Research Article

Goutam Haldar* and Abhijit Banerjee

Characterizations of entire solutions for the system of Fermat-type binomial and trinomial shift equations in $\mathbb{C}^{n\#}$

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Abstract: In this article, we investigate the existence and the precise form of finite-order transcendental entire solutions of some system of Fermat-type quadratic binomial and trinomial shift equations in $\mathbb{C}^n$. Our results are the generalizations of the results of [H. Y. Xu, S. Y. Liu, and Q. P. Li, *Entire solutions for several systems of nonlinear difference and partial differential-difference equations of Fermat-type*, J. Math. Anal. Appl. **483** (2020), 123641, 1–22, DOI: https://doi.org/10.1016/j.jmaa.2019.123641.] and [H. Y. Xu and Y. Y. Jiang, *Results on entire and meromorphic solutions for several systems of quadratic trinomial functional equations with two complex variables*, RACSAM **116** (2022), 8, DOI: https://doi.org/10.1007/s13398-021-01154-9.] to a large extent. Most interestingly, as a consequence of our main result, we have shown that the system of quadratic trinomial shift equation has no solution when it reduces to a system of quadratic trinomial difference equation. In addition, some examples relevant to the content of the article have been exhibited.

Keywords: functions of several complex variables, Fermat-type shift equations, entire solutions, Nevanlinna theory

MSC 2020: 30D35, 39A45

1 Introduction

It is well known that for $m \geq 3$, the Fermat equation $x^m + y^m = 1$ does not admit nontrivial solutions in rational numbers, but it does so for $m = 2$. We refer the reader to take a glance on [1,2]. Using the Nevanlinna theory [3] as a tool, for the Fermat-type functional equation

$$f^m(z) + g^m(z) = 1, \tag{1.1}$$

Montel [4], Iyer [5], and Gross [6] established some remarkable results about the existence of entire and meromorphic solutions of equation (1.1). After that, a number of researchers paid their considerable attentions to study the existence of entire and meromorphic solutions of Fermat-type equation $f^n + g^m = 1$, where f and g are, in general, meromorphic functions and $m, n \in \mathbb{N}$ (see [7–14]).

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* **Corresponding author: Goutam Haldar,** Department of Mathematics, Malda College - 732101, West Bengal, India; Ghani Khan Choudhury Institute of Engineering and Technology, Narayanpur, Malda 732141, West Bengal, India, e-mail: goutamiit1986@gmail.com, goutamiitm@gmail.com
Abhijit Banerjee: Department of Mathematics, University of Kalyani, West Bengal 741235, India, e-mail: abanerjeekal@yahoo.co.in, abanerjeekal@gmail.com

In 2004, for $m = 2$, replacing g by f' in equation (1.1), Yang and Li [14] investigated to find the form of solutions of equation (1.1). They obtained that the transcendental entire solution of $f(z)^2 + f'(z)^2 = 1$ has the form

$$f(z) = \frac{1}{2}\left(Ae^{\alpha z} + \frac{1}{A}e^{-\alpha z}\right),$$

where A and α are nonzero complex constants.

The advent of the difference analogue lemma of the logarithmic derivative (see [15,16]) expedite the research activity to characterize the entire or meromorphic solutions of Fermat-type difference and differential-difference equations (see [17–21]).

With the help of the difference Nevanlinna theory for several complex variables, Cao and Korhonen [22] and Cao and Xu [23] obtained some interesting results on the characterizations of entire and meromorphic solutions for some Fermat-type difference equations and systems of difference equations, which are the extensions from one complex variable to several complex variables. Henceforth, we denote by $z + w = (z_1 + w_1, z_2 + w_2, \ldots, z_n + w_n)$ for any $z = (z_1, z_2, \ldots, z_n)$, $w = (w_1, w_2, \ldots, w_n)$, $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n$, the shift of $f(z)$ is defined by $f(z + c)$, whereas the difference of $f(z)$ is defined by $\Delta_c f(z) = f(z + c) - f(z)$ (see [24]).

2 Solutions to the system of Fermat-type binomial shift equation in $\mathbb{C}^n$

In 2012, Liu et al. [25] proved that the transcendental entire solutions with finite-order of the Fermat-type difference equation $f(z)^2 + f(z + c)^2 = 1$ must satisfy $f(z) = \sin(Az + B)$, where B is a constant and $A = (4k + 1)\pi/2c$, where k is an integer. Xu and Cao [26] have extended the above result to the case of several complex variables as follows.

Theorem A. [26] *Let $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n \setminus \{(0, 0, \ldots, 0)\}$. Then, any nonconstant entire solution $f : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ with finite-order of the Fermat-type difference equation $f(z)^2 + f(z + c)^2 = 1$ has the form of $f(z) = \cos(L(z) + B)$, where L is a linear function of the form $L(z) = a_1z_1 + \cdots + a_nz_n$ on $\mathbb{C}^n$ such that $L(c) = -\pi/2 - 2k\pi$ $(k \in \mathbb{Z})$, and B is a constant on $\mathbb{C}$.*

After that, many researchers have studied some variants of the above equation and obtained some remarkable results in the literature (see [27–30]). In 2016, it was Gao [31] who first investigated the existence and form of entire solutions of the system of differential-difference equation

$$\begin{cases} f_1'(z)^2 + f_2(z + c)^2 = 1, \\ f_2'(z)^2 + f_1(z + c)^2 = 1 \end{cases}$$

in one complex variable and obtained the pair of finite-order transcendental entire solution $(f_1(z), f_2(z))$ that satisfies

$$(f_1(z), f_2(z)) = (\sin(z - ib), \sin(z - ib_1)) \text{ or } (\sin(z + ib), \sin(z + ib_1)),$$

where b and b_1 are constants, and $c = k\pi$, where k is a integer.

Inspired by the above result of Gao [31], Xu et al. [32] in 2020 first converted the above Theorem A into the Fermat-type systems of shift equations and obtained the following result.

Theorem B. [32] *Let $c = (c_1, c_2)$ be a constant in $\mathbb{C}^2$. Then, any pair of transcendental entire solutions with finite-order for the system of Fermat-type difference equations*

$$\begin{cases} f_1(z_1, z_2)^2 + (f_2(z_1 + c_1, z_2 + c_2))^2 = 1, \\ f_2(z_1, z_2)^2 + (f_1(z_1 + c_1, z_2 + c_2))^2 = 1 \end{cases} \tag{2.1}$$

have the following forms:

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+B_1} + e^{-(L(z)+B_1)}}{2}, \frac{A_{21}e^{L(z)+B_1} + A_{22}e^{-(L(z)+B_1)}}{2}\right),$$

where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$, B_1 *is a constant in* $\mathbb{C}$, *and* c, A_{21}, *and* A_{22} *satisfy one of the following cases*:
(*i*) $L(c) = 2k\pi i$, $A_{21} = -i$, *and* $A_{22} = i$, *or* $L(c) = (2k+1)\pi i$, $A_{21} = i$, *and* $A_{22} = -i$, *here and below* k *is an integer*;
(*ii*) $L(c) = (2k + 1/2)\pi i$, $A_{21} = -1$, *and* $A_{22} = -1$, *or* $L(c) = (2k - 1/2)\pi i$, $A_{21} = 1$ *and* $A_{22} = 1$.

From the stand point of Theorem B, it is natural to consider the following system of shift equation, namely the generalized binomial shift equation:

$$\begin{cases} f_1(z)^2 + P(z)^2 f_2(z+c)^2 = Q(z), \\ f_2(z)^2 + P(z)^2 f_1(z+c)^2 = Q(z), \end{cases} \tag{2.2}$$

where $f_j : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ be entire for $j = 1, 2$, $P(z)$, and $Q(z)$ are nonzero polynomials in $\mathbb{C}^n$, and $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n$.

Before we state the main results of this article, let us set the following:

$$\Psi(z) = \sum_{i_1=1}^{{}^nC_2} H_{i_1}^2(s_{i_1}^2) + \sum_{i_2=1}^{{}^nC_3} H_{i_2}^3(s_{i_2}^3) + \sum_{i_3=1}^{{}^nC_4} H_{i_3}^4(s_{i_3}^4) + \cdots + \sum_{i_{n-2}=1}^{{}^nC_{n-1}} H_{i_{n-2}}^{n-1}(s_{i_{n-2}}^{n-1}) + H_{n-1}^n(s_{n-1}^n), \tag{2.3}$$

where $H_{i_1}^2$ is a polynomial in $s_{i_1}^2 \coloneqq d_{i_1 j_1}^2 z_{j_1} + d_{i_1 j_2}^2 z_{j_2}$ with $d_{i_1 j_1}^2 c_{j_1} + d_{i_1 j_2}^2 c_{j_2} = 0$, $1 \le i_1 \le {}^nC_2$, and $1 \le j_1 < j_2 \le n$; $H_{i_2}^3$ is a polynomial in $s_{i_2}^3 \coloneqq d_{i_2 j_1}^3 z_{j_1} + d_{i_2 j_2}^3 z_{j_2} + d_{i_2 j_3}^3 z_{j_3}$ with $d_{i_2 j_1}^3 c_{j_1} + d_{i_2 j_2}^3 c_{j_2} + d_{i_2 j_3}^3 z_3 = 0$, $1 \le i_2 \le {}^nC_3$, and $1 \le j_1 < j_2 < j_3 \le n \ldots$; $H_{i_{n-2}}^{n-1}$ is a polynomial in $s_{i_{n-2}}^{n-1} \coloneqq d_{i_{n-2} j_1} z_{j_1} + d_{i_{n-2} j_2} z_{j_2} + \cdots + d_{i_{n-2} j_{n-1}} z_{j_{n-1}}$ with $d_{i_{n-2} j_1} c_{j_1} + d_{i_{n-2} j_2} c_{j_2} + \cdots + d_{i_{n-2} j_{n-1}} c_{j_{n-1}} = 0$, $1 \le i_{n-2} \le {}^nC_{n-1}$, and $1 \le j_1 < j_2 < \cdots < j_{n-1} \le n$; and H_{n-1}^n is a polynomial in $s_{n-1}^n \coloneqq d_{i_{n-1}1} z_1 + d_{i_{n-1}2} z_2 + \cdots + d_{i_{n-1}n} z_n$ with $d_{i_{n-1}1} c_1 + d_{i_{n-1}2} c_2 + \cdots + d_{i_{n-1}n} c_n = 0$, where for each k, the representation of s_i^k in terms of the conditions of $j_1, j_2, \ldots, j_k$ is unique.

