Research Article

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On the equation $f^n + (f'')^m \equiv 1$

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Abstract: Let n and m be two positive integers, and the second-order Fermat-type functional equation $f^n + (f'')^m \equiv 1$ does not have a nonconstant meromorphic solution in the complex plane, except $(n, m) \in \{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1)\}$. The research gives a ready-to-use scheme to study certain Fermat-type functional differential equations in the complex plane by using the Nevanlinna theory, the complex method, and the Weierstrass factorization theorem.

Keywords: Fermat-type functional equations, meromorphic functions, elliptic functions, Nevanlinna theory, Weierstrass factorization theorem

MSC 2020: 30D35, 34A05

1 Introduction

In this article, a meromorphic function means meromorphic in the finite complex plane. In 1966, Gross [1–4] investigated the Fermat-type functional equation

$$f^n + g^n = 1, \tag{1}$$

and derived the following theorem.

Theorem 1.1. (Gross [1– 4]) *Let f and g be nonconstant meromorphic functions and n be a positive integer. For $n > 3$, the solutions of equation* (1) *do not exist; for $n = 2$ and $n = 3$, there exist the solutions of equation* (1).

Yang [5] studied the following generalized Fermat-type functional equation:

$$f(z)^n + g(z)^m = 1, \tag{2}$$

where n and m are positive integers and obtained the following theorem.

Theorem 1.2. (Yang [5]) *If $\frac{1}{n} + \frac{1}{m} < 1$, then equation* (2) *has no nonconstant entire solutions $f(z)$ and $g(z)$.*

In 2012, Li [6] proved the following results on the functional equation in $\mathbf{C}^n$.

Theorem 1.3. (Li [6]) *Let a_1, a_2, and a_3 be nonzero meromorphic functions in $\mathbf{C}^n$, and m_1 and m_2 be positive integers satisfying $\frac{1}{m_1} + \frac{1}{m_2} < 1$. If f_1 and f_2 are meromorphic solutions of equation $a_1 f_1^{m_1} + a_2 f_2^{m_2} = a_3$ in $\mathbf{C}^n$, then, for $j = 1, 2$,*

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$$T(r,f_j) \le C_j\overline{N}(r,f_j) + o\left\{T\left(r,\frac{a_1}{a_3}\right) + T\left(r,\frac{a_2}{a_3}\right)\right\} + S(r,f_j), \tag{3}$$

where $C_j = \frac{1}{m_j\left(1-\frac{1}{m_1}-\frac{1}{m_2}\right)}$.

Theorem 1.3 shows that when $m_1 \ge 3$ and $m_2 \ge 3$ with $(m_1, m_2) \ne (3,3)$, then $C_j < 1$, the growth of f_j is controlled by the coefficients of equation $a_1 f_1^{m_1} + a_2 f_2^{m_2} = a_3$. Furthermore, if the coefficients a_j are constants, $T(r,f_j) = O(1)$. Therefore, solutions f_j must be constants (see [6]). In other cases, for instance, when $m_1 = 3$ and $m_2 = 3$, there are transcendental meromorphic solutions f and g given by Weierstrass elliptic functions to the equation $f^3 + g^3 = 1$ (see [7]); when $m_1 = 2$ and $m_2 = 2$, it is trivial that the transcendental entire solutions $f = \sin h$ and $g = \cos h$ satisfying the equation $f^2 + g^2 = 1$, where h is a nonconstant entire function; when $m_1 = 2$ and $m_2 > 2$ (or $m_1 > 2$ and $m_2 = 2$), it follows that $C_2 > 1$ (or $C_1 > 1$), hence the nonconstant meromorphic solutions for the equation $f_1^{m_1} + f_2^{m_2} = 1$ may exist; when $m_1 > 4$ and $m_2 = 2$ (or $m_1 = 2$ and $m_2 > 4$), the constant C_j is controlled by the coefficients; further $C_1 < 1$ (or $C_2 < 1$), especially if a_1, a_2, a_3 are constants, the meromorphic solutions of equation $a_1 f_1^{m_1} + a_2 f_2^{m_2} = a_3$ must degenerate to constants when $m_1 > 4$, $m_2 = 2$ (or $m_1 = 2$, $m_2 > 4$) (see [6]).

