\right)e^{wnz}\frac{\mathrm{d}w}{w^{n+1}} = X_l^{n,\alpha}(z) + Y_l^{n,\alpha}(z), \tag{19}$$

where

$$X_l^{n,\alpha}(z) = \frac{n!}{2\pi i}\frac{(1-u)^{\alpha}}{(2\sqrt{\lambda u})^{\alpha}}\int_C f_l(w)e^{wnz}\frac{\mathrm{d}w}{w^{n+1}}, \tag{20}$$

$$Y_l^{n,\alpha}(z) = \frac{(1-u)^{\alpha}}{(2\sqrt{\lambda u})^{\alpha}}\int_C\frac{n!}{2\pi i}\sum_{k=1}^{l}\sum_{j=1}^{\alpha}\frac{r_{k_j}}{(w-w_k)^j}e^{wnz}\frac{\mathrm{d}w}{w^{n+1}}. \tag{21}$$

To evaluate (20), repeat the process used in the last section for $f_l(w)$ instead of $f(w)$, thus we have

$$X_l^{n,\alpha}(z) = \frac{n!}{2\pi i}\frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\int_C f_l(w)e^{n(wz-\log w)}\frac{\mathrm{d}w}{w}. \tag{22}$$

The approximations of $X_l^{n,\alpha}(z)$ can be obtained by expanding $f_l(w)$ around the saddle point z^{-1}. Assume z^{-1} is not a pole of $f_l(w)$, we can expand it as

$$f_l(w) = \sum_{k=0}^{\infty}\frac{f_l^k(z^{-1})}{k!}(w-z^{-1})^k, \; |w-z^{-1}| < r, \tag{23}$$

where r is the distance from z^{-1} to the nearest singularity of $f_l(w)$. Introducing (23) into (20)

$$X_l^{n,\alpha}(z) = (nz)^n\frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\sum_{k=0}^{\infty}\frac{sf_l^k(z^{-1})}{k!}u_k(n,z), \tag{24}$$

where

$$u_k(n,z) = \frac{1}{(nz)^n}\frac{n!}{2\pi i}\int_C (w-z^{-1})^k e^{wnz}\frac{\mathrm{d}w}{w^{n+1}}. \tag{25}$$

Note that the functions (25) can be represented as (Lemma 1, [31])

$$u_k(n,z) = \frac{p_k(n)}{(nz)^k}, \tag{26}$$

where $p_k(n)$ are the polynomials in (11). Thus, (20) may be expanded as the infinite sum

$$X_l^{n,\alpha}(z) = (nz)^n\frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\sum_{k=0}^{\infty}\frac{f_l^k(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k}, \tag{27}$$

valid for $\alpha \in \mathbb{Z}^+$, $z \in \mathbb{C}\backslash\{0\}$ such that $|z^{-1}| < |z^{-1} - w_j|$ for $j = l+1, l+2, \ldots$ given the first $2l$ poles of $f(w)$. It follows from (16) that the kth derivative of $f_l(w)$ is

$$f_l^{(k)}(w) = f^{(k)}(w) - h_l^{(k)}(w),$$

where

$$h_l(w) = -\sum_{k=1}^{l}\sum_{j=1}^{\alpha}\frac{r_{k_j}}{(w-w_k)^j}. \tag{28}$$

Thus, the expansion of $X_l^{n,\alpha}(z)$ in (20) is

$$X_l^{n,\alpha}(z) = (nz)^n\frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\sum_{k=0}^{\infty}\frac{f_l^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k} \tag{29}$$

valid for $|z^{-1}| < |z^{-1} - w_j|, j = l+1, l+2, \ldots$ and $z \neq 0$. The expansion's range of validity is larger than that of the expansion in Theorem (2.1).

Conversely, to obtain an expansion for $Y_l^{n,\alpha}(z)$, we employ similar computations from Corcino et al. [27]. Shifting the integration contour by $w = w_k + t$ in each integral in (21), it follows that $\mathrm{d}w = \mathrm{d}t$ and

$$Y_l^{n,\alpha}(z) = \frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\sum_{k=1}^{l}\sum_{j=1}^{\alpha}e^{w_k nz}r_{k_j}\frac{n!}{2\pi i}\int_{C'}\frac{e^{tnz}}{t^j}\frac{\mathrm{d}t}{(w_k+t)^{n+1}}, \tag{30}$$

where $C' : t = -w_k + Re^{i\theta}, -\pi < \theta \leq \pi$ is a circle with radius R and center at $-w_k$. Note that 0 is not on the w's. This C' is the image of $C : w = Re^{i\theta}$ through the shift $w = w_k + t$. To continue, we use the idea that

$$\frac{e^{tnz}}{t^j} = \int_0^{nz}\frac{e^{tx}}{t^{j-1}} + \frac{1}{t^j}. \tag{31}$$

Substituting (31) into (30), we have

$$Y_l^{n,\alpha}(z) = \frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\sum_{k=1}^{l}\sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j}\frac{n!}{2\pi i}\int_{C'}\left(\int_0^{nz}\frac{e^{tx}}{t^{j-1}} + \frac{1}{t^j}\right)\frac{\mathrm{d}w}{(w_k+t)^{n+1}}. \tag{32}$$

The proceeding parts are expositions of the computations from [27]. Now, we determine

$$\frac{n!}{2\pi i}\int_{C'}\frac{e^{tx}}{t^{j-1}}\frac{\mathrm{d}w}{(w_k+t)^{n+1}} = \frac{\mathrm{d}^n}{\mathrm{d}t^n}e^{tx}t^{-(j-1)}\Bigg|_{t=-w_k}. \tag{33}$$