Now, we state our first result as follows.

Theorem 2.1. *Let* $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n \setminus \{(0, 0, \ldots, 0)\}$. *If* $(f_1(z), f_2(z))$ *be a pair of finite-order transcendental entire solution for the system* (2.2), *then* $P(z)$ *reduces to a constant, say* B *with* $B^2 = 1$; $Q(z) = L_1(z) + \Psi(z) + \xi_1$, *where* $\Psi(z)$ *is a polynomial as defined in equation* (2.3); $L_1(z) = \sum_{=0}^{n} e_j z_j$ *with* $L_1(c) = 0$, $e_j \in \mathbb{C}$, $j = 1, 2, \ldots, n$, *and* $(f_1(z), f_2(z))$ *takes one of the following forms*:

I.

$$f_1(z) = \frac{1}{2}(h_1(z)e^{i(L(z)+\Psi(z)+\xi)} + h_3(z)e^{-i(L(z)+\Psi(z)+\xi)})$$

$$f_2(z) = \frac{1}{2}\left(Ah_1(z)e^{i(L(z)+\Psi(z)+\xi')} + \frac{1}{A}h_3(z)e^{-i(L(z)+\Psi(z)+\xi')}\right),$$

where h_1, *and* h_3 *are polynomials in* s_1 *with* $h_1 h_3 = Q$, $L(z) = \sum_{j=1}^{n} \alpha_j z_j$ *such that* $e^{2iL(c)} = -1$, *and* $e^{2i(\xi-\xi')} = A^2$ *and* $\Phi(z)$ *is a polynomial defined as in equation* (2.3), $A(\neq 0)$, ξ, ξ', $\alpha_j \in \mathbb{C}$.

II.

$$f_1(z) = \frac{1}{2}(h_1(z)e^{i(L(z)+\Psi(z)+\xi)} + h_3(z)e^{-i(L(z)+\Psi(z)+\xi)})$$

$$f_2(z) = \frac{1}{2}\left(\frac{1}{A}h_1(z)e^{i(L(z)+\Psi(z)-\xi')} + Ah_3(z)e^{-i(L(z)+\Psi(z)-\xi')}\right),$$

where h_1 *and* h_3 *are polynomials in* s_1 *with* $h_1 h_3 = Q$, $L(z) = \sum_{j=1}^{n} \alpha_j z_j$ *such that* $e^{2iL(c)} = 1$ *and* $e^{2i(\xi+\xi')} = -1/A^2$ *and* $\Phi(z)$ *is a polynomial defined as in equation* (2.3), $A(\neq 0)$, ξ, ξ', $\alpha_j \in \mathbb{C}$.

Remark 2.2. In Theorem 2.1, let $n = 2$, $P(z) = Q(z) = 1$, $A = 1$, $H(s) \equiv 0$, and $L(z) = iL_1(z)$ with $L_1(z) = a_1z_1 + a_2z_2$, $\xi = i\xi_1$, and $\xi' = i\xi_1'$, where a_1, a_2, ξ_1, and ξ_1' are all constants in $\mathbb{C}$. Then, we easily obtain Theorem B from Theorem 2.1. Hence, our result is more general than Theorem B.

Remark 2.3. Let $Q(z) = k$, where k is a nonzero constant in $\mathbb{C}$. Then, from Theorem 2.1, it follows that h_1 and h_2 are both constants in $\mathbb{C}$.

Now, we exhibit two examples showing that our results are precise.

Example 2.4. Let $L(z) = z_1 + z_2$, $P(z) \equiv 1$, $Q(z) = (2z_1 + 3z_2)^{10}$, $c = (c_1, c_2) = \left((2k+1)\frac{3\pi}{2}, -(2k+1)\pi\right)$, and k being an integer. In addition, let $\Psi(z) = \left(z_1 + \frac{3}{2}z_2\right)^3$, and ξ and ξ' in $\mathbb{C}$ such that $\xi - \xi' = k_1\pi$, $k_1 \in \mathbb{C}$. Then, one can easily verify that (f_1, f_2), where

$$f_1(z) = \frac{1}{2}((2z_1 + 3z_2)^4 e^{i(L(z)+\Psi(z)+\xi)} + (2z_1 + 3z_2)^6 e^{-i(L(z)+\Psi(z)+\xi)}),$$
$$f_2(z) = \frac{1}{2}((2z_1 + 3z_2)^4 e^{i(L(z)+\Psi(z)+\xi')} + (2z_1 + 3z_2)^6 e^{-i(L(z)+\Psi(z)+\xi')}),$$

is a solution to the system (2.2).

Example 2.5. Let $Q(z) = (2z_1 - 3z_2 + z_3)^7$, $\Psi(z) = (z_1 + z_2 - z_3)^3$, $P(z) \equiv 1$, and $c = (c_1, c_2, c_3) = (2, 3, 5)$. Let $L(z) = a_1z_1 + a_2z_2 + a_3z_3$ be such that $2a_1 + 3a_2 + 5a_3 = k\pi$, and choose ξ, ξ' in $\mathbb{C}$ such that $\xi + \xi' = (2k_1 + 1)\pi/2$, k, and k_1 being integers and a_1, a_2, and a_3 are constants in $\mathbb{C}$. Then, one can easily verify that (f_1, f_2), with

$$f_1(z) = \frac{1}{2}[(2z_1 - 3z_2 + z_3)^3 e^{i(L(z)+\Psi(z)+\xi)} + (2z_1 - 3z_2 + z_3)^4 e^{-i(L(z)+\Psi(z)+\xi)}],$$
$$f_2(z) = \frac{1}{2}[(2z_1 - 3z_2 + z_3)^3 e^{i(L(z)+\Psi(z)-\xi')} + (2z_1 - 3z_2 + z_3)^4 e^{-i(L(z)+\Psi(z)-\xi')}],$$

is a solution to the system (2.2).

3 Existence of solutions of quadratic trinomial shift equation in $\mathbb{C}^n$

Let us recall another quadratic trinomial function equation

$$f(z)^2 + 2\alpha f(z)g(z) + g(z)^2 = 1, \tag{3.1}$$

where α is a constant in $\mathbb{C}$. Note that when $\alpha = 0$, equation (3.1) is exactly the equation (1.1) with $m = 2$. So, it will be interesting to investigate the existence and forms of entire and meromorphic solutions of equation (3.1) when $\alpha \neq 0$. In this direction, Saleeby [33] investigated the entire and meromorphic solutions of equation (3.1) on $\mathbb{C}^n$ and discovered that the transcendental entire solutions of equation (3.1), $\alpha^2 \neq 1$, must be of the form $f(z) = \frac{1}{\sqrt{2}}\left(\frac{\cos(h(z))}{\sqrt{1+\alpha}} + \frac{\sin(h(z))}{\sqrt{1-\alpha}}\right)$ and $g(z) = \frac{1}{\sqrt{2}}\left(\frac{\cos(h(z))}{\sqrt{1+\alpha}} - \frac{\sin(h(z))}{\sqrt{1-\alpha}}\right)$, where h is entire $\mathbb{C}^n$. The meromorphic solutions of equation (3.1) must be of the form $f(z) = \frac{\alpha_1 - \alpha_2\beta(z)^2}{(\alpha_1 - \alpha_2)\beta(z)}$ and $g(z) = \frac{1-\beta(z)^2}{(\alpha_1-\alpha_2)\beta(z)}$, where $\beta(z)$ is meromorphic in $\mathbb{C}^n$ and $\alpha_1 = -\alpha + \sqrt{\alpha^2 - 1}$, $\alpha_2 = -\alpha - \sqrt{\alpha^2 - 1}$.

In 2016, Liu and Yang [34] investigated the existence and the form of entire solutions of some quadratic trinomial functional equations in the complex plane $\mathbb{C}$, and obtained the following results.

Theorem C. [34] *If $\alpha \neq 0, \pm 1$, then equation*

$$f(z)^2 + 2\alpha f(z)f'(z) + f'(z)^2 = 1$$

has no transcendental meromorphic solutions.

Theorem D. [34] *If $\alpha \neq 0, \pm 1$, then the transcendental entire solutions with finite-order of the equation*

$$f(z)^2 + 2\alpha f(z)f(z+c) + f(z+c)^2 = 1 \tag{3.2}$$

must be of order 1.

Recently, corresponding to equation (3.2), Xu and his co-authors [35,36] have extended Theorems C and D from the quadratic trinomial shift equation to systems of trinomial difference equations in one as well as several complex variables. We list a result corresponding to several complex variables below.

Theorem E. [36] *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then, any pair of finite-order transcendental entire solutions for the system of trinomial difference equations*

$$\begin{cases} f_1(z)^2 + 2\alpha f_1(z)f_2(z+c) + f_2(z+c)^2 = 1, \\ f_2(z)^2 + 2\alpha f_2(z)f_1(z+c) + f_1(z+c)^2 = 1 \end{cases} \tag{3.3}$$

must be one of the following forms:

I.

$$f_1(z) = \frac{1}{2}\left(\frac{\cos(\gamma(z)+b_1)}{\sqrt{1+\alpha}} + \frac{\sin(\gamma(z)+b_1)}{\sqrt{1-\alpha}}\right),$$

$$f_2(z) = \frac{1}{2}\left(\frac{\cos(\gamma(z)+b_2)}{\sqrt{1+\alpha}} + \frac{\sin(\gamma(z)+b_2)}{\sqrt{1-\alpha}}\right),$$

where $\gamma(z) = L(z) + H(s)$, $L(z) = a_1z_1 + a_2z_2$, $L(c) := a_1c_1 + a_2c_2$, $a_1, a_2, b_1, b_2 \in \mathbb{C}$, $H(s)$ is a polynomial in $s := c_2z_1 - c_1z_2$, and $L(z)$, b_1, and b_2 satisfy

$$e^{2iL(c)} = \frac{-\alpha + \sqrt{\alpha^2-1}}{-\alpha - \sqrt{\alpha^2-1}}, \quad e^{2i(b_1-b_2)} = 1.$$

II.

$$f_1(z) = \frac{1}{2}\left(\frac{\cos(\gamma(z)+b_1)}{\sqrt{1+\alpha}} + \frac{\sin(\gamma(z)+b_1)}{\sqrt{1-\alpha}}\right),$$

$$f_2(z) = \frac{1}{2}\left(\frac{\cos(\gamma(z)+b_2)}{\sqrt{1+\alpha}} - \frac{\sin(\gamma(z)+b_2)}{\sqrt{1-\alpha}}\right),$$

where $\gamma(z)$ is stated as in I, and $L(z)$, b_1, and b_2 satisfy

$$e^{2iL(c)} = 1, \quad e^{2i(b_1-b_2)} = 1.$$

From the above discussion, it is natural to ask the following questions.