In 2013, Deng et al. [8] investigated the Fermat-type differential equation

$$f^n + (f')^m \equiv 1 \tag{4}$$

and proved the existence of meromorphic solutions. They have achieved the following theorem.

Theorem 1.4. (Deng et al. [8]) *Let f be a nonconstant meromorphic function, and let n and m be two positive integers. Then, the solutions of equation* (4) *do not exist, except* $(n, m) \in \{(1,1), (1,2), (2,1), (2,2), (3,2), (4,2)\}$.

In 2018, Dang and Chen [9] extended Deng-Lei-Yang's results and gained the meromorphic solutions for the following Fermat-type differential equation:

$$af^n + b(f')^m \equiv 1. \tag{5}$$

Dang and Chen, in their study [9], purposed the following open problem, which is related to the Fermat-type functional equation $f^n + g^n + h^n = 1$ [10,11].

Problem. (Dang and Chen [9]) Let n, m and k be positive integers. Find out all nonconstant solutions for the Fermat-type functional equation

$$f^n + (f')^m + (f'')^k \equiv 1. \tag{6}$$

In the following, we study the special case of equation (6) as follows:

$$f^n + (f'')^m \equiv 1 \tag{7}$$

and obtain the following result.

Theorem 1.5. *Let n and m be two positive integers. Then, nonconstant meromorphic solutions to equation* (7) *in the complex plane do not exist, except* $(n, m) \in \{(1,1), (1,2), (1,3), (2,1), (3,1)\}$.

2 Some lemmas

For the proof of Theorem 1.5, we require the following concepts and results. We assume that the readers are familiar with the basic concepts and fundamental theorems of Nevanlinna theory of meromorphic functions. In what follows, the notation W [12] stands for a class of meromorphic functions in the complex plane that consists of elliptic functions, rational functions, and rational functions of $e^{\alpha z}$ $(\alpha \in \mathbf{C})$.

Set $m \in \mathbf{N} := \{1, 2, 3, \ldots\}$, $r_j \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $r = (r_0, r_1, \ldots, r_m)$, and $j = 0, 1, \ldots, m$.

Define differential monomial [13] as follows:

$$M_r[w](z) := [w(z)]^{r_0}[w'(z)]^{r_1}[w''(z)]^{r_2}\cdots[w^{(m)}(z)]^{r_m}. \tag{8}$$

Define $\gamma_{M_r} := r_0 + r_1 + \cdots + r_m$ and $\Gamma_{M_r} := r_0 + 2r_1 + \cdots + (n+1)r_m$ as the degree and the weight of the differential monomial $M_r[w]$.

Define differential polynomial as follows:

$$P(w, w', \ldots, w^{(m)}) := \sum_{r \in I} a_r M_r[w], \tag{9}$$

where a_r are constants, and I is a finite index set of multi-indices $r = (r_0, r_1, \ldots, r_m)$. Then, $\deg(P) := \gamma_P := \max_{r \in I}\gamma_{M_r}$ and $\Gamma_P := \max_{r \in I}\Gamma_{M_r}$ are called the total degree and the weight of the differential polynomial P. We say the differential monomial $M_r[w](z)$ is a dominant term of $P(w, w', \ldots, w^{(m)})$ if $\gamma_{M_r} = \gamma_P$.

Consider the following complex ordinary differential equation:

$$P(w, w', \ldots, w^{(m)}) = 0, \tag{10}$$

where P is a polynomial in $w(z), w'(z), \ldots, w^{(n)}(z)$ with constant coefficients.

If there is exactly p distinct formal Laurent series

$$w(z) = \sum_{k=-q}^{\infty} c_k z^k \quad (q > 0, c_{-q} \neq 0), \tag{11}$$

which satisfies equation (10), we say equation (10) satisfies the $\langle p, q\rangle$ condition [13]. If we only determine p distinct principle parts $\sum_{k=-q}^{-1} c_k z^k$ $(q > 0, c_{-q} \neq 0)$, we say equation (10) satisfies the weak $\langle p, q\rangle$ condition. If equation (10) satisfies the $\langle p, q\rangle$ condition, we say equation (10) satisfies the finiteness property: it has only a finitely formal Laurent series with a finite principal part admitting equation (10).

