Research Article

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Approximate multi-variable bi-Jensen-type mappings

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Abstract: In this study, we obtained the stability of the multi-variable bi-Jensen-type functional equation:

$$n^2 f\left(\frac{x_1 + \cdots + x_n}{n}, \frac{y_1 + \cdots + y_n}{n}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i, y_j).$$

Keywords: linear 2-normed space, quasi-normed space, multi-variable bi-Jensen-type mapping

MSC 2020: 39B52, 39B82

1 Introduction

In 1940, Ulam [1] suggested the stability problem of functional equations concerning the stability of group homomorphisms:

> Let $\mathcal{G}$ be a group and let $\mathcal{H}$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : \mathcal{G} \to \mathcal{H}$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in \mathcal{G}$ then there is a homomorphism $H : \mathcal{G} \to \mathcal{H}$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in \mathcal{G}$?

Hyers-Ulam stability is a mathematical result that deals with the variation of approximations of a function under small perturbations. Research on the stability of functional equations has been continuously conducted, and rich results are coming out [2–8].

Throughout this article, let $\mathcal{X}$ and $\mathcal{Y}$ be the vector spaces.

Definition 1. A mapping $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is called a *bi-Jensen mapping* if f satisfies the system of equations:

$$2f\left(\frac{x+y}{2}, z\right) = f(x,z) + f(y,z), \quad \text{and} \quad 2f\left(x, \frac{y+z}{2}\right) = f(x,y) + f(x,z). \tag{1}$$

Let $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be a mapping. In 2006, Bae and Park [9] obtained the general solution of the bi-Jensen functional equation

$$4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w) \tag{2}$$

and proved its stability. Subsequent articles have been published since 2008 by several authors [10–13].

For an integer n greater than 1, consider the multi-variable bi-Jensen-type functional equation:

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$$n^2 f\left(\frac{x_1 + \cdots + x_n}{n}, \frac{y_1 + \cdots + y_n}{n}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i, y_j). \tag{3}$$

Equation (2) is a special case of equation (3).

In 2011, Park [14] investigated the approximate additive, Jensen, and quadratic mappings in 2-Banach spaces. In 2018, EL-Fassi [15] investigated the generalized hyperstability of bi-Jensen functional equation in (2, β)-Banach spaces.

In this study, we solved the solution and investigated the stability of the multi-variable bi-Jensen-type functional equation (3) in 2-Banach spaces and quasi-Banach spaces.

2 Main results

We introduce some definitions on 2-Banach spaces [16,17].

Definition 2. Let $\mathcal{X}$ be a real vector space with $\dim\mathcal{X} \geq 2$ and $\|\cdot,\cdot\| : \mathcal{X}^2 \to \mathbb{R}$ be a function. Then, $(\mathcal{X}, \|\cdot,\cdot\|)$ is called a *linear 2-normed space* if the following conditions hold:
(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(b) $\|x, y\| = \|y, x\|$,
(c) $\|\alpha x, y\| = |\alpha|\|x, y\|$,
(d) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$,

for all $\alpha \in \mathbb{R}$ and $x, y, z \in \mathcal{X}$. In this case, the function $\|\cdot,\cdot\|$ is called a *2-norm* on $\mathcal{X}$.

Definition 3. Let $\{x_n\}$ be a sequence in a linear 2-normed space $\mathcal{X}$. The sequence $\{x_n\}$ is said to be *convergent* in $\mathcal{X}$ if there exists an element $x \in \mathcal{X}$ such that

$$\lim_{n\to\infty}\|x_n - x, y\| = 0$$

for all $y \in \mathcal{X}$. In this case, we say that the sequence $\{x_n\}$ converges to the limit x, simply, denoted by $\lim_{n\to\infty} x_n = x$.

Definition 4. A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a *Cauchy sequence* if there are two linearly independent points $y, z \in \mathcal{X}$ such that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\|x_m - x_n, y\| < \varepsilon$ and $\|x_m - x_n, z\| < \varepsilon$. A *2-Banach space* is defined to be a linear 2-normed space in which every Cauchy sequence is convergent.

In the following lemma, we obtain some basic properties in a linear 2-normed space, which will be used to prove the stability results.