Using the Leibniz rule for differentiation, (33) can be written as

$$\frac{n!}{2\pi i}\int_{C'}\frac{e^{tx}}{t^{j-1}}\frac{\mathrm{d}w}{(w_k+t)^{n+1}} = \sum_{s=0}^{n}\binom{n}{s}x^{n-s}e^{-w_k x}(-1)^{(j-1)}\langle j-1\rangle_s(w_k)^{-(j-1+s)}, \tag{34}$$

where $\langle j-1\rangle_s$ denote the rising factorial of $j-1$ with increment s. It can also be computed that

$$\frac{n!}{2\pi i}\int_{C'}t^{-j}\frac{\mathrm{d}w}{(w_k+t)^{n+1}} = \frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^{-j})\Bigg|_{t=w_k} = \frac{(-1)^j\langle j\rangle_n}{w_k^{j+n}}. \tag{35}$$

Consider the incomplete gamma function

$$\Gamma(n-s+1, w_k z) = \int_{w_k z}^{\infty} e^{-t}t^{n-s}\mathrm{d}t. \tag{36}$$

Let $\eta = \frac{t}{w_k}$. Then $t = \eta w_k$ and $w_k\mathrm{d}\eta = \mathrm{d}t$. Moreover, $t = \infty \Leftrightarrow \eta = \infty$; $t = w_k z \Leftrightarrow \eta = z$. Thus, (36) becomes

$$\Gamma(n-s+1, w_k z) = \int_z^{\infty} e^{-w_k\eta}(w_k\eta)^{n-s}w_k\mathrm{d}\eta. \tag{37}$$

From this, it can be shown that

$$\int_0^z e^{-w_k\eta}\eta^{n-s}\mathrm{d}\eta = \int_0^{\infty} e^{-w_k\eta}\eta^{n-s}\mathrm{d}\eta - \frac{\Gamma(n-s+1, w_k z)}{w_k^{n-s+1}}. \tag{38}$$

Note that $z \mapsto nz$. Then

$$\int_0^{nz} e^{-w_k\eta}\eta^{n-s}\mathrm{d}\eta = \int_0^{\infty} e^{-w_k\eta}\eta^{n-s}\mathrm{d}\eta - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}}. \tag{39}$$

Substituting (34) and (35) into (32) gives

$$\begin{aligned} Y_l^{n,\alpha}(z) &= \frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\sum_{k=1}^{l}\sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j}\left[\left(\int_0^{nz}\sum_{s=0}^{n}\binom{n}{s}x^{n-s}e^{-w_k x}(-1)^{(j-1)}\langle j-1\rangle_s(w_k)^{-(j-1+s)}\right)\mathrm{d}x + \frac{(-1)^j\langle j\rangle_n}{w_k^{j+n}}\right] \\ &= \frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\sum_{k=1}^{l}\sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j}\left[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle j-1\rangle_s(w_k)^{-(j-1+s)}\left(\int_0^{nz}x^{n-s}e^{-w_k x}\mathrm{d}x\right) + \frac{(-1)^j\langle j\rangle_n}{w_k^{j+n}}\right]. \end{aligned} \tag{40}$$

Using (39) in (40), we obtain

$$\begin{aligned} Y_l^{n,\alpha}(z) = \frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\sum_{k=1}^{l}\sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j}&\left[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle j-1\rangle_s(w_k)^{-(j-1+s)}\right. \\ &\left.\times\left(\int_0^{\infty}e^{-w_k t}t^{n-s}\mathrm{d}t - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}}\right) + \frac{(-1)^j\langle j\rangle_n}{w_k^{j+n}}\right]. \end{aligned} \tag{41}$$

Note that for $n \geq s$,

$$\int_0^\infty t^{n-s} e^{-w_k t} \mathrm{d}t = \frac{(n-s)!}{w_k^{n-s+1}}. \tag{42}$$

Hence, (41) can be written as

$$\begin{aligned} Y_l^{n,\alpha}(z) = \frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha} \sum_{k=1}^{l} \sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j} &\left[\sum_{s=0}^{n} \binom{n}{s} (-1)^{(j-1)} \langle j-1 \rangle_s (w_k)^{-(j-1+s)} \left(\frac{(n-s)!}{w_k^{n-s+1}} - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}} \right) \right. \\ &\left. + \frac{(-1)^j \langle j \rangle_n}{w_k^{j+n}} \right]. \end{aligned} \tag{43}$$

Substituting (32) and (43) into (21), we have

$$\begin{aligned} H_n^\alpha\left(nz + \frac{\alpha}{2}; u; \lambda\right) = \frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha} &\left\{ \sum_{k=1}^{l} \sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j} \left[\sum_{s=0}^{n} \binom{n}{s} (-1)^{(j-1)} \langle j-1 \rangle_s (w_k)^{-(j-1+s)} \left(\frac{(n-s)!}{w_k^{n-s+1}} \right. \right. \right. \\ &\left. \left. \left. - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}} \right) + \frac{(-1)^j \langle j \rangle_n}{w_k^{j+n}} \right] + (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k} \right\} \end{aligned}$$

valid for $\alpha \in \mathbb{Z}^+$ such that for all $k = l+1, l+2, \ldots$, where the polynomials $p_k(n)$ are given in (11) and $h_l^{(k)}$ is the kth derivative of $h_l(w)$ given by (28). □

The accuracy of the asymptotic formula obtained in (6) and (15) is shown in Figure 5.

The following corollary gives the approximation with enlarged region of validity for the Frobenius-Euler polynomials.