Question 3.1. Can we extend Theorem E from $\mathbb{C}^2$ to $\mathbb{C}^n$ for any positive integer n?

We also note that the all the authors in this specific field used the term “difference” equations where as all equations are mainly governed by the shift of the function and there is hardly any presence of the difference. Hence, the next question is most relevant.

Question 3.2. What can be said about the pair of entire solutions of the trinomial difference equation

$$\begin{cases} f_1(z)^2 + 2\alpha f_1(z)\Delta_c f_2 + (\Delta_c f_2)^2 = 1 \\ f_2(z)^2 + 2\alpha f_2(z)\Delta_c f_1 + (\Delta_c f_1)^2 = 1? \end{cases} \tag{3.4}$$

Let $\tilde{L}(f) = a_1 f(z+c) + a_2 f(z)$ and $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n$, where $a_1(\neq 0), \alpha, b, h, a_2 \in \mathbb{C}$. Motivated by the above questions, let us consider the following system of quadratic trinomial shift equation:

$$\begin{cases} af_1(z)^2 + 2hf_1(z)\tilde{L}(f_2) + b\tilde{L}(f_2)^2 = 1 \\ af_2(z)^2 + 2hf_2(z)\tilde{L}(f_1) + b\tilde{L}(f_1)^2 = 1. \end{cases} \tag{3.5}$$

Motivated by Theorem E, we consider equation (3.5), which is more general setting than equation (3.3) to obtain our main result, which provides the answers of Questions 3.1 and 3.2 in a compact and convenient manner.

Before stating the result, let us first set that

$$A_1 := \frac{1}{\sqrt{1+\alpha}} + \frac{i}{\sqrt{1-\alpha}} \quad \text{and} \quad A_2 := \frac{1}{\sqrt{1+\alpha}} - \frac{i}{\sqrt{1-\alpha}},$$

where $\alpha = h/\sqrt{ab}\,(\neq 0, \pm 1)$ is a constant in $\mathbb{C}$. Now, we state our result as follows.

Theorem 3.3. *Let a, b, and h be constants in $\mathbb{C}$ such that $ab \neq 0$ and $h \neq 0, \pm\sqrt{ab}$. Then, any pair of finite-order transcendental entire solutions $(f_1(z), f_2(z))$ of the system* (3.5) *must be one of the following forms*:

I.

$$f_1(z) = \frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{\frac{1}{4}}\left[\frac{\cos(L(z)+\Psi(z))}{\sqrt{\sqrt{ab}+h}} + \frac{\sin(L(z)+\Psi(z))}{\sqrt{\sqrt{ab}-h}}\right],$$

$$f_2(z) = \frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{\frac{1}{4}}\left[\frac{\cos(L(z)+\Psi(z)+\eta)}{\sqrt{\sqrt{ab}+h}} + \frac{\sin(L(z)+\Psi(z)+\eta)}{\sqrt{\sqrt{ab}-h}}\right],$$

where $L(z) = \sum_{j=1}^{n} a_j z_j$, and $\Psi(z)$ is a polynomial defined as in equation (2.3), *$a_j \in \mathbb{C}$ such that*

$$e^{2i\eta} = 1, \quad (a + ba_2^2 - ba_1^2)e^{i\eta} + 2a_2 h = 0,$$

$\eta \in \mathbb{C}$, and $L(z)$ satisfies the relation

$$e^{iL(c)} = \frac{\sqrt{b}\,a_1 A_2 e^{i\eta}}{\sqrt{a}A_1 - \sqrt{b}\,a_2 A_2 e^{i\eta}} = \frac{\sqrt{b}\,a_1 A_2 e^{-i\eta}}{\sqrt{a}A_1 - \sqrt{b}\,a_2 A_2 e^{-i\eta}}$$
$$= \frac{\sqrt{a}A_2 e^{i\eta} - \sqrt{b}\,a_2 A_1}{\sqrt{b}\,a_1 A_1} = \frac{\sqrt{a}A_2 - \sqrt{b}\,a_2 A_1 e^{i\eta}}{\sqrt{b}\,a_1 A_1 e^{i\eta}}.$$

II.

$$f_1(z) = \frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{\frac{1}{4}}\left[\frac{\cos(L(z)+\Psi(z)+\xi)}{\sqrt{\sqrt{ab}+h}} + \frac{\sin(L(z)+\Psi(z)+\xi)}{\sqrt{\sqrt{ab}-h}}\right],$$

$$f_2(z) = \frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{\frac{1}{4}}\left[\frac{\cos(L(z)+\Psi(z)+\xi-\eta)}{\sqrt{\sqrt{ab}+h}} + \frac{\sin(L(z)+\Psi(z)+\xi-\eta)}{\sqrt{\sqrt{ab}-h}}\right],$$

where $L(z) = \sum_{j=1}^{n} a_j z_j$, and $\Psi(z)$ is a polynomial defined as in equation (2.3), *$a_j \in \mathbb{C}$ such that*

$$e^{2i\eta} = 1, \quad a + ba_2^2 - ba_1^2 = 2\sqrt{ab}\,a_2 e^{i\eta},$$

$\xi, \eta \in \mathbb{C}$, and $L(z)$ satisfies the relation

$$e^{iL(c)} = \frac{\sqrt{a} - \sqrt{b}\,a_2 e^{i\eta}}{\sqrt{b}\,a_1 e^{i\eta}} = \frac{\sqrt{b}\,a_1 e^{i\eta}}{\sqrt{a} - \sqrt{b}\,a_2 e^{i\eta}} = \frac{\sqrt{b}\,a_1 e^{-i\eta}}{\sqrt{a} - \sqrt{b}\,a_2 e^{-i\eta}} = \frac{\sqrt{a} - \sqrt{b}\,a_2 e^{-i\eta}}{\sqrt{b}\,a_1 e^{-i\eta}}.$$

Remark 3.4. In Theorem 2.1, let $n = 2$, $a = b = 1$, $a_1 = 1$, $a_2 = 0$, and $c = (c_1, c_2) \in \mathbb{C}^2$ with $d_1 = c_2$ and $d_2 = c_1$. Choose $\eta \in \mathbb{C}$ be such that $e^{2i\eta} = 1$. Then, one can easily obtain Theorem E from Theorem 3.3. Therefore, Theorem 3.3 is more general than Theorem E.

Remark 3.5. Let $a = b = 1$, $a_1 = 1$, $a_2 = -1$, and $\alpha^2 \neq 0, 1$, where $\alpha \in \mathbb{C}$. Then, equation (3.5) becomes equation (3.4). Then, one can easily obtain the following Corollary.

Corollary 3.6. *The system of trinomial difference equation* (3.4) *does not possess any pair of finite-order transcendental entire solutions.*

The following examples show that our result is precise.

Example 3.7. In Theorem 3.3, let $a = b = 1$, $h = 2$, $a_1 = \sqrt{6}$, $a_2 = 1$, and $e^{2i\eta} = 1$. Suppose $L(z) = z_1 + 2z_2 + 3z_3$ and $\Psi(z) = (z_1 - z_2 + z_3)^{10}$. Choose $c = (c_1, c_2, c_3) \in \mathbb{C}^3$ be such that

$$\begin{cases} c_1 + 2c_2 + 3c_3 = -i\log\left(\dfrac{\sqrt{3}+1}{-\sqrt{2}}\right) \\ c_1 - c_2 + c_3 = 0. \end{cases}$$

Then, one can easily verify that (f_1, f_2) is a pair of transcendental entire solution of equation (3.5), where

$$f_1(z) = \frac{1}{\sqrt{2}}\left(\frac{\cos(L(z)+\Psi(z))}{\sqrt{3}} - i\sin(L(z)+\Psi(z))\right)$$

and

$$f_2(z) = \frac{1}{\sqrt{2}}\left(\frac{\cos(L(z)+\Psi(z)+\eta)}{\sqrt{3}} - i\sin(L(z)+\Psi(z)+\eta)\right).$$

Example 3.8. In Theorem 3.3, let $a = 2$, $b = 1$, $h = 2$, $a_1 = \sqrt{2}$, $a_2 = 4$, and $e^{2i\eta} = -1$. Suppose $L(z) = 2z_1 + 3z_2 + 4z_3$ and $\Psi(z) = (c_2c_3z_1 - 2c_3c_1z_2 + c_1c_2z_3)^{11}$. Choose $c = (c_1, c_2, c_3) \in \mathbb{C}^3$ be such that

$$e^{iL(c)} = -\frac{\sqrt{\sqrt{2}+1} + \sqrt{\sqrt{2}-1}}{(2\sqrt{2}+1)\sqrt{\sqrt{2}-1} + (2\sqrt{2}-1)\sqrt{\sqrt{2}+1}}.$$

Then, one can easily verify that (f_1, f_2) is a pair of transcendental entire solution of equation (3.5), where

$$f_1(z) = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)^{\frac{1}{4}}\left(\frac{\cos(L(z)+\Psi(z))}{\sqrt{2+\sqrt{2}}} - \frac{i\sin(L(z)+\Psi(z))}{\sqrt{2-\sqrt{2}}}\right)$$

and

$$f_2(z) = -\frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)^{\frac{1}{4}}\left(\frac{\cos(L(z)+\Psi(z))}{\sqrt{2+\sqrt{2}}} - \frac{i\sin(L(z)+\Psi(z))}{\sqrt{2-\sqrt{2}}}\right).$$

Remark 3.9. If $h = \pm\sqrt{ab}$, then equation (3.5) reduces to $[\sqrt{a}f_1(z) \pm \sqrt{b}\tilde{L}(f_2)]^2 = 1$ and $[\sqrt{a}f_2(z) - \sqrt{b}\tilde{L}(f_1)]^2 = 1$. If $\sqrt{a}f_1(z) - \sqrt{b}\tilde{L}(f_2) = 1$ and $\sqrt{a}f_2(z) \pm \sqrt{b}\tilde{L}(f_1) = 1$, then we must have

$$\begin{cases} (a - ba_2^2)f_1(z) = \sqrt{a} + \sqrt{b}(a_1 + a_2) + 2ba_1a_2f_1(z+c) + ba_1^2f_1(z+2c), \\ (a - ba_2^2)f_2(z) = \sqrt{a} + \sqrt{b}(a_1 + a_2) + 2ba_1a_2f_2(z+c) + ba_1^2f_2(z+2c). \end{cases}$$

Then, it is easy to find the transcendental entire solutions with finite or infinite-order when $a_2 = 0$. For example,

$$(f_1(z), f_2(z)) = \left(\frac{ba_1^2}{a}(e^{(z_1-2z_2)\pi i} + z_1 + z_2), \frac{ba_1^2}{a}(-e^{(z_1-2z_2)\pi i} + z_1 + z_2)\right),$$

$$(f_1(z), f_2(z)) = \left(\frac{ba_1^2}{a}(\sin(e^{(z_1-2z_2)\pi i}) + z_1 + z_2), \frac{ba_1^2}{a}(-\sin(e^{(z_1-2z_2)\pi i}) + z_1 + z_2)\right)$$

are the solutions of equation (3.5).