Lemma 1. [14] *Let $(\mathcal{X}, \|\cdot,\cdot\|)$ be a linear 2-normed space and $x \in \mathcal{X}$.*
(*a*) *If $\|x, y\| = 0$ for all $y \in X$, then $x = 0$.*
(*b*) *$|\|x, z\| - \|y, z\|| \leq \|x - y, z\|$ for all $x, y, z \in \mathcal{X}$.*
(*c*) *If a sequence $\{x_n\}$ is convergent in $\mathcal{X}$, then $\lim_{n\to\infty}\|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$ for all $y \in \mathcal{X}$.*

Let $\mathcal{X}$ be a normed space and $\mathcal{Y}$ be a 2-Banach space.

Lemma 2. *Let $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfy* (3). *And let $g_x, g'_y : \mathcal{X} \to \mathcal{Y}$ be given by $g_x(y) \coloneqq f(x, y) - f(x, 0)$ and $g'_y(x) \coloneqq f(x, y) - f(0, y)$ for all $x, y \in \mathcal{X}$. Then, g_x is additive for all $x \in \mathcal{X}$ and g'_y is additive for all $y \in \mathcal{X}$.*

Proof. Letting $x_1 = x_2 = \cdots = x_n = x$, $y_1 = y$, and $y_2 = \cdots = y_n = 0$ in (3), we have

$$nf\left(x, \frac{y}{n}\right) = f(x,y) + (n-1)f(x,0)$$

for all $x, y \in \mathcal{X}$. So, we have

$$ng_x\left(\frac{y}{n}\right) = nf\left(x, \frac{y}{n}\right) - nf(x,0) = f(x,y) - f(x,0) = g_x(y) \tag{4}$$

for all $x, y \in \mathcal{X}$. Putting $x_1 = x_2 = \cdots = x_n = x$ and $y_3 = \cdots = y_n = 0$ in (3), we have

$$nf\left(x, \frac{y_1 + y_2}{n}\right) = f(x,y_1) + f(x,y_2) + (n-2)f(x,0)$$

for all $x, y_1, y_2 \in \mathcal{X}$. So, we have

$$ng_x\left(\frac{y_1 + y_2}{n}\right) = nf\left(x, \frac{y_1 + y_2}{n}\right) - nf(x,0) = f(x,y_1) + f(x,y_2) - 2f(x,0) = g_x(y_1) + g_x(y_2)$$

for all $x, y_1, y_2 \in \mathcal{X}$. By equation (4) and the aforementioned equation, we know that g_x is additive for all $x \in \mathcal{X}$. Similarly, we also know that g'_y is additive for all $y \in \mathcal{X}$. □

Theorem 1. *A mapping* $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ *satisfies* (1) *if and only if it satisfies* (3).

Proof. First, we assume that f satisfies (1). By [9], there exist a bi-additive mapping $B : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ and two additive mappings $A, A' : \mathcal{X} \to \mathcal{Y}$ such that $f(x,y) = B(x,y) + A(x) + A'(y) + f(0,0)$ for all $x, y \in \mathcal{X}$. Thus, we have

$$\begin{aligned}
n^2 f\left(\frac{x_1 + \cdots + x_n}{n}, \frac{y_1 + \cdots + y_n}{n}\right) &= n^2 B\left(\frac{1}{n}\sum_{i=1}^n x_i, \frac{1}{n}\sum_{j=1}^n y_j\right) + n^2 A\left(\frac{1}{n}\sum_{i=1}^n x_i\right) + n^2 A'\left(\frac{1}{n}\sum_{j=1}^n y_j\right) + n^2 f(0,0) \\
&= \sum_{i=1}^n \sum_{j=1}^n B(x_i, y_j) + n\sum_{i=1}^n A(x_i) + n\sum_{j=1}^n A'(y_j) + n^2 f(0,0) \\
&= \sum_{i=1}^n \sum_{j=1}^n B(x_i, y_j) + \sum_{i=1}^n \sum_{j=1}^n A(x_i) + \sum_{i=1}^n \sum_{j=1}^n A'(y_j) + \sum_{i=1}^n \sum_{j=1}^n f(0,0) \\
&= \sum_{i=1}^n \sum_{j=1}^n f(x_i, y_j)
\end{aligned}$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{X}$, i.e., f satisfies (3).