Corollary 3.2. *For $u \in \mathbb{C} \backslash \{0, 1\}$, $\alpha, n \in \mathbb{Z}^+$, and $z \in \mathbb{C}$ such that $|z^{-1}| < |z^{-1} - w_k|$ for all $|z^{-1}|$ for all $k = l+1, l+2, \ldots$, the Frobenius-Euler polynomials of order α satisfy*

$$\begin{aligned} H_n^\alpha\left(nz + \frac{\alpha}{2}; u\right) = \frac{(1-u)^\alpha}{(2\sqrt{u})^\alpha} &\left\{ \sum_{k=1}^{l} \sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j} \left[\sum_{s=0}^{n} \binom{n}{s} (-1)^{(j-1)} \langle j-1 \rangle_s (w_k)^{-(j-1+s)} \left(\frac{(n-s)!}{w_k^{n-s+1}} - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}} \right) \right. \right. \\ &\left. \left. + \frac{(-1)^j \langle j \rangle_n}{w_k^{j+n}} \right] + (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k} \right\}, \end{aligned} \tag{44}$$

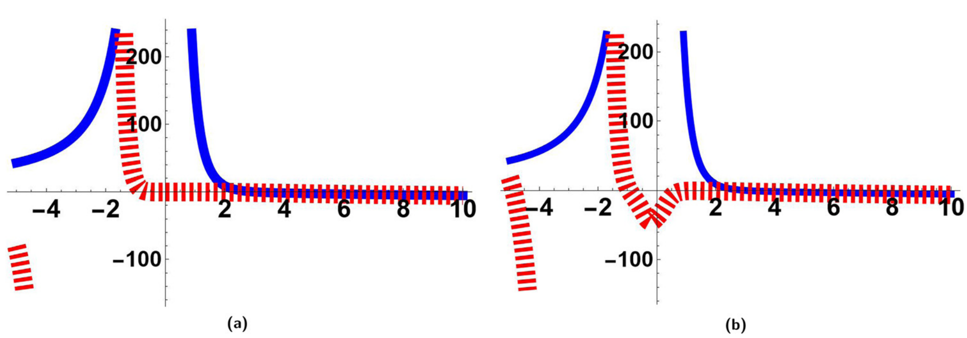

Figure 5: Solid lines in (a) and (b) represent $H_n^\alpha(nz + \frac{\alpha}{2}; u; \lambda)$, whereas dashed lines in (a) and (b) represent the right-hand side of (6) and (15) for $n = 3$, $\alpha = 3$, $u = 4$, and $\lambda = 6$, respectively, with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^1$ where we choose $\sigma = 0.5$.

Table 1: Exact and approximate values of $H_4^3(4z + 3/2;\ 2;\ 5)$

z	Exact values $H_4^3(4z + 3/2;\ 2;\ 5)$	Approximate values using Theorem 2.1	Relative error
3	-2.39241×10^2	-2.42462×10^2	1.34628×10^{-2}
4	-1.00774×10^3	-1.01673×10^3	8.91731×10^{-3}
5	-2.93002×10^3	-2.94714×10^3	5.84244×10^{-3}
6	-6.83097×10^3	-6.8580×10^3	3.9568×10^{-3}
7	-1.3763×10^4	-1.38013×10^4	2.78004×10^{-3}
8	-2.50062×10^4	-2.50567×10^4	2.01911×10^{-3}
9	-4.2068×10^4	-4.21315×10^4	1.50901×10^{-3}
10	-6.66836×10^4	-6.67607×10^4	1.15571×10^{-3}
11	-1.00816×10^5	-1.00907×10^5	9.0386×10^{-4}
12	-1.46654×10^5	-1.4676×10^5	7.19781×10^{-4}

where the polynomials $p_k(n)$ are given in (11) *and h_l^k is the kth derivative of the function $h_l(w)$ given by* (28) *and*

$$\sum_{j=1}^{m} \frac{r_{k_j}}{(w - w_k)^j},$$

are the given principal parts of the Laurent series corresponding to the poles $w_k = 2\pi i + \nu$, where $\nu = \log(u)$ and the entire function $h_l(w)$ is determined by $f(w) = \operatorname{csch}^{\alpha}\left(\frac{w-\nu}{2}\right)$.

Proof. This follows from Theorem 3.1 by taking $\lambda = 1$. □

The following table shows the exact and approximate values of the Apostol-Frobenius-Euler polynomials of order α, $H_n^{\alpha}(nz + \alpha/2;\ u;\ \lambda)$, where $n = 4$, $\alpha = 3$, $u = 2$, and $\lambda = 5$ for some increasing values of $z = x$.

It can be observed from Table 1 that for increasing values of z, the relative error decreases.

4 Generalized Apostol-type Frobenius-Euler polynomials

For parameters $\lambda, u, \in \mathbb{C}, u \neq \lambda$ and $a, b, c \in \mathbb{R}^+$ with $a \neq b$, the generalized Apostol-type Frobenius-Euler polynomials of order α, denoted by $\mathcal{H}_n^{\alpha}(z;\ u;\ a, b, c, \lambda)$, are defined by means of the following generating function [24]:

$$\left(\frac{a^w - u}{\lambda b^w - u}\right)^{\alpha} c^{zw} = \sum_{n=0}^{\infty} \mathcal{H}_n^{\alpha}(z;\ u;\ a, b, c, \lambda)\frac{w^n}{n!}, |w| < \left|\frac{\log\left(\frac{\lambda}{u}\right)}{\ln b}\right|. \tag{45}$$

On setting $a = 1$, $b = e^m$, $c = e$, and $\lambda = 1$ in (45), Belbachir and Souddi [34] gives the generalized Frobenius-Euler polynomials of order α with parameter m, $\mathcal{H}_n^{\alpha}(z;\ u;\ m)$, defined by the following generating function:

$$\left(\frac{1 - u}{e^{mw} - u}\right)^{\alpha} c^{zw} = \sum_{n=0}^{\infty} \mathcal{H}_n^{\alpha}(z;\ u;\ m)\frac{w^n}{n!}, \quad |w| < 2\pi. \tag{46}$$

When $a = 1$, $b = e^m$, $c = e$, $u = -1$, and $\lambda = 1$ in (45), this results in the generalized Euler polynomials of order α with parameter m, $E_n^{\alpha}(z;\ m)$, defined by the generating function [34]

$$\left(\frac{2}{e^{mw} + 1}\right)^{\alpha} e^{zw} = \sum_{n=0}^{\infty} E_n^{\alpha}(z;\ m)\frac{w^n}{n!}, |w| < 2\pi. \tag{47}$$

In this section, the methods used to obtain the approximations in Theorem 2.1 and 3.1 are applied to the case of the generalized Apostol-type Frobenius-Euler polynomials of order α with parameters a, b, and c. Approximations for the generalized Frobenius-Euler polynomials and generalized Euler polynomials of order α with parameter m are given as corollaries.