The function defined by the expression

$$\wp(z) = \begin{cases} \dfrac{1}{z^2} + \displaystyle\sum_{\omega \in L\setminus\{0\}} \left(\dfrac{1}{(z-\omega)^2} - \dfrac{1}{\omega^2}\right), & \text{for } z \notin L, \\ \infty, & \text{for } z \in L, \end{cases} \tag{12}$$

is called Weierstrass $\wp(z, g_2, g_3)$ function [14] with two periods $2\omega_1$ and $2\omega_2$ for the lattice L and solves equation $(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, where g_2 and g_3 are elliptic invariant defined by

$$g_2 = \sum_{(m,n)\neq(0,0)} \frac{60}{(2m\omega_1 + 2n\omega_2)^4},\ g_3 = \sum_{(m,n)\neq(0,0)} \frac{140}{(2m\omega_1 + 2n\omega_2)^6}. \tag{13}$$

The Weierstrass $\wp(z)$ function has the Laurent series $\wp(z) = \frac{1}{z^2} + \frac{g_2 z^2}{20} + \frac{g_3 z^4}{28} + \cdots$, and $\wp'(z) = \frac{-2}{z^3} + \cdots$, where the dots indicate terms of higher order. Furthermore, $\wp'(-z) = -\wp'(z)$, $2\wp''(z) = 12\wp^2(z) - g_2$, $\wp'''(z) = 12\wp(z)\wp'(z), \ldots$, any kth derivatives of $\wp$ can be deduced by the identities one by one. Each elliptic function with periods $2\omega_1$ and $2\omega_2$ is a rational function of $\wp$ and $\wp'$. The addition formula [14] reads

$$\wp(z - z_0) = -\wp(z) - \wp(z_0) + \frac{1}{4}\left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}\right]^2. \tag{14}$$

Lemma 2.1. (Conte and Musette [15]) *Two successive degeneracies and addition formula of Weierstrass elliptic functions* $\wp(z) := \wp(z, g_2, g_3)$ *are*

(1) *Degeneracy to simply periodic functions* (*i.e., rational functions of one exponential* e^{kz}) *according to*

$$\wp(z, 3d^2, -d^3) = 2d - \frac{3d}{2}\coth^2\left(\sqrt{\frac{3d}{2}}\,z\right) \tag{15}$$

if one root e_j *is double* $(\Delta(g_2, g_3) := g_2^3 - 27g_3^2 = 0)$.

(2) *Degeneracy to rational functions of z according to*

$$\wp(z,0,0)=\frac{1}{z^2}\tag{16}$$

if one root e_j is triple ($g_2=g_3=0$).

Lemma 2.2. (Chang and Yang [16]) *Let f and g are two meromorphic functions. If $c_1f+c_2g=1$, then*

$$T(r,f)\le\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,f)+S(r,f).\tag{17}$$

Lemma 2.3. (Chang and Yang [16]) *Let f be a meromorphic function, and let k be a positive integer. Then,*

$$T(r,f^{(k)})\le(k+1)T(r,f)+S(r,f).\tag{18}$$

Lemma 2.4. (Chang and Yang [16]) *Let h be a meromorphic function, n be a positive integer, and $a_i(1\le i\le n)$ be complex constants such that $a_n\ne0$. Then,*

$$T(r,a_nf^n+a_{n-1}f^{n-1}+\cdots+a_1f)=nT(r,f).\tag{19}$$

Lemma 2.5. (Chang and Yang [16]) *Let $f(z)$ be a nonconstant entire function, and, $f(z)=e^{h(z)}$. Then,*
(1) $T(r,h)=o(T(r,f))(r\to\infty)$,
(2) $T(r,h')=S(r,f)$.

Lemma 2.6. (Wittich [17]) *If the algebraic differential equation $P(z,f)=0$ has only one dominant term, where $P(z,f)$ is a differential polynomial in f with polynomial coefficients, then the equation has no transcendental entire solutions.*

Lemma 2.7. (Eremenko [18], Eremenko et al. [19]) *Let $k\in\mathbf{N}$, then any meromorphic solution $w(z)$ of k-order Briot-Bouquet equations*

$$F(w^{(k)},w)=\sum_{i=0}^{n}P_i(w)(w^{(k)})^i=0\tag{20}$$

belong to the class W, where $P_i(w)$ are polynomials with constant coefficients and $w(z)$ has at least one pole.