Conversely, assume that f satisfies (3). Define $g_x, g'_y : \mathcal{X} \to \mathcal{Y}$ by $g_x(y) \coloneqq f(x,y) - f(x,0)$ and $g'_y(x) \coloneqq f(x,y) - f(0,y)$ for all $x, y \in \mathcal{X}$. By Lemma 2, g_x is additive for all $x \in \mathcal{X}$ and g'_y is additive for all $y \in \mathcal{X}$. Thus, we obtain

$$2g_x\left(\frac{y+z}{2}\right) = g_x(y) + g_x(z)$$

and

$$2f\left(x, \frac{y+z}{2}\right) = 2g_x\left(\frac{y+z}{2}\right) + 2f(x,0) = f(x,y) + f(x,z)$$

for all $x, y, z \in \mathcal{X}$. Similarly, we obtain

$$2f\left(\frac{x+y}{2}, z\right) = f(x,z) + f(y,z)$$

for all $x, y, z \in \mathcal{X}$, i.e., f satisfies (1). □

The following theorem proves the stability of equation (3) in 2-Banach spaces.

Theorem 2. *Let* $r \in (0,2)$, $\varepsilon > 0$, $\delta, \eta \geq 0$, *and let* $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ *be a surjection satisfying* $f(x,0) = 0$ *such that*

$$\left\| n^2 f\left(\frac{x_1 + \cdots + x_n}{n}, \frac{y_1 + \cdots + y_n}{n}\right) - \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i, y_j), f(s,t) \right\| \leq \varepsilon + \delta\left(\sum_{i=1}^{n} \|x_i\|^r + \sum_{j=1}^{n} \|y_j\|^r\right) + \eta(\|s\| + \|t\|) \tag{5}$$

for all $x, x_1, \dots, x_n, y_1, \dots, y_n, s, t \in \mathcal{X}$. *Then, there exists a unique bi-additive mapping* $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(x,y) - f(0,y) - F(x,y), f(s,t)\| \leq \frac{2}{n^2 - 1}[\varepsilon + \eta(\|s\| + \|t\|)] + \frac{n^r \delta}{n^2 - n^r}(\|x\|^r + 2\|y\|^r) \tag{6}$$

for all $x, y, s, t \in \mathcal{X}$.

Proof. Let $g(x,y) \coloneqq f(x,y) - f(0,y)$ for all $x, y \in \mathcal{X}$. Then, $g(0,y) = 0$ for all $y \in \mathcal{X}$. By (5), g satisfies

$$\left\| n^2 g\left(\frac{x_1 + \cdots + x_n}{n}, \frac{y_1 + \cdots + y_n}{n}\right) - \sum_{i=1}^{n}\sum_{j=1}^{n} g(x_i, y_j), f(s,t) \right\| \leq 2\varepsilon + \delta\left(\sum_{i=1}^{n} \|x_i\|^r + 2\sum_{j=1}^{n} \|y_j\|^r\right) + 2\eta(\|s\| + \|t\|) \tag{7}$$

for all $x, x_1, \dots, x_n, y_1, \dots, y_n, s, t \in \mathcal{X}$. Putting $x_1 = n^{k+1}x$, $x_2 = \cdots = x_n = 0$, $y_1 = n^{k+1}y$, $y_2 = \cdots = y_n = 0$ in (7), we gain

$$\left\| \frac{1}{n^{2k}} g(n^k x, n^k y) - \frac{1}{n^{2(k+1)}} g(n^{k+1}x, n^{k+1}y), f(s,t) \right\| \leq \frac{1}{n^{2(k+1)}}[2\varepsilon + \delta n^{r(k+1)}(\|x\|^r + 2\|y\|^r) + 2\eta(\|s\| + \|t\|)] \tag{8}$$

for all $x, y, s, t \in \mathcal{X}$ and all k. Thus, we have

$$\left\| \frac{1}{n^{2l}} g(n^l x, n^l y) - \frac{1}{n^{2m}} g(n^m x, n^m y), f(s,t) \right\| \leq \sum_{k=l}^{m-1} \frac{1}{n^{2(k+1)}}[2\varepsilon + \delta n^{r(k+1)}(\|x\|^r + 2\|y\|^r) + 2\eta(\|s\| + \|t\|)] \tag{9}$$

for all integers l, m $(0 \leq l < m)$ and all $x, y, s, t \in \mathcal{X}$. By (9), the sequence $\{\frac{1}{n^{2k}} g(n^k x, n^k y)\}$ is a Cauchy sequence for each $x, y \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\{\frac{1}{n^{2k}} g(n^k x, n^k y)\}$ converges for each $x, y \in \mathcal{X}$.