4.1 Uniform expansions

Uniform approximations for the generalized Apostol-type Frobenius-Euler polynomials are obtained using the saddle point method. The following theorem describes the said approximation.

Theorem 4.1. *For $n, \alpha \in \mathbb{Z}^+$, $a, b, c \in \mathbb{R}^+$, $u, \lambda \in \mathbb{C}\backslash\{0, 1\}$, and $z \in \mathbb{C}\backslash\{0\}$ such that $|\mathrm{Im}\, z^{-1}| < \frac{2\pi}{\beta} - \mathrm{Arg}\left(\frac{\delta}{\beta}\right)$ or $|z^{-1}| < |z^{-1} - \left(\frac{2\pi i - \delta}{\beta}\right)|$, the generalized Apostol-type Frobenius-Euler polynomials of order α satisfy*

$$\begin{aligned}\mathcal{H}_n^{\alpha}\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; u; a, b, c, \lambda\right) = \frac{(nz)^n \left(\frac{c}{b}\right)^{\frac{\alpha}{2z}} (a^{1/z} - u)^{\alpha}}{(2\sqrt{\lambda u})^{\alpha} \sinh^{\alpha}\left(\frac{z\delta + \beta}{2z}\right)} \Bigg\{1 - \frac{\alpha}{2nz^2}\Bigg[\frac{\log(a) a^{1/z}}{(a^{1/z} - u)}\left(\log(a) + \alpha \log\left(\frac{c}{b}\right)\right. \\ \left. - \alpha\beta \coth\left(\frac{z\delta + \beta}{2z}\right)\right) + \frac{\alpha\beta}{2} \coth\left(\frac{z\delta + \beta}{2z}\right)\left(\frac{\beta}{2} \coth\left(\frac{z\delta + \beta}{2z}\right)\right. \\ \left. - \log\left(\frac{c}{b}\right)\right) + \frac{1}{4}\left(\beta^2 \mathrm{csch}^2\left(\frac{z\delta + \beta}{2z}\right) + \alpha \log^2\left(\frac{c}{b}\right)\right)\Bigg] + O\left(\frac{1}{n^2}\right)\Bigg\},\end{aligned} \tag{48}$$

where $\delta = \log\left(\frac{\lambda}{u}\right)$, $\beta = \log(b)$, $\gamma = \log(c)$, and the logarithm is taken to be the principal branch.

Proof. Applying the Cauchy integral formula to (45),

$$\mathcal{H}_n^{\alpha}(z; u; a, b, c, \lambda) = \frac{n!}{2\pi i} \int_C \frac{1}{u^{\alpha}} \frac{(a^w - u)^{\alpha}}{(e^{\delta + w\beta} - 1)^{\alpha}} e^{zw\gamma} \frac{\mathrm{d}w}{w^{n+1}}, \tag{49}$$

where C is a circle 0 with radius lesser than $|\frac{2\pi i - \delta}{\beta}|$ and $\delta = \log\left(\frac{\lambda}{u}\right)$, $\beta = \log(b)$, and $\gamma = \log(c)$ are logarithms taken to be the principal branch.

With $\left(2\left(\frac{\lambda b^w}{u}\right)^{\frac{1}{2}} \sinh\left(\frac{\delta + w\beta}{2}\right)\right)^{\alpha} = (e^{\delta + w\beta} - 1)^{\alpha}$, (49) becomes

$$\mathcal{H}_n^{\alpha}(z; u; a, b, c, \lambda) = \frac{n!}{2\pi i} \int_C \frac{1}{(2\sqrt{\lambda u})^{\alpha}} \frac{(a^w - u)^{\alpha}}{(b)^{\frac{\alpha w}{2}} \sinh^{\alpha}\left(\frac{\delta + w\beta}{2}\right)} e^{zw\gamma} \frac{\mathrm{d}w}{w^{n+1}}. \tag{50}$$

The shifting of the variable $z \to \frac{z}{\gamma} + \frac{\alpha}{2}$ from (50) gives

$$\mathcal{H}_n^{\alpha}\left(\frac{z}{\gamma} + \frac{\alpha}{2}; u; a, b, c, \lambda\right) = \frac{n!}{2\pi i} \frac{1}{(2\sqrt{\lambda u})^{\alpha}} \int_C f(w) e^{zw} \frac{\mathrm{d}w}{w^{n+1}}, \tag{51}$$

where

$$f(w) = \left(\frac{c}{b}\right)^{\alpha w} \frac{(a^w - u)^{\alpha}}{\sinh^{\alpha}\left(\frac{\delta + w\beta}{2}\right)} \tag{52}$$

is a meromorphic function with simple poles of order α at the zeros of $\sinh^{\alpha}\left(\frac{\delta+w\beta}{2}\right)$, which are given by $w_j = \frac{2j\pi i-\delta}{\beta}, j = \pm 1, \pm 2, \ldots.$

Taking $z \to nz$ and letting $nz \to \infty$ with fixed z,

$$\mathcal{H}_n^{\alpha}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};\ u;\ a,b,c,\lambda\right) = \frac{n!}{2\pi i}\frac{1}{(2\sqrt{\lambda u})^{\alpha}}\int_C f(w)e^{n(zw-\log w)}\frac{\mathrm{d}w}{w}. \tag{53}$$