4 Proofs of the main results

Before we starting the proof of the main results, we present here some necessary lemmas which will play key role to prove the main results of this article.

Lemma 4.1. [37] *Let $f_j \not\equiv 0 (j = 1, 2 \ldots, m;\ m \geq 3)$ be meromorphic functions on $\mathbb{C}^n$ such that $f_1, \ldots, f_{m-1}$ are not constants and $f_1 + f_2 + \cdots + f_m = 1$, as well as*

$$\sum_{j=1}^{m}\left\{N_{n-1}\left(r, \frac{1}{f_j}\right) + (m-1)\overline{N}(r, f_j)\right\} < \lambda T(r, f_j) + O(\log^+ T(r, f_j))$$

holds for $j = 1, \ldots, m-1$ and all r outside possibly a set with finite logarithmic measure, where $\lambda < 1$ is a positive number. Then, $f_m = 1$.

Lemma 4.2. [37] *Let $f_j \not\equiv 0\ (j = 1, 2, 3)$ be meromorphic functions on $\mathbb{C}^n$ such that f_1 are not constant and $f_1 + f_2 + f_3 = 1$, as well as*

$$\sum_{j=1}^{3}\left\{N_2\left(r, \frac{1}{f_j}\right) + 2\overline{N}(r, f_j)\right\} < \lambda T(r, f_j) + O(\log^+ T(r, f_j))$$

holds for all r outside possibly a set with finite logarithmic measure, where $\lambda < 1$ is a positive number. Then, either $f_2 = 1$ or $f_3 = 1$.

Lemma 4.3. [38–40] *For an entire function F on $\mathbb{C}^n$, $F(0) \not\equiv 0$ and put $\rho(n_F) = \rho < \infty$. Then, there exist a canonical function f_F and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_F(z)e^{g_F(z)}$. For the special case $n = 1$, f_F is the canonical product of Weierstrass.*

Lemma 4.4. [41] *If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite-order, then there are only two possible cases:*

(*i*) *the internal function h is a polynomial and the external function g is of finite-order; or*
(*ii*) *the internal function h is not a polynomial but a function of finite-order, and the external function g is of zero order.*

Proof of Theorem 2.1. Let (f_1, f_2) be a pair of finite-order transcendental entire solution of system (2.2). First, we write equation (2.2) as follows:

$$\begin{cases}(f_1(z) + iP(z)f_2(z+c))(f_1(z) - iP(z)f_2(z+c)) = Q(z)\\(f_2(z) + iP(z)f_1(z+c))(f_2(z) - iP(z)f_1(z+c)) = Q(z).\end{cases} \tag{4.1}$$

In view of equation (4.1), Lemmas 4.3 and 4.4, it follows that there exist nonconstant polynomials $h_1(z)$ and $h_2(z)$, and any nonzero polynomials $h_{11}(z)$, $h_{12}(z)$, $h_{21}(z)$, and $h_{22}(z)$ in $\mathbb{C}^n$ with $h_{11}(z)h_{12}(z) = h_{21}(z)h_{22}(z) = Q(z)$ such that

$$\begin{cases}f_1(z) + iP(z)f_2(z+c) = h_{11}(z)e^{ih_1(z)}\\f_1(z) - iP(z)f_2(z+c) = h_{12}(z)e^{-ih_1(z)}\\f_2(z) + iP(z)f_1(z+c) = h_{21}(z)e^{ih_2(z)}\\f_2(z) - iP(z)f_1(z+c) = h_{22}(z)e^{-ih_2(z)},\end{cases}$$

which yield that

$$\begin{cases} f_1(z) = \frac{1}{2}[h_{11}(z)e^{ih_1(z)} + h_{12}(z)e^{-ih_1(z)}]; \\ f_2(z+c) = \frac{1}{2iP(z)}[h_{11}(z)e^{ih_1(z)} - h_{12}(z)e^{-ih_1(z)}]; \\ f_2(z) = \frac{1}{2}[h_{21}(z)e^{ih_2(z)} + h_{22}(z)e^{-ih_2(z)}]; \\ f_1(z+c) = \frac{1}{2iP(z)}[h_{21}(z)e^{ih_2(z)} - h_{22}(z)e^{-ih_2(z)}]. \end{cases} \tag{4.2}$$

Now, after some easy calculations, we obtain from equation (4.2) that

$$\frac{iP(z)h_{11}(z+c)}{-h_{22}(z)}e^{i(h_1(z+c)+h_2(z))} + \frac{iP(z)h_{12}(z+c)}{-h_{22}(z)}e^{-i(h_1(z+c)-h_2(z))} + \frac{h_{21}(z)}{h_{22}(z)}e^{2ih_2(z)} = 1, \tag{4.3}$$

and

$$\frac{iP(z)h_{21}(z+c)}{-h_{12}(z)}e^{i(h_2(z+c)+h_1(z))} + \frac{iP(z)h_{22}(z+c)}{-h_{12}(z)}e^{-i(h_2(z+c)-h_1(z))} + \frac{h_{11}(z)}{h_{12}(z)}e^{2ih_1(z)} = 1. \tag{4.4}$$

By Lemma 4.1, it follows from equations (4.3) and (4.4) that either

$$\frac{iP(z)h_{11}(z+c)}{-h_{22}(z)}e^{i(h_1(z+c)+h_2(z))} = 1 \text{ or } \frac{iP(z)h_{12}(z+c)}{-h_{22}(z)}e^{-i(h_1(z+c)-h_2(z))} = 1,$$

and

$$\frac{iP(z)h_{21}(z+c)}{-h_{12}(z)}e^{i(h_2(z+c)+h_1(z))} = 1 \text{ or } \frac{iP(z)h_{22}(z+c)}{-h_{12}(z)}e^{-i(h_2(z+c)-h_1(z))} = 1.$$

Now, we consider the following four possible cases:

Case 1. Let

$$\begin{cases} \frac{iP(z)h_{11}(z+c)}{-h_{22}(z)}e^{i(h_1(z+c)+h_2(z))} = 1, \\ \frac{iP(z)h_{21}(z+c)}{-h_{12}(z)}e^{i(h_2(z+c)+h_1(z))} = 1. \end{cases} \tag{4.5}$$

Next equations (4.3), (4.4), and (4.5) yield

$$\begin{cases} \frac{iP(z)h_{12}(z+c)}{h_{21}(z)}e^{-i(h_1(z+c)+h_2(z))} = 1, \\ \frac{iP(z)h_{22}(z+c)}{h_{11}(z)}e^{-i(h_2(z+c)+h_1(z))} = 1. \end{cases} \tag{4.6}$$

Since $h_1(z)$ and $h_2(z)$ are two nonconstant polynomials, it yields from equation (4.5) that $h_1(z+c) + h_2(z) = \eta_1$ and $h_2(z+c) + h_1(z) = \eta_2$, where η_1 and η_2 are constants in $\mathbb{C}$. Thus, we assume that $h_1(z) = L(z) + \Psi(z) + \xi$ and $h_2 = -(L(z) + \Psi(z)) + \xi'$, where $L(z) = \sum_{j=1}^{n} a_j z_j$, and $\Psi(z)$ is a polynomial defined as in equation (2.3), and a_j, ξ, ξ' are all constants in $\mathbb{C}$.

From equations (4.5) and (4.6), we obtain $P(z)^2 Q(z+c) = Q(z)$. This implies that $P(z)$ must be constant, say B such that $B^2 = 1$. Thus, we must have $Q(z+c) = Q(z)$, which implies that $Q(z) = L_1(z) + \Psi(z) + \xi_1$, where $\Psi(z)$ is a polynomial as defined in equation (2.3), $L_1(z) = \sum_{=0}^{n} e_j z_j$ with $L_1(c) = 0$ and $e_j \in \mathbb{C}$, $j = 1, 2, \dots, n$. As $Q(z)$ is a periodic function of period c, and $Q(z) = h_{11}(z)h_{12}(z) = h_{21}(z)h_{22}(z)$, we have h_{11}, h_{12}, h_{21}, and h_{22} as c-periodic functions.

Therefore, equations (4.5) and (4.6) give

$$\begin{cases} -iBK_1e^{i(L(c)+\xi+\xi')} = 1; \\ -iBK_2e^{i(-L(c)+\xi+\xi')} = 1; \\ \dfrac{iB}{K_2}e^{-i(L(c)+\xi+\xi')} = 1; \\ \dfrac{iB}{K_1}e^{-i(-L(c)+\xi+\xi')} = 1, \end{cases} \tag{4.7}$$

where $K_1 = h_{11}(z)/h_{22}(z)$ and $K_2 = h_{21}(z)/h_{12}(z)$, which implies that $K_1 = K_2 = A$, a nonzero constant in $\mathbb{C}$. Now, from equation (4.7), we can easily obtain the following.

$$e^{2i(\xi+\xi')} = -1/A^2, \quad \text{say}, \quad \text{and} \quad e^{2iL(c)} = 1.$$

Thus, from equation (4.2), we obtain

$$f_1 = \frac{1}{2}(h_1(z)e^{i(L(z)+\Psi(z)+\xi)} + h_3(z)e^{-i(L(z)+\Psi(z)+\xi)}),$$

and

$$f_2 = \frac{1}{2}\left(\frac{1}{A}h_1(z)e^{i(L(z)+\Psi(z)-\xi')} + Ah_3(z)e^{-i(L(z)+\Psi(z)-\xi')}\right).$$

Case 2. Let

$$\begin{cases} \dfrac{iP(z)h_{11}(z+c)}{-h_{22}(z)}e^{i(h_1(z+c)+h_2(z))} = 1, \\ \dfrac{iP(z)h_{22}(z+c)}{-h_{12}(z)}e^{-i(h_2(z+c)-h_1(z))} = 1. \end{cases} \tag{4.8}$$

As $h_1(z)$ and $h_2(z)$ are polynomials in $\mathbb{C}^n$, from equation (4.8), we have $h_1(z+c) + h_2(z) = \eta_1$ and $-h_2(z+c) + h_1(z) = \eta_2$, where η_1 and η_2 are constants in $\mathbb{C}$. This implies that $h_1(z+2c) + h_1(z) = \eta_1 + \eta_2$, which yields that $h_1(z)$ is constant, a contradiction.