Lemma 2.8. (Yuan et al. [13]) *Let $p,l,m,n\in\mathbf{N}$,* deg $P(w,w^{(m)})<n$. *Suppose that the mth order Briot-Bouquet equation*

$$P(w^{(m)},w)=bw^n+c\tag{21}$$

satisfies the weak $\langle p,q\rangle$ condition, then all meromorphic solutions w belong to the class W. Furthermore, all nonconstant meromorphic solutions must be one of the following three forms:
(1) *Each elliptic solution with pole at $z=0$ can be written as follows:*

$$\begin{aligned}w(z)=&\sum_{i=1}^{l-1}\sum_{j=2}^{q}\frac{(-1)^jc_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\left(\frac{1}{4}\left[\frac{\wp'(z)+B_i}{\wp(z)-A_i}\right]^2-\wp(z)\right)\\&+\sum_{i=1}^{l-1}\frac{c_{-i1}}{2}\frac{\wp'(z)+B_i}{\wp(z)-A_i}+\sum_{j=2}^{q}\frac{(-1)^jc_{-lj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z)+c_0,\end{aligned}\tag{22}$$

where c_{-ij} are given by series (11), $B_i^2=4A_i^3-g_2A_i-g_3$, *and* $\sum_{i=1}^{l}c_{-i1}=0$, $c_0\in\mathbf{C}$.
(2) *Each rational function solution $w:=R(z)$ is of the form*

$$R(z)=\sum_{i=1}^{l}\sum_{j=1}^{q}\frac{c_{ij}}{(z-z_i)^j}+c_0,\tag{23}$$

with $l(\le p)$ distinct poles of multiplicity q.

(3) *Each simply periodic solution is a rational function $R(\xi)$ of $\xi = e^{\alpha z}(\alpha \in \mathbf{C})$. $R(\xi)$ has $l(\le p)$ distinct poles of multiplicity q, and is of the form*

$$R(\xi) = \sum_{i=1}^{l}\sum_{j=1}^{q} \frac{c_{ij}}{(\xi - \xi_i)^j} + c_0. \tag{24}$$

By former discussion, we will apply the complex method (see [13,20–24]) to investigate the exact solutions of equation (7) by using the following two steps:

(1) Substitute series (11) into equation (7) to determine that the weak $\langle p, q\rangle$ condition holds.
(2) By indeterminant relations (22)–(24), find the elliptic solutions, rational solutions, and simply periodic solutions $f(z)$ of equation (7).

3 The Proof of Theorem 5

Proof. According to Theorem 1.3, also utilizing Remark 1 (b) and (c) in [6], it is clear that equation (7) has no nonconstant meromorphic solutions when $n \ge 3$, $m \ge 3$ with $(n, m) \ne (3, 3)$, and $n > 4$, $m = 2$ (or $n = 2$, $m > 4$). For $(n, m) = (n, 1)$, by Lemma 2.3, we have $T(r, f^n) = T(r, 1 - f'') \le 3T(r, f) + S(r, f)$. Therefore, this implies that $n \le 3$. Therefore, we only need to consider the following cases for (n, m): (1, 1), (1, m) ($m \ge 2$), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 2).

Case 1. $(n, m) = (1, 1)$, $f + f'' \equiv 1$.

Consider the homogeneous differential equation $f + f'' \equiv 0$. The characteristic equation is $\lambda^2 + 1 = 0$, $\lambda = \pm i$, then the general solution for the equation $f + f'' \equiv 0$ be $f = C_1 \sin z + C_2 \cos z$, where C_1 and C_2 are arbitrary. We assume that the special solution for equation $f + f'' \equiv 1$ is $f = B$, then $B = 1$. Therefore, $f = C_1 \sin z + C_2 \cos z + 1$ satisfies equation $f + f'' \equiv 1$, where C_1 and C_2 are arbitrary.

Case 2. $(n, m) = (1, m)$, $m \ge 2$, $f + (f'')^m \equiv 1$.

By Lemma 2.6, f is not a transcendental entire function. If a meromorphic function f with at least one pole with multiplicity q satisfies the equation, and $q = m(q + 2)$, then $m = \frac{q}{q+2} < 1$, but it contradicts that m is a positive integer. For this purpose, we only need to build rational solution. Assuming that $f = a_0 z^p + a_1 z^{p-1} + a_2 z^{p-2} + \cdots + a_p$, $a_0 \ne 0$ satisfying the equations, then $(f'')^m = p^m(p-1)^m a_0^m z^{m(p-2)} + \cdots$, in the case we have $p = m(p - 2)$ and $p = 2m/(m - 1)$. By simply computing, we have $(m, p) = (2, 4)$, $(m, p) = (3, 3)$, $(m, p) = (4, 8/3)$, and $(m, p) = (5, 5/2)$. We only consider the two cases: $(m, p) = (2, 4)$ and $(m, p) = (3, 3)$.