Define $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ by:

$$F(x,y) \coloneqq \lim_{k\to\infty} \frac{1}{n^{2k}} g(n^k x, n^k y) \tag{10}$$

for all $x, y \in \mathcal{X}$. By (7), we have

$$\frac{1}{n^{2k}} \left\| n^2 g\left(n^{k-1}\sum_{i=1}^{n} x_i, n^{k-1}\sum_{j=1}^{n} y_j\right) - \sum_{i=1}^{n}\sum_{j=1}^{n} g(n^k x_i, n^k y_j), f(s,t) \right\| \leq \frac{1}{n^{2k}}\left[2\varepsilon + n^{kr}\delta\left(\sum_{i=1}^{n} \|x_i\|^r + 2\sum_{j=1}^{n} \|y_j\|^r\right) + 2\eta(\|s\| + \|t\|)\right]$$

for all $x_1, \dots, x_n, y_1, \dots, y_n, s, t \in \mathcal{X}$ and all k. Letting $k \to \infty$ in the aforementioned inequality, we obtain that F satisfies (3). By Theorem 1, F is a bi-Jensen mapping. Setting $l = 0$ and taking $m \to \infty$ in (9), one can obtain inequality (6).

Define $G_x, G'_y : \mathcal{X} \to \mathcal{Y}$ by $G_x(y) \coloneqq F(x,y) - F(x,0)$ and $G'_y(x) \coloneqq F(x,y) - F(0,y)$ for all $x, y \in \mathcal{X}$. By Lemma 2, G_x is additive for all $x \in \mathcal{X}$ and G'_y is additive for all $y \in \mathcal{X}$. Since $F(x,0) = F(0,y) = 0$ for all $x, y \in \mathcal{X}$, we have $G_x(y) = G'_y(x) = F(x,y)$ for all $x, y \in \mathcal{X}$. Hence, F is bi-additive.

Let $G : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be another bi-additive mapping satisfying (6). Then, we have

$$\begin{aligned} \|F(x,y) - G(x,y), f(s,t)\| &= \lim_{k\to\infty} \frac{1}{n^{2k}} \|f(n^k x, n^k y) - f(0, n^k y) - G(n^k x, n^k y), f(s,t)\| \\ &\leq \lim_{k\to\infty} \frac{1}{n^{2k}}\left(\frac{2}{n^2 - 1}[\varepsilon + \eta(\|s\| + \|t\|)] + \frac{n^{r(k+1)}\delta}{n^2 - n^r}(\|x\|^r + 2\|y\|^r)\right) = 0 \end{aligned}$$

for all $x, y, s, t \in \mathcal{X}$. So $F = G$. □

In [18–20], one can find the concept of quasi-Banach spaces.

Definition 5. Let $\mathcal{X}$ be a real vector space. A *quasi-norm* is a real-valued function on $\mathcal{X}$ satisfying the following:
(i) $\|x\| \geq 0$ for all $x \in \mathcal{X}$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in \mathcal{X}$.
(iii) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in \mathcal{X}$.

The pair $(\mathcal{X}, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on $\mathcal{X}$. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in \mathcal{X}$. In this case, a quasi-Banach space is called a *p-Banach space*.

From now on, assume that $\mathcal{X}$ is a quasi-normed space with quasi-norm $\|\cdot\|$ and that $\mathcal{Y}$ is a p-Banach space with p-norm $\|\cdot\|_{\mathcal{Y}}$. Let K be the modulus of concavity of $\|\cdot\|_{\mathcal{Y}}$.

We will use the following lemma in the proof of the next theorem.

Lemma 3. [21] *Let $0 \leq p \leq 1$ and let $x_1, x_2, \ldots, x_n$ be non-negative real numbers. Then,*

$$(x_1 + x_2 + \cdots + x_n)^p \leq x_1^p + x_2^p + \cdots + x_n^p.$$

The following theorem proves the stability of equation (3) in quasi-Banach spaces.