Case 3. Let

$$\begin{cases} \dfrac{iP(z)h_{12}(z+c)}{-h_{22}(z)}e^{i(-h_1(z+c)+h_2(z))} = 1, \\ \dfrac{iP(z)h_{21}(z+c)}{-h_{12}(z)}e^{i(h_2(z+c)+h_1(z))} = 1. \end{cases}$$

Then, by similar arguments as in Case 2, we obtain a contradiction.

Case 4. Let

$$\begin{cases} \dfrac{iP(z)h_{12}(z+c)}{-h_{22}(z)}e^{i(-h_1(z+c)+h_2(z))} = 1, \\ \dfrac{iP(z)h_{22}(z+c)}{-h_{12}(z)}e^{i(-h_2(z+c)+h_1(z))} = 1. \end{cases} \tag{4.9}$$

From equation (4.3), (4.4), and (4.9), we obtain

$$\begin{cases} \dfrac{iP(z)h_{11}(z+c)}{h_{21}(z)}e^{i(h_1(z+c)-h_2(z))} = 1, \\ \dfrac{iP(z)h_{21}(z+c)}{h_{11}(z)}e^{i(h_2(z+c)-h_1(z))} = 1. \end{cases} \tag{4.10}$$

Since $h_1(z)$ and $h_2(z)$ are two nonconstant polynomials in $\mathbb{C}^n$, equation (4.9) yields that $-h_1(z+c) + h_2(z) = \eta_1$ and $-h_2(z+c) + h_1(z) = \eta_2$, where η_1 and η_2 are constants in $\mathbb{C}$. This implies that $-h_1(z+2c) + h_1(z) = \eta_1 + \eta_2$ and $-h_2(z+2c) + h_2(z) = \eta_1 + \eta_2$. Thus, we assume that $h_1(z) = L(z) + \Psi(z) + \xi$ and

$h_2 = L(z) + \Psi(z) + \xi'$, where $L(z) = \sum_{j=1}^{n} a_j z_j$ and $\Psi(z)$ is a polynomial defined as in equation (2.3), $a_j, \xi, \xi' \in \mathbb{C}$. Then, by similar arguments as in Case 1 of Theorem 2.1, we obtain the conclusion II of Theorem 2.1. □

Proof of Theorem 3.3. Let us make a transformation

$$\sqrt{a} f_1(z) = U, \quad \sqrt{b} L(f_2)(z) = V. \tag{4.11}$$

Then, the first equation of equation (3.5) reduces to

$$U^2 + 2\alpha UV + V^2 = 1, \tag{4.12}$$

where $\alpha = h/\sqrt{ab}$. Let

$$U = \frac{1}{\sqrt{2}}(U_1 - V_1), \quad V = \frac{1}{\sqrt{2}}(U_1 + V_1). \tag{4.13}$$

Then, equation (4.12) becomes

$$(1 + \alpha)U_1^2 + (1 - \alpha)V_1^2 = 1. \tag{4.14}$$

Now, in view of Lemmas 4.3 and 4.4, we obtain from (4.14) that

$$\begin{cases} \sqrt{1+\alpha}\, U_1 + i\sqrt{1-\alpha}\, V_1 = e^{ih_1(z)} \\ \sqrt{1+\alpha}\, U_1 - i\sqrt{1-\alpha}\, V_1 = e^{-ih_1(z)}, \end{cases}$$

where $h_1(z)$ is a nonconstant polynomial, from which we can obtain that

$$U_1 = \frac{e^{ih_1(z)} + e^{-ih_1(z)}}{2\sqrt{1+\alpha}}, \quad V_1 = \frac{e^{ih_1(z)} - e^{-ih_1(z)}}{2i\sqrt{1-\alpha}}. \tag{4.15}$$

Therefore, equations (4.11), (4.13) and (4.15) together imply

$$\begin{cases} f_1(z) = \dfrac{1}{2\sqrt{2a}}(A_1 e^{ih_1(z)} + A_2 e^{-ih_1(z)}), \\ \tilde{L}(f_2) = \dfrac{1}{2\sqrt{2b}}(A_2 e^{ih_1(z)} + A_1 e^{-ih_1(z)}). \end{cases} \tag{4.16}$$

Similarly, from the second equation of equation (4.1), we can obtain

$$\begin{cases} f_2(z) = \dfrac{1}{2\sqrt{2a}}(A_1 e^{ih_2(z)} + A_2 e^{-ih_2(z)}) \\ \tilde{L}(f_1) = \dfrac{1}{2\sqrt{2b}}(A_2 e^{ih_2(z)} + A_1 e^{-ih_2(z)}), \end{cases} \tag{4.17}$$

where $h_2(z)$ is a nonconstant polynomial in $\mathbb{C}^n$.

After simple computations, from equations (4.16) and (4.17), we obtain that

$$\begin{aligned} &\sqrt{b}\, a_1 A_1 e^{i(h_1(z+c)+h_2(z))} + \sqrt{b}\, a_1 A_2 e^{i(-h_1(z+c)+h_2(z))} + \sqrt{b}\, a_2 A_1 e^{i(h_2(z)+h_1(z))} + \sqrt{b}\, a_2 A_2 e^{i(h_2(z)-h_1(z))} - \sqrt{a} A_2 e^{2ih_2(z)} \\ &= \sqrt{a} A_1 \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} &\sqrt{b}\, a_1 A_1 e^{i(h_2(z+c)+h_1(z))} + \sqrt{b}\, a_1 A_2 e^{i(-h_2(z+c)+h_1(z))} + \sqrt{b}\, a_2 A_1 e^{i(h_1(z)+h_2(z))} + \sqrt{b}\, a_2 A_2 e^{i(h_1(z)-h_2(z))} - \sqrt{a} A_2 e^{2ih_1(z)} \\ &= \sqrt{a} A_1. \end{aligned} \tag{4.19}$$

Now, we discuss two possible cases.

Case 1. Let $a_2 \neq 0$

Subcase 1.1. Let $h_2(z) - h_1(z) = \eta$, where η is a constant in $\mathbb{C}$. Then, it can be easily seen that $h_1(z) + h_2(z)$ and $h_2(z) + h_1(z + c)$ are nonconstant polynomials in $\mathbb{C}^n$.

Therefore, equations (4.18) and (4.19), respectively, can be rewritten as follows:

$$\sqrt{b}\,a_1A_1e^{i\eta}e^{i(h_1(z+c)+h_1(z))}+\sqrt{b}\,a_1A_2e^{i\eta}e^{i(h_1(z)-h_1(z+c))}+(\sqrt{b}\,a_2A_1-\sqrt{a}\,A_2e^{i\eta})e^{i\eta}e^{2ih_1(z)}=\sqrt{a}\,A_1-\sqrt{b}\,a_2A_2e^{i\eta} \tag{4.20}$$

and

$$\sqrt{b}\,a_1A_1e^{i\eta}e^{i(h_1(z+c)+h_1(z))}+\sqrt{b}\,a_1A_2e^{-i\eta}e^{i(h_1(z)-h_1(z+c))}+(\sqrt{b}\,a_2A_1e^{i\eta}-\sqrt{a}\,A_2)e^{2ih_1(z)}=\sqrt{a}\,A_1-\sqrt{b}\,a_2A_2e^{-i\eta}. \tag{4.21}$$

Note that equation (4.20) implies $\sqrt{b}\,a_2A_1-\sqrt{a}\,A_2e^{i\eta}$ and $\sqrt{a}\,A_1-\sqrt{b}\,a_2A_2e^{i\eta}$ cannot be zero at the same time. Otherwise, from equation (4.20), we obtain that $A_1e^{2ih_1(z+c)}=-A_2$, which implies that $h_1(z+c)$, and hence $h_1(z)$ is constant, a contradiction.

Let $\sqrt{b}\,a_2A_1-\sqrt{a}\,A_2e^{i\eta}\neq 0$ and $\sqrt{a}\,A_1-\sqrt{b}\,a_2A_2e^{i\eta}=0$. Then, we can write equation (4.20) as

$$\sqrt{b}\,a_1A_1e^{i(h_1(z+c)-h_1(z))}+\sqrt{b}\,a_1A_2e^{-i(h_1(z+c)+h_1(z))}=\sqrt{b}\,a_2A_1-\sqrt{a}\,A_2e^{i\eta}.$$

By the second main theorem of Nevanlinna for several complex variables, we obtain

$$\begin{aligned}T(r,e^{-i(h_1(z+c)+h_1(z))})&\leq\overline{N}(r,e^{-i(h_1(z+c)+h_1(z))})+\overline{N}\left(r,\frac{1}{e^{-i(h_1(z+c)+h_1(z))}}\right)+\overline{N}\left(r,\frac{1}{e^{-i(h_1(z+c)+h_1(z))}-w}\right)\\&\quad+S(r,e^{-i(h_1(z+c)+h_1(z))})\\&\leq\overline{N}\left(r,\frac{1}{e^{i(h_1(z+c)-h_1(z))}}\right)+S(r,e^{-i(h_1(z+c)+h_1(z))})\\&\leq S(r,e^{-i(h_1(z+c)+h_1(z))})+S(r,e^{-i(h_1(z+c)-h_1(z))}),\end{aligned}$$

where $w=(\sqrt{b}\,a_2A_1-\sqrt{a}\,A_2e^{i\eta})/\sqrt{b}\,a_1A_2$. This implies that $h_1(z+c)+h_1(z)$, and hence $h_1(z)$ is constant, a contradiction.