Subcase 2.1 When $(m, p) = (2, 4)$, substituting $f = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$ into the equation $f + (f'')^2 \equiv 1$, we obtain the solution $f = -\frac{1}{144}(z - z_0)^4 + C\ (z - z_0)^3 - 54\ C^2(z - z_0)^2 + 1296\ C^3(z - z_0) - 11{,}664\ C^4 + 1$, where C is arbitrary.

Subcase 2.2 When $(m, p) = (3, 3)$, substituting $f = a_0 z^3 + a_1 z^2 + a_2 z + a_3$ into the equation $f + (f'')^3 \equiv 1$, then the solutions of the mentioned equation satisfy $f = \frac{\sqrt{6}i}{36}(z - z_0)^3 + C(z - z_0)^2 - 2\sqrt{6}C^2 i(z - z_0) - 8C^3 + 1$ and $f = -\frac{\sqrt{6}i}{36}(z - z_0)^3 + C(z - z_0)^2 + 2\sqrt{6}C^2 i(z - z_0) - 8C^3 + 1$, where z_0 and C are arbitrary.

Case 3. $(n, m) = (2, 1)$, $f^2 + f'' \equiv 1$.

By $2\wp''(z) = 12\wp^2(z) - g_2$, we have $(-6\wp(z))^2 + (-6\wp(z))'' - 3g_2 = 0$, and g_3 is arbitrary. Let $f = -6\wp(z, g_2, g_3)$, $g_2 = \frac{1}{3}$, and in this case $f^2 + f'' \equiv 1$, we obtain $f(z) = -6\wp\left(z - z_0, \frac{1}{3}, g_3\right)$.

Case 4. $(n, m) = (2, 2)$, $f^2 + (f'')^2 \equiv 1$.

It is easy to know that the equation does not have any meromorphic solution with at least one pole. If f has a pole, then the pole order of f'' is higher than f and thus the equation $f^2 + (f'')^2 \equiv 1$ cannot be held. It is clear that the equation $f^2 + (f'')^2 \equiv 1$ does not have any polynomial solution. Because if f is a polynomial, then $T(r, f) = \deg(f) \log r = p \log r$ and $2p \log r = T(r, f^2) = \deg(f'')^2 \log r = 2(p - 2) \log r = (2p - 4) \log r$, hence it is a contradiction. Thus, f must be entire. Since $f^2 + (f'')^2 \equiv 1$, we have $(f + if'')(f - if'') \equiv 1$. By the Weierstrass factorization theorem, we can assume that $f + if'' = e^{ih}$, h is entire, and $f - if'' = e^{-ih}$, then

$$f = \frac{e^{ih} + e^{-ih}}{2} = \cos h, \tag{25}$$

$$f'' = \frac{e^{ih} - e^{-ih}}{2i} = \sin h. \tag{26}$$

Following equation (25), we have $f' = -h' \sin h$ and $f'' = -h'' \sin h - (h')^2 \cos h$. Following equation (26), we have $\sin h = -h'' \sin h - (h')^2 \cos h$, then

$$(1 + h'') \sin h = -(h')^2 \cos h. \tag{27}$$

We assert that h is a constant. Otherwise, by equation (27), $\cot h = -\frac{1+h''}{h'}$, and by Lemma 2.5, we have $T(r, \cot h) = T(r, -\frac{1+h''}{h'}) = O(T(r, h)) = S(r, \cot h)$, hence it is a contradiction. This shows that h must be a constant. Then, by equation (25), f is a constant. Therefore, the equation $f^2 + (f'')^2 \equiv 1$ does not have any nonconstant meromorphic solution.

Case 5. $(n, m) = (2, 3)$, $f^2 + (f'')^3 \equiv 1$.