Theorem 3. *Let $r \in (0, 2)$, $\varepsilon > 0$, $\delta \geq 0$ and let $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(x, 0) = 0$ such that*

$$\left\| n^2 f\left(\frac{x_1 + \cdots + x_n}{n}, \frac{y_1 + \cdots + y_n}{n}\right) - \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i, y_j)\right\|_{\mathcal{Y}} \leq \varepsilon + \delta\left(\sum_{i=1}^{n}\|x_i\|^r + \sum_{j=1}^{n}\|y_j\|^r\right) \tag{11}$$

for all $x, x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{X}$. Then, there exists a unique bi-additive mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x, y) - f(0, y) - F(x, y)\|_{\mathcal{Y}} \leq \left[\frac{2\varepsilon^p}{n^{2p} - 1} + \frac{n^{pr}\delta^p}{n^{2p} - n^{pr}}(\|x\|^{pr} + 2\|y\|^{pr})\right]^{\frac{1}{p}} \tag{12}$$

for all $x, y \in \mathcal{X}$.

Proof. Letting $x_1 = n^{k+1}x, x_2 = \cdots = x_n = 0, y_1 = n^{k+1}y, y_2 = \cdots = y_n = 0$ in (11), we gain

$$\begin{aligned} &\left\| \frac{1}{n^{2k}} f(n^k x, n^k y) - \frac{1}{n^{2(k+1)}} f(n^{k+1}x, n^{k+1}y) - \frac{n-1}{n^{2(k+1)}} f(0, n^{k+1}y)\right\|_{\mathcal{Y}} \\ &\leq \frac{1}{n^{2(k+1)}}[\varepsilon + \delta n^{r(k+1)}(\|x\|^r + \|y\|^r)] \end{aligned}$$

for all $x, y \in \mathcal{X}$ and all k. Putting $x = 0$ in the aforementioned inequality, we obtain

$$\left\| \frac{1}{n^{2k}} f(0, n^k y) - \frac{n}{n^{2(k+1)}} f(0, n^{k+1}y)\right\|_{\mathcal{Y}} \leq \frac{1}{n^{2(k+1)}}[\varepsilon + \delta n^{r(k+1)}\|y\|^r]$$

for all $y \in \mathcal{X}$ and all k. By the aforementioned two inequalities, we have

$$\begin{aligned} &\left\| \frac{1}{n^{2k}}[f(n^k x, n^k y) - f(0, n^k y)] - \frac{1}{n^{2(k+1)}}[f(n^{k+1}x, n^{k+1}y) - f(0, n^{k+1}y)]\right\|_{\mathcal{Y}}^p \\ &\leq \frac{1}{n^{2p(k+1)}}[2\varepsilon^p + \delta^p n^{pr(k+1)}(\|x\|^{pr} + 2\|y\|^{pr})] \end{aligned} \tag{13}$$

for all $x, y \in \mathcal{X}$ and all k. Thus, we have

$$\left\| \frac{1}{n^{2l}}[f(n^l x, n^l y) - f(0, n^l y)] - \frac{1}{n^{2m}}[f(n^m x, n^m y) - f(0, n^m y)] \right\|_{\mathcal{Y}}^p \leq \sum_{k=l}^{m-1} \frac{1}{n^{2p(k+1)}}[2\varepsilon^p + \delta^p n^{pr(k+1)}(\|x\|^{pr} + 2\|y\|^{pr})] \tag{14}$$

for all integers l, m $(0 \leq l < m)$ and all $x, y \in \mathcal{X}$. By (14), the sequence $\{\frac{1}{n^{2k}}[f(n^k x, n^k y) - f(0, n^k y)]\}$ is a Cauchy sequence for all $x, y \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\{\frac{1}{n^{2k}}[f(n^k x, n^k y) - f(0, n^k y)]\}$ converges for all $x, y \in \mathcal{X}$.