It was observed using the saddle point method in Theorem 2.1 that the main contribution of the integrand in (53) originates at the saddle point of the argument of the exponential. Thus, expanding $f(w)$ around the saddle point $w = z^{-1}$ gives the approximations of $\mathcal{H}_n^{\alpha}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};\ u;\ a,b,c,\lambda\right)$. Consequently, it follows from Lemma 1, Lemma 2, and Theorem 1 of [31] that

$$\mathcal{H}_n^{\alpha}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};\ u;\ a,b,c,\lambda\right) = \frac{(nz)^n}{(2\sqrt{\lambda u})^{\alpha}}\sum_{k=0}^{\infty}\frac{f^{(k)}(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k}, \tag{54}$$

where $p_k(n)$ are the polynomials in (11). Solving for the derivatives $f^{(k)}(z^{-1})$ for $k = 0,1,2$ yields

$$f^{(0)}(z^{-1}) = f(w) = \frac{\left(\frac{c}{b}\right)^{\frac{\alpha}{2z}}(a^{1/z}-u)^{\alpha}}{\sinh^{\alpha}\left(\frac{z\delta+\beta}{2z}\right)}$$

$$f^{(1)}(z^{-1}) = \frac{\left(\frac{c}{b}\right)^{\frac{\alpha}{2z}}(a^{1/z}-u)^{\alpha}}{\sinh^{\alpha}\left(\frac{z\delta+\beta}{2z}\right)}\left[\frac{\log\left(\frac{c}{b}\right)}{2}+\frac{a^{1/z}\log(a)}{a^{1/z}-u}-\frac{\beta}{2}\coth\left(\frac{z\delta+\beta}{2z}\right)\right]$$

$$\begin{aligned}f^{(2)}(z^{-1}) = {}& \frac{\alpha\left(\frac{c}{b}\right)^{\frac{\alpha}{2z}}(a^{1/z}-u)^{\alpha}}{\sinh^{\alpha}\left(\frac{z\delta+\beta}{2z}\right)}\left[\frac{\log(a)a^{1/z}}{(a^{1/z}-u)}\left(\log(a)+\alpha\log\left(\frac{c}{b}\right)-\alpha\beta\coth\left(\frac{z\delta+\beta}{2z}\right)\right)\right.\\ &\left.+\frac{\alpha\beta}{2}\coth\left(\frac{z\delta+\beta}{2z}\right)\left(\frac{\beta}{2}-\log\left(\frac{c}{b}\right)\right)+\frac{1}{4}\left(\beta^2\operatorname{csch}^2\left(\frac{z\delta+\beta}{2z}\right)+\alpha\log^2\left(\frac{c}{b}\right)\right)\right].\end{aligned}$$

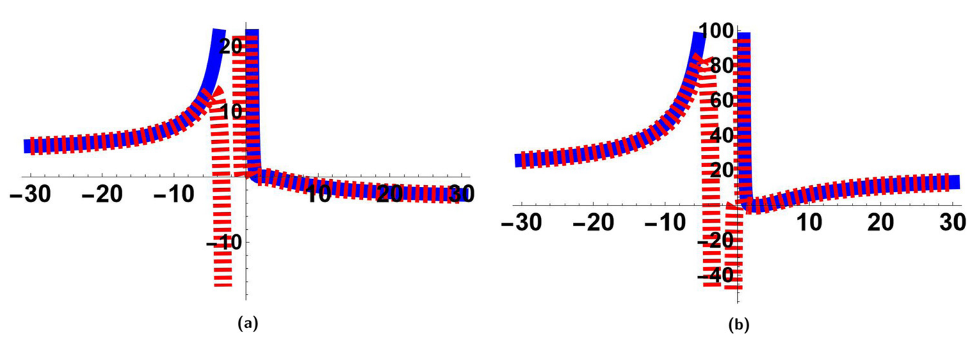

Figure 6: Solid lines represent the generalized Apostol-type Frobenius-Euler polynomials of order α, $\mathcal{H}_n^{\alpha}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};\ u;\ a,b,c,\lambda\right)$ for several values of n, whereas dashed lines represent the right-hand side of (48) with $z \equiv x$, both normalized by the factor $(1+|\frac{x}{\sigma}|^n)^1$ where we choose $\sigma = 0.5$. (a) $n = 5$, $\alpha = 3$, $u = 2$, $a = 3$, $b = 4$, $c = 4$, $\lambda = 5$ and (b) $n = 6$, $\alpha = 4$, $u = 3$, $a = 5$, $b = 4$, $c = 3$, $\lambda = 8$.

Expanding the sum in (54) and keeping only the first three terms give (Figure 6)

$$\begin{aligned}\mathcal{H}_n^{\alpha}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};\,u;\,a,b,c,\lambda\right) &= \frac{(nz)^n}{(2\sqrt{\lambda u})^{\alpha}}\left[\frac{f^{(0)}(z^{-1})}{0!}+\frac{f^{(1)}(z^{-1})}{1!}\frac{p_1(n)}{nz}+\frac{f^{(2)}(z^{-1})}{2!}\frac{p_2(n)}{(nz)^2}+O\left(\frac{1}{n^2}\right)\right]\\ &= \frac{(nz)^n\left(\frac{c}{b}\right)^{\frac{\alpha}{2z}}(a^{1/z}-u)^{\alpha}}{(2\sqrt{\lambda u})^{\alpha}\sinh^{\alpha}\left(\frac{z\delta+\beta}{2z}\right)}\left\{1-\frac{\alpha}{2nz^2}\left[\frac{\log(a)a^{1/z}}{(a^{1/z}-u)}\left(\log(a)+\alpha\log\left(\frac{c}{b}\right)\right.\right.\right.\\ &\quad \left.-\alpha\beta\coth\left(\frac{z\delta+\beta}{2z}\right)\right)+\frac{\alpha\beta}{2}\coth\left(\frac{z\delta+\beta}{2z}\right)\left(\frac{\beta}{2}-\log\left(\frac{c}{b}\right)\right)\\ &\quad \left.\left.+\frac{1}{4}\left(\beta^2\operatorname{csch}^2\left(\frac{z\delta+\beta}{2z}\right)+\alpha\log^2\left(\frac{c}{b}\right)\right)\right]+O\left(\frac{1}{n^2}\right)\right\}.\end{aligned}$$

□

Remark 4.2. Taking $a = 1$, $b = e$, and $c = e$, Theorem 4.1 gives a uniform approximation which is similar to that obtained in Theorem 2.1 for the Apostol Frobenius-Euler polynomials of order α.

The following corollaries give the uniform approximations for the generalized Frobenius-Euler polynomials and generalized Euler polynomials of order α with parameter m.