Similarly, we can obtain a contradiction for the case $\sqrt{b}\,a_2A_1-\sqrt{a}\,A_2e^{i\eta}=0$ and $\sqrt{a}\,A_1-\sqrt{b}\,a_2A_2e^{i\eta}\neq 0$.

Therefore, $\sqrt{b}\,a_2A_1-\sqrt{a}\,A_2e^{i\eta}\neq 0$ and $\sqrt{a}\,A_1-\sqrt{b}\,a_2A_2e^{i\eta}\neq 0$. By similar arguments, we obtain that $\sqrt{b}\,a_2A_1e^{i\eta}-\sqrt{a}\,A_2\neq 0$ and $\sqrt{a}\,A_1-\sqrt{b}\,a_2A_2e^{-i\eta}\neq 0$.

By Lemma 4.1, equations (4.20) and (4.21) yield

$$\begin{cases}\sqrt{b}\,a_1A_2e^{i\eta}e^{-ih_1(z+c)+ih_1(z)}=\sqrt{a}\,A_1-\sqrt{b}\,a_2A_2e^{i\eta}\\\sqrt{b}\,a_1A_2e^{-i\eta}e^{-ih_1(z+c)+ih_1(z)}=\sqrt{a}\,A_1-\sqrt{b}\,a_2A_2e^{-i\eta}.\end{cases} \tag{4.22}$$

Taking into account equations (4.20), (4.21), and (4.22), we obtain

$$\begin{cases}\sqrt{b}\,a_1A_1e^{ih_1(z+c)-ih_1(z)}=\sqrt{a}\,A_2e^{i\eta}-\sqrt{b}\,a_2A_1\\\sqrt{b}\,a_1A_1e^{i\eta}e^{ih_1(z+c)-ih_1(z)}=\sqrt{a}\,A_2-\sqrt{b}\,a_2A_1e^{i\eta}.\end{cases} \tag{4.23}$$

From (4.22), we conclude that $h_1(z+c)-h_1(z)$ is constant, and hence we assume that $h_1(z)=L(z)+\Psi(z)+\xi$, where $L(z)=\sum_{j=1}^{n}a_jz_j$ and $\Psi(z)$ is a polynomial as defined in equation (2.3), $a_j,\xi\in\mathbb{C}$ for $j=1,2,\ldots,n$. Therefore, in view of equations (4.22) and (4.23), we easily obtain

$$\begin{cases}\sqrt{b}\,a_1A_2e^{i\eta}e^{-iL(c)}=\sqrt{a}\,A_1-\sqrt{b}\,a_2A_2e^{i\eta}\\\sqrt{b}\,a_1A_2e^{-i\eta}e^{-iL(c)}=\sqrt{a}\,A_1-\sqrt{b}\,a_2A_2e^{-i\eta}\\\sqrt{b}\,a_1A_1e^{iL(c)}=\sqrt{a}\,A_2e^{i\eta}-\sqrt{b}\,a_2A_1\\\sqrt{b}\,a_1A_1e^{i\eta}e^{iL(c)}=\sqrt{a}\,A_2-\sqrt{b}\,a_2A_1e^{i\eta}.\end{cases} \tag{4.24}$$

Therefore, it follows from equation (4.24) that

$$e^{iL(c)}=\frac{\sqrt{b}\,a_1A_2e^{i\eta}}{\sqrt{a}\,A_1-\sqrt{b}\,a_2A_2e^{i\eta}}=\frac{\sqrt{b}\,a_1A_2e^{-i\eta}}{\sqrt{a}\,A_1-\sqrt{b}\,a_2A_2e^{-i\eta}}=\frac{\sqrt{a}\,A_2e^{i\eta}-\sqrt{b}\,a_2A_1}{\sqrt{b}\,a_1A_1}=\frac{\sqrt{a}\,A_2-\sqrt{b}\,a_2A_1e^{i\eta}}{\sqrt{b}\,a_1A_1e^{i\eta}}.$$

From the last equation, we can obtain that

$$e^{2i\eta}=1,\quad (a+ba_2^2-ba_1^2)e^{i\eta}+2a_2h=0.$$

Hence, from equations (4.16) and (4.17), we obtain

$$f_1 = \frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{\frac{1}{4}}\left[\frac{\cos(L(z)+\Psi(z))}{\sqrt{\sqrt{ab}+h}} + \frac{\sin(L(z)+\Psi(z))}{\sqrt{\sqrt{ab}-h}}\right],$$

$$f_2 = \frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{\frac{1}{4}}\left[\frac{\cos(L(z)+\Psi(z)+\eta)}{\sqrt{\sqrt{ab}+h}} + \frac{\sin(L(z)+\Psi(z)+\eta)}{\sqrt{\sqrt{ab}-h}}\right].$$

Subcase 1.2. Let $h_2(z) - h_1(z)$ be nonconstant. Observe that $h_2(z) - h_1(z+c)$ is also nonconstant.

Subcase 1.2.1. Let $h_2(z) + h_1(z) = \eta$, where η is a constant in $\mathbb{C}$.

Therefore, equations (4.18) and (4.19), respectively, yield

$$\sqrt{b}\,a_1A_1e^{i(h_1(z+c)+h_2(z))} + \sqrt{b}\,a_1A_2e^{i(-h_1(z+c)+h_2(z))} + \sqrt{b}\,a_2A_2e^{i(h_2(z)-h_1(z))} - \sqrt{a}\,A_2e^{2ih_2(z)} = \sqrt{a}\,A_1 - \sqrt{b}\,a_2A_1e^{i\eta} \tag{4.25}$$

and

$$\sqrt{b}\,a_1A_1e^{i(h_2(z+c)+h_1(z))} + \sqrt{b}\,a_1A_2e^{i(-h_2(z+c)+h_1(z))} + \sqrt{b}\,a_2A_2e^{i(h_1(z)-h_2(z))} - \sqrt{a}\,A_2e^{2ih_1(z)} = \sqrt{a}\,A_1 - \sqrt{b}\,a_2A_1e^{i\eta}. \tag{4.26}$$

Let $\sqrt{a} - \sqrt{b}\,a_2e^{i\eta} = 0$. From equation (4.25), we obtain

$$\sqrt{b}\,a_1A_1e^{i(h_1(z+c)-h_2(z))} + \sqrt{b}\,a_1A_2e^{-i(h_1(z+c)+h_2(z))} = \sqrt{a}\,A_2 - \sqrt{b}\,a_2A_2e^{-i\eta}. \tag{4.27}$$

Now, from equation (4.27), we claim $\sqrt{a}\,A_2 - \sqrt{b}\,a_2A_2e^{-i\eta} \neq 0$. Otherwise, we must obtain $e^{2ih_1(z+c)} = -A_2/A_1$, which implies that $h_1(z+c)$, and hence $h_1(z)$ is constant, which is impossible. Now, by second main theorem of Nevanlinna for several complex variables, we obtain

$$\begin{aligned} T(r, e^{i(h_1(z+c)+h_1(z))}) &\leq \overline{N}(r, e^{i(h_1(z+c)+h_1(z))}) + \overline{N}\left(r, \frac{1}{e^{i(h_1(z+c)+h_1(z))}}\right) + \overline{N}\left(r, \frac{1}{e^{i(h_1(z+c)+h_1(z))} - w_1}\right) + S(r, e^{i(h_1(z+c)+h_1(z))}) \\ &\leq \overline{N}\left(r, \frac{1}{e^{i(h_1(z)-h_1(z+c))}}\right) + S(r, e^{i(h_1(z+c)+h_1(z))}) + S\left(r, \frac{1}{e^{i(h_1(z)-h_1(z+c))}}\right) + S(r, e^{i(h_1(z+c)+h_1(z))}), \end{aligned}$$

where $w_1 = (\sqrt{a}\,A_2e^{i\eta} - \sqrt{b}\,a_2A_2)/\sqrt{b}\,a_1A_2$. This implies that $h_1(z+c) + h_1(z)$, and hence $h_1(z)$ is constant, a contradiction.

Hence, $\sqrt{a} - \sqrt{b}\,a_2e^{i\eta} \neq 0$. Therefore, Lemma 4.1 together with equations (4.25) and (4.26) give

$$\begin{cases} \sqrt{b}\,a_1A_1e^{i(h_1(z+c)+h_2(z))} = \sqrt{a}\,A_1 - \sqrt{b}\,a_2A_1e^{i\eta} \\ \sqrt{b}\,a_1A_1e^{i(h_2(z+c)+h_1(z))} = \sqrt{a}\,A_1 - \sqrt{b}\,a_2A_1e^{i\eta}. \end{cases} \tag{4.28}$$

Next, equations (4.25), (4.26), and (4.28) yield

$$\begin{cases} \sqrt{b}\,a_1e^{-i(h_1(z+c)+h_2(z))} = \sqrt{a} - \sqrt{b}\,a_2e^{-i\eta} \\ \sqrt{b}\,a_1e^{-i(h_2(z+c)+h_1(z))} = \sqrt{a} - \sqrt{b}\,a_2e^{-i\eta}. \end{cases} \tag{4.29}$$

In view of equation (4.28), we have $h_1(z+c) + h_2(z) = \eta_1$ and $h_2(z+c) + h_1(z) = \eta_2$, where η_1 and η_2 are constants in $\mathbb{C}$. Thus, we assume that $h_1(z) = L(z) + \Psi(z) + \xi$ and $h_2(z) = -(L(z) + \Psi(z)) + \xi'$, where $L(z) = \sum_{=1}^{n}\alpha_jz_j$ and $\Psi(z)$ is a polynomial as defined in equation (2.3), $\alpha_j, \xi, \xi' \in \mathbb{C}$ and $\eta = \xi + \xi'$.