Define $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ by:

$$F(x, y) := \lim_{k\to\infty} \frac{1}{n^{2k}}[f(n^k x, n^k y) - f(0, n^k y)]$$

for all $x, y \in \mathcal{X}$. Setting $x_1 = \cdots = x_n = 0$ in (11), we gain

$$\left\| n^2 f\left(0, \frac{1}{n}\sum_{j=1}^n y_j\right) - n\sum_{j=1}^n f(0, y_j) \right\|_{\mathcal{Y}} \leq \varepsilon + \delta \sum_{j=1}^n \|y_j\|^r$$

for all $y_1, \ldots, y_n \in \mathcal{X}$. By (11), the aforementioned inequality and Lemma 3, we have

$$\begin{aligned}
&\frac{1}{n^{2pk}} \left\| n^2 f\left(n^{k-1}\sum_{i=1}^n x_i, n^{k-1}\sum_{j=1}^n y_j\right) - n^2 f\left(0, n^{k-1}\sum_{j=1}^n y_j\right) - \sum_{i=1}^n\sum_{j=1}^n [f(n^k x_i, n^k y_j) - f(0, n^k y_j)] \right\|_{\mathcal{Y}}^p \\
&\leq \frac{1}{n^{2pk}}\left[2\varepsilon^p + n^{krp}\delta^p\left(\sum_{i=1}^n \|x_i\|^{pr} + 2\sum_{j=1}^n \|y_j\|^{pr}\right)\right] \\
&= 2\varepsilon^p\left(\frac{1}{n^{2p}}\right)^k + n^{(r-2)pk}\delta^p\left(\sum_{i=1}^n \|x_i\|^{pr} + 2\sum_{j=1}^n \|y_j\|^{pr}\right)
\end{aligned}$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{X}$ and all k. Letting $k \to \infty$ in the aforementioned inequality, we obtain that F satisfies (3).

Define $G_x, G'_y : \mathcal{X} \to \mathcal{Y}$ by $G_x(y) := F(x, y) - F(x, 0)$ and $G'_y(x) := F(x, y) - F(0, y)$ for all $x, y \in \mathcal{X}$. By Lemma 2, G_x is additive for all $x \in \mathcal{X}$ and G'_y is additive for all $y \in \mathcal{X}$. Since $F(x, 0) = F(0, y) = 0$ for all $x, y \in \mathcal{X}$, we have $G_x(y) = G'_y(x) = F(x, y)$ for all $x, y \in \mathcal{X}$. Hence, F is bi-additive. Setting $l = 0$ and taking $m \to \infty$ in (14), one can obtain inequality (12).

To prove the uniqueness of F, let $G : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be another bi-additive mapping satisfying (12). Then, we have

$$\begin{aligned}
\|F(x, y) - G(x, y)\|_{\mathcal{Y}} &= \lim_{k\to\infty} \frac{1}{n^{2k}} \|f(n^k x, n^k y) - f(0, n^k y) - G(n^k x, n^k y)\|_{\mathcal{Y}} \\
&\leq \lim_{k\to\infty} \frac{1}{n^{2k}}\left[\frac{2\varepsilon^p}{n^{2p} - 1} + \frac{n^{pr(k+1)}\delta^p}{n^{2p} - n^{pr}}(\|x\|^{pr} + 2\|y\|^{pr})\right]^{\frac{1}{p}} = 0
\end{aligned}$$

for all $x, y \in \mathcal{X}$. So $F = G$. □

Taking $n = 2$ and $\delta = 0$ in Theorem 3, we obtain the following corollary. The result coincides with the one of Corollary 4 in [22].

Corollary 1. *Let $\varepsilon > 0$ be fixed. Suppose that $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(x, 0) = 0$ such that*

$$\left\| 4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) - f(x, z) - f(x, w) - f(y, z) - f(y, w) \right\|_{\mathcal{Y}} \leq \varepsilon$$

for all $x, y, z, w \in \mathcal{X}$. Then, there exists a unique bi-additive mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(x,y) - f(0,y) - F(x,y)\|_{\mathcal{Y}} \leq \varepsilon \left(\frac{2}{4^p - 1}\right)^{\frac{1}{p}}$$

for all $x, y \in \mathcal{X}$.

3 Conclusion

We demonstrated the stability of the multi-variable bi-Jensen functional equation (3) as the duplicative fusion equation of the multi-variable Jensen functional equation:

$$nf\left(\frac{x_1 + \cdots + x_n}{n}\right) = f(x_1) + \cdots + f(x_n).$$

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