It is easy to know that the equation does not have any meromorphic solution with at least one pole. If f has a pole, then the pole order of $(f'')^3$ is higher than f^2, and thus the equation $f^2 + (f'')^3 \equiv 1$ cannot be held. Furthermore, by Lemma 2.6, the equation does not have any transcendental entire solution. Since $p = 6$, we assume that $f = \sum_{k=0}^{6} a_k z^{6-k}$, and by using Maple, substituting this series into $f^2 + (f'')^3 \equiv 1$, collecting the terms and solving the algebraic equation, we can deduce the following coefficient terms: $a_0 = -1/27{,}000$, $a_2 = -11{,}250a_1^2$, $a_3 = 67{,}500{,}000a_1^3$, $a_4 = -227{,}812{,}500{,}000a_1^4$, $a_5 = 410{,}062{,}500{,}000{,}000\,a_1^5$, $a_6 = -307{,}546{,}875{,}000{,}000{,}000\,a_1^6$, and a_1 is an arbitrary constant. But, in fact, under these conditions of coefficients, f satisfies $f^2 + (f'')^3 = 0$. Hence, $f^2 + (f'')^3 \equiv 1$ does not have any nonconstant meromorphic solution.

Case 6. $(n, m) = (2, 4)$, $f^2 + (f'')^4 \equiv 1$.

It is easy to know the equation does not have any meromorphic solution with at least one pole. If f has a pole, then the pole order of $(f'')^4$ is higher than f^2, and thus the equation $f^2 + (f'')^4 \equiv 1$ cannot be held. Furthermore, by Lemma 2.6, the equation does not have any transcendental entire solution. Since $p = 4$, we can assume $f = \sum_{k=0}^{4} a_k z^{4-k}$, and by substituting this series into the equation $f^2 + (f'')^4 \equiv 1$, collecting all terms, and solving the algebraic equation, we can deduce the following coefficients: $a_0 = i/144$, $a_2 = -54i\,a_1^2$, $a_3 = -1{,}296a_1^3$, $a_4 = 11{,}664ia_1^4$, or $a_0 = -i/144$, $a_2 = 54ia_1^2$, $a_3 = -1{,}296a_1^3$, $a_4 = -11{,}664ia_1^4$, and a_1 is an arbitrary constant. But, in fact, under these conditions of coefficients, f satisfies $f^2 + (f'')^4 = 0$. Hence, $f^2 + (f'')^4 \equiv 1$ does not have any nonconstant meromorphic solution.

Case 7. $(n, m) = (3, 1)$, $f^3 + f'' \equiv 1$.

It is easy to know that the equation $f^3 + f'' \equiv 1$ does not have any polynomial solution. Because if we assume that polynomial f with degree $p > 0$ satisfies the equation, we have $T(r, f^3) = 3T(r, f) = 3\deg(f)\log r = 3p\log r = T(r, f'') = \deg(f'')\log r = (p - 2)\log r$, so $p = -1$, which is a contradiction to $p > 0$. Furthermore, by Lemma 2.6, the equation does not have any transcendental entire solution. The equation is a second-order Briot-Bouquet equation, and by Lemma 2.7, all meromorphic solutions belong to the class W. Assuming $f(z)$ be a meromorphic solution of the equation, and $f(z)$ has a movable pole z_0, then in a neighborhood of z_0, the Laurent series of w is in the form of $\sum_{k=-1}^{\infty} c_k (z - z_0)^k (c_{-1} \neq 0)$, and the weak $<p, q>$ condition holds. By simply computing, we have $f = \frac{\sqrt{-2}}{z} - \frac{z^2}{4} + c_3 z^3 - \frac{3\sqrt{-2}}{224} z^5 + \cdots$ and $f = -\frac{\sqrt{-2}}{z} - \frac{z^2}{4} - c_3 z^3 + \frac{3\sqrt{-2}}{224} z^5 + \cdots$.

By Lemma 2.8, we infer the indeterminant relations of elliptic solutions of $f^3 + f'' \equiv 1$ with pole at $z = z_0 \in \mathbf{C}$

$$f = c_{-1} \frac{\wp'(z - z_0, g_2, g_3) + B}{\wp(z - z_0, g_2, g_3) - A} + c_0, \tag{28}$$

where A and B are constants, and $B^2 = 4A^3 - g_2 A - g_3$.