Therefore, from equations (4.28) and (4.29), we obtain

$$\begin{cases} \sqrt{b}\,a_1e^{i(L(c)+\eta)} = \sqrt{a} - \sqrt{b}\,a_2e^{i\eta} \\ \sqrt{b}\,a_1e^{i(-L(c)+\eta)} = \sqrt{a} - \sqrt{b}\,a_2e^{i\eta} \\ \sqrt{b}\,a_1e^{-i(L(c)+\eta)} = \sqrt{a} - \sqrt{b}\,a_2e^{-i\eta} \\ \sqrt{b}\,a_1e^{-i(-L(c)+\eta)} = \sqrt{a} - \sqrt{b}\,a_2e^{-i\eta}, \end{cases} \tag{4.30}$$

which yield

$$e^{iL(c)} = \frac{\sqrt{a} - \sqrt{b}\,a_2e^{i\eta}}{\sqrt{b}\,a_1e^{i\eta}} = \frac{\sqrt{b}\,a_1e^{i\eta}}{\sqrt{a} - \sqrt{b}\,a_2e^{i\eta}}$$

$$= \frac{\sqrt{b}\,a_1e^{-i\eta}}{\sqrt{a} - \sqrt{b}\,a_2e^{-i\eta}} = \frac{\sqrt{a} - \sqrt{b}\,a_2e^{-i\eta}}{\sqrt{b}\,a_1e^{-i\eta}},$$

$$e^{2i\eta} = 1, \quad 2\sqrt{ab}\,a_2e^{i\eta} = a - ba_1^2 + ba_2^2.$$

Again, from equations (4.16) and (4.17), we obtain

$$f_1 = \frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{\frac{1}{4}}\left[\frac{\cos(L(z) + \Psi(z) + \xi)}{\sqrt{\sqrt{ab} + h}} + \frac{\sin(L(z) + \Psi(z) + \xi)}{\sqrt{\sqrt{ab} - h}}\right],$$

$$f_2 = \frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{\frac{1}{4}}\left[\frac{\cos(L(z) + \Psi(z) + \xi - \eta)}{\sqrt{\sqrt{ab} + h}} - \frac{\sin(L(z) + \Psi(z) + \xi - \eta)}{\sqrt{\sqrt{ab} - h}}\right].$$

Subcase 1.2.2. Let $h_2(z) + h_1(z)$ be nonconstant.

If $h_1(z + c) + h_2(z)$ is nonconstant, then by Lemma 4.1, we obtain from equation (4.18)

$$\sqrt{b}\,a_1A_2e^{i(-h_1(z+c)+h_2(z))} = \sqrt{a}A_1,$$

which implies that $-h_1(z + c) + h_2(z)$ is a constant. Let $-h_1(z + c) + h_2(z) = \eta_1$, where $\eta_1 \in \mathbb{C}$. If $h_2(z + c) + h_1(z) = \eta_2$, a complex constant in $\mathbb{C}$, then we must have $h_2(z + 2c) + h_2(z) = \eta_1 + \eta_2$, which implies that $h_2(z)$ is constant, a contradiction. Thus, $h_2(z + c) + h_1(z)$ is nonconstant. Then, by Lemma 4.1, we obtain from equation (4.19)

$$\sqrt{b}\,a_1A_2e^{i(-h_2(z+c)+h_1(z))} = \sqrt{a}A_1.$$

This implies that $-h_2(z + c) + h_1(z) = \text{const.} = \eta_3$. But, we must have $-h_1(z + c) + h_1(z) = \eta_1 + \eta_3$. Thus, we conclude that $h_1(z) = L(z) + \Psi(z) + \xi$, where $L(z) = \sum_{j=1}^{n} a_jz_j$, and $\Psi(z)$ is a polynomial as defined in equation (2.3), a_j, and ξ are all constants in $\mathbb{C}$, and therefore, $h_2(z) = L(z) + \Psi(z) + \xi + \eta_1 + L(c)$. This implies $h_2(z) - h_1(z) = \eta_1 + L(c) = \text{constant}$, a contradiction. Hence, $h_1(z + c) + h_2(z) = \eta_4$, a constant in $\mathbb{C}$. Now, if $h_2(z + c) + h_1(z) = \eta_5$, a constant in $\mathbb{C}$, then we must have $h_1(z + 2c) - h_1(z) = \eta_4 - \eta_5 = \text{constant}$. This implies that $h_1(z) = L(z) + \Psi(z) + \xi$, where $L(z), \Psi(z)$, and ξ are just defined above. Therefore, $h_1(z) = -(L(z) + \Psi(z) + \xi) + \eta_4 - L(c) = \text{constant}$. Then, $h_2(z) + h_1(z) = \eta_4 - L(c)$=constant, a contradiction. Thus, $h_2(z + c) + h_1(z)$ is nonconstant.

Then, by Lemma 4.1 and from equation (4.19), we obtain

$$\sqrt{b}\,a_1A_2e^{i(-h_2(z+c)+h_1(z))} = \sqrt{a}A_1,$$

which yields that $-h_2(z + c) + h_1(z) = \text{constant} = \eta_6$. Then, we must have $h_1(z + 2c) + h_1(z) = \eta_4 + \eta_6 = \text{constant}$. This implies that $h_1(z)$ is constant, which is not possible.

Case 2. Let $a_2 = 0$. Then, (4.18) and (4.19), respectively, reduces to

$$\sqrt{b}\,a_1A_1e^{i(h_1(z+c)+h_2(z))} + \sqrt{b}\,a_1A_2e^{i(-h_1(z+c)+h_2(z))} - \sqrt{a}A_2e^{2ih_2(z)} = \sqrt{a}A_1, \tag{4.31}$$

and

$$\sqrt{b}\,a_1A_1e^{i(h_2(z+c)+h_1(z))} + \sqrt{b}\,a_1A_2e^{i(-h_2(z+c)+h_1(z))} - \sqrt{a}A_2e^{2ih_1(z)} = \sqrt{a}A_1. \tag{4.32}$$

By Lemma 4.1, we obtain from (4.31) that either

$$\sqrt{b}\,a_1A_1e^{i(h_1(z+c)+h_2(z))} = \sqrt{a}A_1 \text{ or } \sqrt{b}\,a_1A_1e^{i(-h_1(z+c)+h_2(z))} = \sqrt{a}A_1.$$

In a similar manner, from equation (4.32), we obtain that either

$$\sqrt{b}\,a_1A_1e^{i(h_2(z+c)+h_1(z))} = \sqrt{a}A_1 \text{ or } \sqrt{b}\,a_1A_1e^{i(-h_2(z+c)+h_1(z))} = \sqrt{a}A_1.$$

Now, we consider the following four possible cases:

Subcase 2.1. Let

$$\begin{cases}\sqrt{b}\,a_1A_1e^{i(h_1(z+c)+h_2(z))} = \sqrt{a}\,A_1\\ \sqrt{b}\,a_1A_1e^{i(h_2(z+c)+h_1(z))} = \sqrt{a}\,A_1.\end{cases} \tag{4.33}$$

It follows from equation (4.33) that $h_1(z + c) + h_2(z) = \eta_1$ and $h_2(z + c) + h_1(z) = \eta_1$, where η_1 and η_2 are constants in $\mathbb{C}$. This implies that $h_1(z) = L(z) + \Psi(z) + \xi$ and $h_2(z) = -(L(z) + \Psi(z)) + \xi'$, $L(z), \Psi(z)$ are defined as in Theorem 3.3 and ξ and ξ' are constants in $\mathbb{C}$.

Again, in view of equation (4.31), (4.32), and (4.33), it follows that

$$\begin{cases}\sqrt{b}\,a_1A_2e^{-i(h_1(z+c)+h_2(z))} = \sqrt{a}\,A_2\\ \sqrt{b}\,a_1A_2e^{-i(h_2(z+c)+h_1(z))} = \sqrt{a}\,A_2.\end{cases} \tag{4.34}$$

From equation (4.33) and (4.34), we obtain

$$\begin{cases}\sqrt{b}\,a_1e^{i(L(c)+\xi+\xi')} = \sqrt{a}\\ \sqrt{b}\,a_1e^{i(-L(c)+\xi+\xi')} = \sqrt{a}\\ \sqrt{b}\,a_1e^{-i(L(c)+\xi+\xi')} = \sqrt{a}\\ \sqrt{b}\,a_1e^{i(-L(c)+\xi+\xi')} = \sqrt{a},\end{cases} \tag{4.35}$$

from which we can easily deduce the following:

$$e^{2iL(c)} = 1,\quad e^{2i\eta} = 1,\quad ba_1^2 = a \quad \text{with} \quad \eta = \xi + \xi'.$$

Therefore, from equations (4.16) and (4.17), we obtain

$$f_1 = \frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{\frac{1}{4}}\left[\frac{\cos(L(z) + \Psi(z) + \xi)}{\sqrt{\sqrt{ab} + h}} + \frac{\sin(L(z) + \Psi(z) + \xi)}{\sqrt{\sqrt{ab} - h}}\right],$$

$$f_2 = \frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{\frac{1}{4}}\left[\frac{\cos(L(z) + \Psi(z) + \xi - \eta)}{\sqrt{\sqrt{ab} + h}} - \frac{\sin(L(z) + \Psi(z) + \xi - \eta)}{\sqrt{\sqrt{ab} - h}}\right],$$

which is the conclusion II of Theorem 3.3 with $a_2 = 0$.

Subcase 2.2.

Let

$$\begin{cases}\sqrt{b}\,a_1A_1e^{i(h_1(z+c)+h_2(z))} = \sqrt{a}\,A_1\\ \sqrt{b}\,a_1A_2e^{i(-h_2(z+c)+h_1(z))} = \sqrt{a}\,A_1.\end{cases}$$

From the above equations, we obtain $h_1(z + c) + h_2(z) = \eta_1$ and $-h_2(z + c) + h_1(z) = \eta_2$, where η_1 and η_2 are two constants in $\mathbb{C}$. This implies $h_1(z + 2c) + h_1(z) = \eta_1 + \eta_2$, which yields that $h_1(z)$ is constant, a contradiction.

Subcase 2.3.

Let

$$\begin{cases}\sqrt{b}\,a_1A_2e^{i(-h_1(z+c)+h_2(z))} = \sqrt{a}\,A_1\\ \sqrt{b}\,a_1A_1e^{i(h_2(z+c)+h_1(z))} = \sqrt{a}\,A_1.\end{cases}$$

Similar to the arguments of Subcase 2.2, we obtain a contradiction.

Subcase 2.4.