Corollary 4.3. *For $n, \alpha \in \mathbb{Z}^+$, $m \in R^+$, $u \in \mathbb{C}\backslash\{0,1\}$, and $z \in \mathbb{C}\backslash\{0\}$ such that $|\mathrm{Im}\, z^{-1}| < \frac{2\pi+\mathrm{Arg}(u)}{m}$ or $|z^{-1}| < |z^{-1} - \frac{(2\pi i+v)}{m}|$, the generalized Frobenius-Euler polynomials of order α with parameter m satisfy*

$$\begin{aligned}\mathcal{H}_n^{\alpha}\left(nz+\frac{\alpha}{2};\,u;\,m\right) &= \frac{(nz)^n(e^{1-m})^{\frac{\alpha}{2z}}(1-u)^{\alpha}}{(2\sqrt{u})^{\alpha}\sinh^{\alpha}\left(\frac{m-zv}{2z}\right)}\left\{1-\frac{\alpha}{8nz^2}\left[\alpha m\coth\left(\frac{m-zv}{2z}\right)\left(m\coth\left(\frac{m-zv}{2z}\right)\right.\right.\right.\\ &\quad \left.\left.\left.-2(1-m)\right)+m^2\operatorname{csch}^2\left(\frac{m-zv}{2z}\right)+\alpha(1-m)^2\right]+O\left(\frac{1}{n^2}\right)\right\},\end{aligned} \tag{55}$$

where $v = \log(u)$ and the logarithm is taken to be the principal branch.

Proof. This follows from Theorem 4.1 by taking $a = 1$, $b = e^m$, $c = e$, and $\lambda = 1$. □

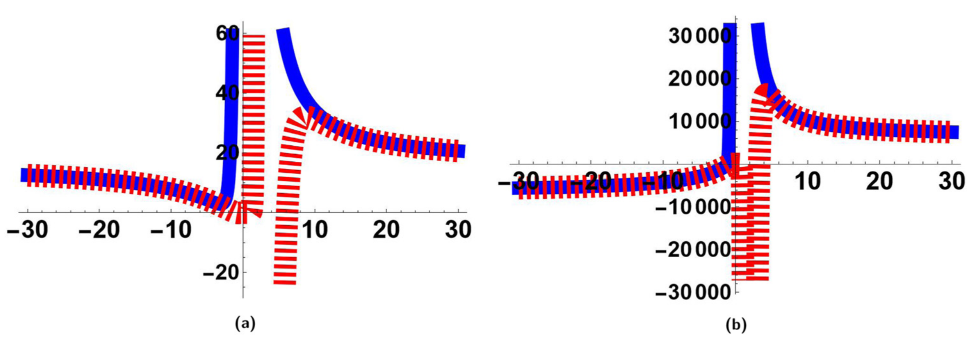

Figure 7: Solid lines represent the generalized Frobenius-Euler polynomials of order α with parameter m, $\mathcal{H}_n^{\alpha}\left(nz+\frac{\alpha}{2};\,u;\,m\right)$ for several values of n, whereas dashed lines represent the right-hand side of (55) with $z \equiv x$, both normalized by the factor $(1+|\frac{x}{\sigma}|^n)^1$ where we choose $\sigma = 0.5$. (a) $n = 4$, $\alpha = 5$, $m = 6$, $u = 5$ and (b) $n = 7$, $\alpha = 4$, $m = 3$, $u = 6$.

Corollary 4.4. *For $n, \alpha \in \mathbb{Z}^+$, $m \in \mathbb{R}^+$, and $z \in C\backslash 0$ such that $|\mathrm{Im}\, z^{-1}| < \frac{\pi}{m}$ or $|z^{-1}| < |z^{-1} - \frac{\pi}{m}|$, the generalized Euler polynomials of order α with parameter m satisfy*

$$E_n^{\alpha}\left(nz + \frac{\alpha}{2}; m\right) = (nz)^n (e^{1-m})^{\frac{\alpha}{2z}} \operatorname{sech}\left(\frac{m}{2z}\right)\left\{1 - \frac{\alpha}{8nz^2}\left[\alpha m \tanh\left(\frac{m}{2z}\right)\left(m \tanh\left(\frac{m}{2z}\right) - 2(1-m)\right) - m^2 \operatorname{sech}^2\left(\frac{m}{2z}\right) + \alpha(1-m)^2\right] + O\left(\frac{1}{n^2}\right)\right\}. \tag{56}$$

Proof. This follows from Theorem 4.1 by taking $a = 1$, $b = e^m$, $c = e$, $u = -1$, and $\lambda = 1$. □

The graphs in Figures 7 and 8 show the approximations of Corollaries 4.3 and 4.4

The following table shows the exact and approximate values of the generalized Apostol-type Frobenius-Euler polynomials of order α, $\mathcal{H}_n^{\alpha}\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; u, a, b, c; \lambda\right)$, where $n = 4$, $\alpha = 3$, $u = 2$, $a = 2$, $b = 3$, $c = 4$, $\lambda = 5$, and $\gamma = \log c = \log 4$ for some increasing values of $z = x$.

It can also be observed from Table 2 that for increasing values of z, the relative error decreases.

4.2 Enlarged region of validity

In Section 3, approximation of the Apostol-Frobenius-Euler polynomials of order α is obtained using the method of contour integration that introduced the incomplete gamma function in the formula. Using the same technique, approximation for the generalized Apostol-type Frobenius-Euler polynomials of order α with parameters a, b, and c is given in the following theorem.

Theorem 4.5. *For $n, \alpha \in \mathbb{Z}^+$, a, b, and $c \in \mathbb{R}^+$, $u, \lambda \in \mathbb{C}\backslash\{0, 1\}$, and $z \in \mathbb{C}$ such that $|z^{-1}| < |z^{-1} - w_k|$ for all $k = l+1, l+2, \ldots,$ the generalized Apostol-type Frobenius-Euler polynomials of order α satisfy*

$$\begin{aligned}\mathcal{H}_n^{\alpha}\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; u; a, b, c, \lambda\right) = \frac{1}{(2\sqrt{\lambda u})^{\alpha}}&\left\{\sum_{k=1}^{l}\sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j}\left[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle j-1\rangle_s (w_k)^{-(j-1+s)}\right.\right.\\ &\times\left.\left(\frac{(n-s)!}{w_k^{n-s+1}} - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}}\right) + \frac{(-1)^j\langle j\rangle_n}{w_k^{j+n}}\right]\\ &\left.+ (nz)^n \sum_{k=0}^{\infty}\frac{f^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k}\right\},\end{aligned} \tag{57}$$

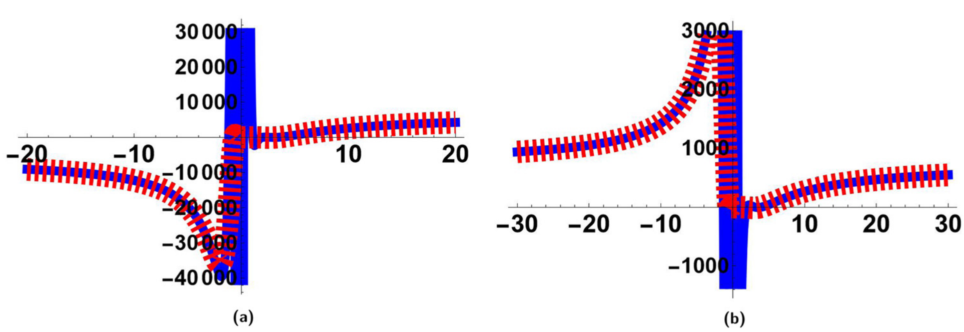

Figure 8: Solid lines represent the generalized Euler polynomials of order α with parameter m, $\mathcal{E}_n^{\alpha}\left(nz + \frac{\alpha}{2}; m\right)$ for several values of n, whereas dashed lines represent the right-hand side of (56) with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^1$ where we choose $\sigma = 0.5$. (a) $n = 4$, $\alpha = 5$, $m = 6$, $u = 5$ and (b) $n = 7$, $\alpha = 4$, $m = 3$, $u = 6$.

Table 2: Exact and approximate values of $\mathcal{H}_4^3\left(\frac{4z}{\log 4} + \frac{3}{2};2;2,3,4;5\right)$

z	Exact values $\mathcal{H}_4^3\left(\frac{4z}{\log 4} + \frac{3}{2};2,3,4;5\right)$	Approximate values using Theorem 4.1	Relative error
3	-6.22627×10	-8.05204×10	2.93238×10
4	-4.54829×10^2	-4.99541×10^2	9.83059×10^{-2}
5	-1.6595×10^3	-1.74272×10^3	5.01423×10^{-2}
6	-4.3878×10^3	-4.52054×10^3	3.02532×10^{-2}
7	-9.57876×10^3	-9.77113×10^3	2.00824×10^{-2}
8	-1.8399×10^4	-1.86604×10^4	1.42037×10^{-2}
9	-3.22427×10^4	-3.25819×10^4	1.05171×10^{-2}
10	-5.27317×10^4	-5.31569×10^4	8.06453×10^{-3}
11	-8.17151×10^4	-8.22346×10^4	6.35733×10^{-3}
12	-1.2127×10^5	-1.21891×10^5	5.12551×10^{-3}

where the polynomials $p_k(n)$ are given in (11) *and h_l^k is the kth derivative of the function $h_l(w)$ given by* (28) *and*

$$\sum_{j=1}^{m} \frac{r_{k_j}}{(w - w_k)^j},$$

are the given principal parts of the Laurent series corresponding to the poles $w_k = (2k\pi i - \delta)/\beta$, where $\delta = \log\left(\frac{\lambda}{u}\right)$ and $\beta = \log(b)$ and the entire function $h(w)$ is determined by $f(w) = \frac{\left(\frac{c}{b}\right)^{aw}(a^w - u)^\alpha}{\sinh^\alpha\left(\frac{\delta + w\beta}{2}\right)}$.

Proof. Recall from (53),

$$\mathcal{H}_n^\alpha\left(\frac{nz}{\gamma} + \frac{\alpha}{2};\, u;\, a, b, c, \lambda\right) = \frac{n!}{2\pi i} \frac{1}{(2\sqrt{\lambda u})^\alpha} \int_C f(w) e^{n(zw - \log w)} \frac{\mathrm{d}w}{w}, \tag{58}$$

where

$$f(w) = \frac{\left(\frac{c}{b}\right)^{aw}(a^w - u)^\alpha}{\sinh^\alpha\left(\frac{\delta + w\beta}{2}\right)},$$

with $\delta = \log\left(\frac{\lambda}{u}\right)$ and $\beta = \log(b)$. Substituting (16) into (58) gives

$$\mathcal{H}_n^\alpha\left(\frac{nz}{\gamma} + \frac{\alpha}{2};\, u;\, a, b, c, \lambda\right) = S_l^{n,\alpha}(z) + T_l^{n,\alpha}(z), \tag{59}$$

where

$$S_l^{n,\alpha}(z) = \frac{n!}{2\pi i} \frac{1}{(2\sqrt{\lambda u})^\alpha} \int_C f_l(w) e^{wnz} \frac{\mathrm{d}w}{w^{n+1}}, \tag{60}$$

$$T_l^{n,\alpha}(z) = \frac{1}{(2\sqrt{\lambda u})^\alpha} \int_C \frac{n!}{2\pi i} \sum_{k=1}^{l} \sum_{j=1}^{\alpha} \frac{r_{k_j}}{(w - w_k)^j} e^{wnz} \frac{\mathrm{d}w}{w^{n+1}}. \tag{61}$$

Repeating the process used in Theorem 3.1 to expand $f_l(w)$ around the saddle point z^{-1} gives the expansion of $S_l^{n,\alpha}(z)$ as

$$S_l^{n,\alpha}(z) = (nz)^n \frac{1}{(2\sqrt{\lambda u})^\alpha} \sum_{k=0}^{\infty} \frac{f_l^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k} \tag{62}$$

valid for $|z^{-1}| < |z^{-1} - w_j|, j = l + 1, l + 2, \ldots$ and $z \neq 0$.