Let

$$\begin{cases}\sqrt{b}\,a_1A_2e^{i(-h_1(z+c)+h_2(z))} = \sqrt{a}\,A_1\\ \sqrt{b}\,a_1A_2e^{i(-h_2(z+c)+h_1(z))} = \sqrt{a}\,A_1.\end{cases} \tag{4.36}$$

In view of equations (4.31), (4.32), and (4.36), we deduce

$$\begin{cases} \sqrt{b}\,a_1 A_1 e^{i(h_1(z+c)-h_2(z))} = \sqrt{a}\,A_2 \\ \sqrt{b}\,a_1 A_1 e^{i(h_2(z+c)-h_1(z))} = \sqrt{a}\,A_2. \end{cases} \tag{4.37}$$

Now, from equation (4.36), we know $-h_1(z+c)+h_2(z)=\eta_1$ and $-h_2(z+c)+h_1(z)=\eta_2$, where $\eta_1,\eta_2\in\mathbb{C}$. This implies $h_1(z)-h_1(z+2c)=\eta_1+\eta_2=h_2(z)-h_2(z+2c)$. Therefore, we assume that $h_1(z)=L(z)+\Psi(z)+\xi$ and $h_2(z)=L(z)+\Psi(z)+\xi'$, where $L(z)$ and $\Psi(z)$ are defined as in Theorem 3.3 and $\xi,\xi'\in\mathbb{C}$.

Next, from equations (4.36) and (4.37), we obtain

$$\begin{cases} \sqrt{b}\,a_1 A_2 e^{i(-L(c)+\xi'-\xi)} = \sqrt{a}\,A_1 \\ \sqrt{b}\,a_1 A_2 e^{i(-L(c)+\xi-\xi')} = \sqrt{a}\,A_1 \\ \sqrt{b}\,a_1 A_1 e^{i(L(c)+\xi-\xi')} = \sqrt{a}\,A_2 \\ \sqrt{b}\,a_1 A_1 e^{i(L(c)+\xi'-\xi)} = \sqrt{a}\,A_2, \end{cases} \tag{4.38}$$

from which we can easily obtain the following:

$$e^{2i\eta}=1,\quad ba_1^2=a,\quad e^{2iL(c)}=\frac{A_2^2}{A_1^2},\quad \text{where}\quad \xi-\xi'=\eta.$$

In this case, the form of the pair of finite-order transcendental entire solutions $(f_1(z),f_2(z))$ will be same as I of Theorem 3.3 with $a_2=0$. □

Proof of Corollary 3.6. Suppose $a=b=a_1=1$ and $a_2=-1$. Then, from the conclusion I of Theorem 3.3, we obtain that

$$e^{iL(c)}=\frac{A_2e^{i\eta}}{A_1+A_2e^{i\eta}}=\frac{A_2e^{-i\eta}}{A_1+A_2e^{-i\eta}}=\frac{A_2e^{i\eta}+A_1}{A_1}=\frac{A_2+A_1e^{i\eta}}{A_1e^{i\eta}}. \tag{4.39}$$

From first and second of equation (4.39), we obtain that $e^{2i\eta}=1$. From first and fourth of equation (4.39), we obtain $A_1^2+A_2^2=0$, i.e., $a=0$, which is a contradiction.

Again, from the conclusion II of Theorem 3.3, we obtain that $4=1$, which is a contradiction.

Hence, the system of equation (3.4) does not possess any pair of finite-order transcendental entire solutions. □

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References

[1] R. Taylor and A. Wiles, *Ring-theoretic properties of certain Hecke algebra*, Ann. Math. **141** (1995), 553–572.

[2] A. Wiles, *Modular elliptic curves and Fermat's last theorem*, Ann. Math. **141** (1995), 443–551.

[3] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[4] P. Montel, *Lecons sur les familles de nomales fonctions analytiques et leurs applications*, Gauthier-Viuars Paris, (1927), 135–136.

[5] G. Iyer, *On certain functional equations*, J. Indian. Math. Soc. **3** (1939), 312–315.

[6] F. Gross, *On the equation* $f^n(z) + g^n(z) = 1$, Bull. Amer. Math. Soc. **72** (1966), 86–88.

[7] T. B. Cao, *The growth, oscillation and fixed points of solutions of complex linear differential equations in the unit disc*, J. Math. Anal. Appl. **352** (2009), no. 2, 739–748, DOI: https://doi.org/10.1016/j.jmaa.2008.11.033.

[8] G. G. Gundersen, K. Ishizaki, and N. Kimura, *Restrictions on meromorphic solutions of Fermat type equations*, Proc. Edinburgh Math. Soc. **63** (2020), no. 3 654–665, DOI: https://doi.org/10.1017/S001309152000005X.

[9] P. C. Hu and Q. Wang, *On meromorphic solutions of functional equations of Fermat type*, Bull. Malays. Math. Sci. Soc. **42** (2019), 2497–2515, DOI: https://doi.org/10.1007/s40840-018-0613-1.

[10] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin/Newyork, 1993.

[11] C. C. Yang, *A generalization of a theorem of P. Montel on entire functions*, Proc. Amer. Math. Soc. **26** (1970), 332–334.

[12] L. Z. Yang and J. L. Zhang, *Non-existence of meromorphic solutions of Fermat type functional equations*, Aequationes Math. **76** (2008), 140–150.

[13] H. Y. Yi and L. Z. Yang, *On meromorphic solutions of Fermat type functional equations*, Sci. China Ser. A **41**(2011), 907–932.

[14] C. C. Yang and P. Li, *On the transcendental solutions of a certain type of non-linear differential equations*, Arch. Math. **82** (2004), 442–448.

[15] Y. M. Chiang and S. J. Feng, *On the Nevanlinna characteristic of* $f(z + \eta)$ *and difference equations in the complex plane*, Ramanujan J. **16** (2008), 105–129, DOI: https://doi.org/10.1007/s11139-007-9101-1.

[16] R. G. Halburd and R. J. Korhonen, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl. **314** (2006), 477–487.

[17] G. Dang and J. Cai, *Entire solutions of the second-order Fermat-type differential-difference equation*, J. Math. **2020** (2020), Article ID 4871812, DOI: https://doi.org/10.1155/2020/4871812.

[18] R. G. Halburd and R. J. Korhonen, *Finite-order meromorphic solutions and the discrete Painleve equations*, Proc. Lond. Math. Soc. **94** (2007), no. 2, 443–474.

[19] P. Li and C. C. Yang, *On the nonexistence of entire solutions of certain type of nonlinear differential equations*, J. Math. Anal. Appl. **320** (2006), 827–835.

[20] L. W. Liao, C. C. Yang, and J. J. Zhang, *On meromorphic solutions of certain type of non-linear differential equations*, Ann. Acad. Sci. Fenn. Math. **38** (2013), 581–593.

[21] K. Liu and X. J. Dong, *Fermat type differential and difference equations*, Electron. J. Differential Equations **2015** (2015), no. 159, 1–10.

[22] T. B. Cao and R. J. Korhonen, *A new version of the second main theorem for meromorphic mappings intersecting hyperplanes in several complex variables*, J. Math. Anal. Appl. **444** (2016), no. 2, 1114–1132, DOI: https://doi.org/10.1016/j.jmaa.2016.06.050.

[23] T. B. Cao and L. Xu, *Logarithmic difference lemma in several complex variables and partial difference equations*, Ann. Math. Pure Appl. **199** (2020), 767–794, DOI: https://doi.org/10.1007/s10231-019-00899-w.

[24] R. J. Korhonen, *A difference Picard theorem for meromorphic functions of several variables*, Comput. Methods Funct. Theory **12** (2012), no. 1, 343–361.

[25] K. Liu, T. B. Cao, and H. Z. Cao, *Entire solutions of Fermat type differential-difference equations*, Arch. Math. **99** (2012), 147–155.

[26] L. Xu and T. B. Cao, *Solutions of complex Fermat-type partial difference and differential-difference equations*, Mediterr. J. Math. **15** (2018), 1–14.

[27] X.M. Zheng and X.Y. Xu, *Entire solutions of some Fermat type functional equations concerning difference and partial differential in* $\mathbb{C}^2$, Anal. Math. **48** (2022), 199–226, DOI: https://doi.org/10.1007/s10476-021-0113-7.

[28] G. Haldar, *Solutions of Fermat-type partial differential difference equations in* $\mathbb{C}^2$, Mediterr. J. Math. **20** (2023), 50, DOI: https://doi.org/10.1007/s00009-022-02180-6.

[29] G. Haldar, and M. B. Ahamed, *Entire solutions of several quadratic binomial and trinomial partial differential-difference equations in* $\mathbb{C}^2$, Anal. Math. Phys. **12** (2022), Article number 113, DOI: https:10.1007/s13324-022-00722-5.

[30] H. Y. Xu, and G. Haldar, *Solutions of complex nonlinear functional equations including second order partial differential and difference equations in* $\mathbb{C}^2$, Electron. J. Differential Equations. **43** (2023), 1–18.

[31] L. Y. Gao, *Entire solutions of two types of systems of complex differential-difference equations*, Acta Math. Sinica (Chin. Ser.) **59** (2016), 677–685.

[32] H. Y. Xu, S. Y. Liu, and Q. P. Li, *Entire solutions for several systems of nonlinear difference and partial differential-difference equations of Fermat-type*, J. Math. Anal. Appl. **483** (2020), 123641, 1–22, DOI: https://doi.org/10.1016/j.jmaa.2019.123641.

[33] E. G. Saleeby, *On complex analytic solutions of certain trinomial functional and partial differential equations*, Aequationes Math. **85** (2013), 553–562.

[34] K. Liu and L. Z. Yang, *A note on meromorphic solutions of Fermat types equations*, An. Stiint. Univ. Al. I. Cuza Lasi Mat. (N. S.). **1** (2016), 317–325.

[35] H. Y. Xu, H. Li, and X. Ding, *Entire and meromorphic solutions for systems of the differential difference equations*, Demonstr. Math. **55** (2022), 676–694, DOI: https://doi.org/10.1515/dema-2022-0161.

[36] H. Y. Xu and Y. Y. Jiang, *Results on entire and meromorphic solutions for several systems of quadratic trinomial functional equations with two complex variables*, RACSAM **116** (2022), 8, DOI: https://doi.org/10.1007/s13398-021-01154-9.

[37] P. C. Hu, P. Li, and C. C Yang, *Unicity of Meromorphic Mappings, Advances in Complex Analysis and its Applications*, vol. 1, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.

[38] P. Lelong, *Fonctionnelles Analytiques et Fonctions Entières (n variables)*, Presses de L'Universit´e de Montr´eal, 1968.

[39] L. I. Ronkin, *Introduction to the Theory of Entire Functions of Several Variables*, Moscow: Nauka 1971 (Russian), American Mathematical Society, Providence, 1974.

[40] W. Stoll, *Holomorphic Functions of Finite Order in Several Complex Variables*, American Mathematical Society, Providence, 1974.

[41] G. Polya, *On an integral function of an integral function*, Math. Soc. **1** (1926), 12–15.