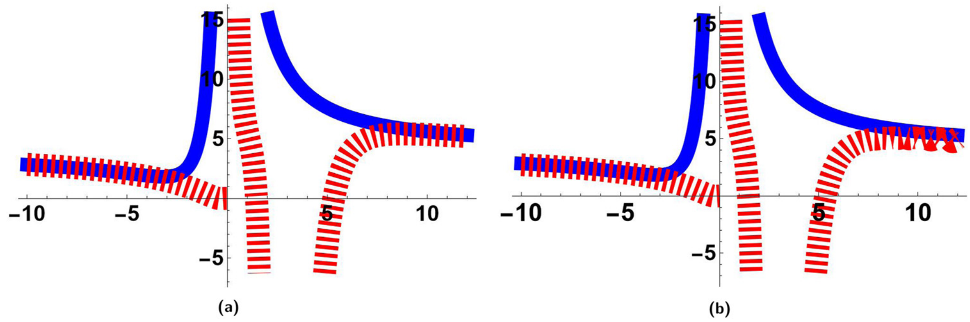

Figure 9: Solid lines in (a) and (b) represent $\mathcal{H}_n^{\alpha}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};\, u;\, a, b, c, \lambda\right)$ whereas dashed lines in (a) and (b) represent the right-hand side of (6) and (15) for $n = 2$, $\alpha = 2$, $u = 3$, $a = 5$, $b = 3$, $c = 2$, and $\lambda = 2$, respectively, with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^1$ where we choose $\sigma = 0.5$. (a) $n = 3$, $\alpha = 3$, $u = 4$, $\lambda = 6$ and (b) $n = 3$, $\alpha = 3$, $u = 4$, $\lambda = 6$.

Conversely, the expansion of $T_l^{n,\alpha}$ can be obtained by employing the same method to evaluate the integral of $Y_l^{n,\alpha}$ in (21). The method of contour integration involves shifting the integration contour by $w = w_k + t$, which resulted in an expansion with incomplete gamma function. Thus, the expansion is given as

$$\begin{aligned} T_l^{n,\alpha}(z) = \frac{1}{(2\sqrt{\lambda u})^{\alpha}} \sum_{k=1}^{l}\sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j}\Bigg[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle j-1\rangle_s (w_k)^{-(j-1+s)} \\ \times\left(\frac{(n-s)!}{w_k^{n-s+1}} - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}}\right) + \frac{(-1)^j\langle j\rangle_n}{w_k^{j+n}}\Bigg]. \end{aligned} \tag{63}$$

Substituting (62) and (63) into (59) gives

$$\begin{aligned} \mathcal{H}_n^{\alpha}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};\, u;\, a, b, c, \lambda\right) = \frac{1}{(2\sqrt{\lambda u})^{\alpha}}\Bigg\{ \sum_{k=1}^{l}\sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j}\Bigg[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle j-1\rangle_s (w_k)^{-(j-1+s)} \\ \times\left(\frac{(n-s)!}{w_k^{n-s+1}} - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}}\right) + \frac{(-1)^j\langle j\rangle_n}{w_k^{j+n}}\Bigg] \\ + (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k}\Bigg\}, \end{aligned} \tag{64}$$

where the polynomials $p_k(n)$ are given in (11) and h_l^k is the kth derivative of the function $h_l(w)$ given by (28) and

$$\sum_{j=1}^{m}\frac{r_{k_j}}{(w-w_k)^j}$$

are the given principal parts of the Laurent series corresponding to the poles $w_k = (2k\pi i - \delta)/\beta$, where $\delta = \log\left(\frac{\lambda}{u}\right)$ and $\beta = \log(b)$ and the entire function $h(w)$ is determined by $f(w) = \frac{\left(\frac{c}{b}\right)^{\alpha w}(a^w - u)^{\alpha}}{\sinh^{\alpha}\left(\frac{\delta + w\beta}{2}\right)}$. □

The comparison of the accuracy of the asymptotic formula obtained in (48) and (57) is shown in Figure 9.

5 Conclusion

The study of special functions, particularly the investigation of Apostol-type polynomials, has proven to be a captivating area in mathematics. These special functions exhibit unique properties and have a wide range of applications in various fields of knowledge. The development of topics in special functions involves exploring combinations, extensions, and generalizations of classical polynomials such as Bernoulli, Euler, Genocchi, and tangent polynomials.

This article focused on the Apostol-Frobenius-Euler polynomials of order a, denoted by $H_n^a(z;\, u;\, \lambda)$, and their higher-order approximations. Building upon the saddle point method, Corcino et al. [27] derived approximations for the higher-order tangent polynomials and introduced a new method to extend the validity region of these approximations. The findings of this study indicate that these methods can also be applied to the higher-order Apostol-Frobenius-Euler polynomials.

Consequently, the article obtained approximations of the higher-order Apostol-Frobenius-Euler polynomials using hyperbolic functions, specifically for large values of the parameter n. Additionally, uniform approximations with an enlarged region of validity were derived. Furthermore, the same methods were applied to obtain approximations of the generalized Apostol-type Frobenius-Euler polynomials of order α with parameters a, b, and c.

To assess the accuracy of the exact values and the corresponding approximations of these polynomials, the article presented tables of values and graphs for specific values of the parameters. These tables and graphs provide visual evidence of the effectiveness of the derived approximations.

In summary, the research presented in this article contributes to the field of special functions by expanding our understanding of Apostol-type polynomials, specifically the Apostol-Frobenius-Euler polynomials. The developed methods for obtaining higher-order approximations and enlarging the validity region demonstrate their applicability and effectiveness. The obtained results and graphical illustrations highlight the accuracy and usefulness of these approximations, providing valuable insights for further research and practical applications in various disciplines.

For future research work, it is recommended to explore the applicability of the methods used in this article to Apostol-Frobenius-Genocchi polynomials of the higher order.

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