Submitting f into $f^3 + f'' \equiv 1$, and equating the coefficients, we have the following elliptic solutions:

$$f = \frac{\sqrt{-2}}{2} \times \frac{\wp'\left(z + c, g_2, \frac{1}{8}\right) + \frac{\sqrt{-2}}{4}}{\wp\left(z + c, g_2, \frac{1}{8}\right)}, \tag{29}$$

$$f = -\frac{\sqrt{-2}}{2} \times \frac{\wp'\left(z + c, g_2, \frac{1}{8}\right) - \frac{\sqrt{-2}}{4}}{\wp\left(z + c, g_2, \frac{1}{8}\right)}, \tag{30}$$

where g_2 and c are arbitrary. By Lemma 2.1 and according to the degeneration of Weierstrass elliptic function, we know that the equation does not have any rational solution. If we assume that $g_2 = \frac{3}{4}$, we have $g_2^3 - 27g_3^2 = 0$, and by the degeneration of Weierstrass elliptic function, we can obtain

$$f = \frac{-2 + 3\sqrt{-6}\cot\left(\frac{\sqrt{3}z}{2}\right)\csc^2\left(\frac{\sqrt{3}z}{2}\right)}{6\csc^2\left(\frac{\sqrt{3}z}{2}\right) - 2}, \tag{31}$$

$$f = \frac{-2 - 3\sqrt{-6}\cot\left(\frac{\sqrt{3}z}{2}\right)\csc^2\left(\frac{\sqrt{3}z}{2}\right)}{6\csc^2\left(\frac{\sqrt{3}z}{2}\right) - 2}. \tag{32}$$

Then, we assert that the equation does not have simply periodic solutions in the form of $f = \frac{b}{e^{az} - \xi} + h$, where $\xi, a, h, b(\neq 0)$ are constants. Substituting f into the equation, we have

$$\frac{2ba^2(e^{az})^2 + b^3}{(e^{az} - \xi)^3} + \frac{3b^2h - ba^2e^{az}}{(e^{az} - \xi)^2} + \frac{3bh^2}{e^{az} - \xi} + h^3 \equiv 1. \tag{33}$$

Then, we obtain the following algebraic equation:

$$\begin{cases} 2ba^2(e^{az})^2 + b^3 = 0 \\ 3b^2h - ba^2e^{az} = 0 \\ 3bh^2 = 0 \\ h^3 = 1. \end{cases} \tag{34}$$

Obviously, the above equation has no solution.

Therefore, $f^3 + f'' \equiv 1$ has nonconstant meromorphic solutions.

Case 8. $(n, m) = (3, 2)$, $f^3 + (f'')^2 \equiv 1$.

The equation $f^3 + (f'')^2 \equiv 1$ does not have any polynomial solution, because if we assume that polynomial f with degree $p > 0$ satisfying the equation, we have $T(r, f^3) = 3T(r, f) = 3\deg(f)\log r = 3p\log r = T(r, (f'')^2) = 2T(r, f'') = 2\deg(f'')\log r = 2(p - 2)\log r$, so $p = -4$, it is a contradiction to $p > 0$. Furthermore, by Lemma 2.6, the equation does not have any transcendental entire solution. Now we only consider the meromorphic solutions with a pole. The equation $f^3 + (f'')^2 \equiv 1$ is a second-order Briot-Bouquet equation, and by Lemma 2.7, all meromorphic solutions belong to the class W. If we assume that $f(z)$ is a meromorphic solution of the equation, and $f(z)$ has a movable pole z_0, then, in a neighborhood of z_0, the Laurent series of w is in the form of $\sum_{k=-4}^{\infty} c_k(z - z_0)^k (c_{-4} \neq 0)$, and the weak $<p, q>$ condition holds. By simply computing, we know the equation only admits the following one Laurent series in a neighborhood of $z = 0$: $f = -400z^{-4} + c_6z^6 + \cdots$, c_6 is arbitrary.

By Lemma 2.8, we infer that the indeterminant relations of elliptic solutions of equation (7) with pole at $z = 0$ is

$$f=\sum_{j=2}^{4}\frac{(-1)^{j}c_{-j}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z)+c_0=\frac{(-1)^{4}c_{-4}}{(4-1)!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\wp(z)+c_0=\frac{-400}{6}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\wp(z). \tag{35}$$

Noting that the Laurent series of equation (35) is $f=-400z^{-4}-\frac{20g_2}{3}-\frac{200g_3}{7}z^2+O(z^4)$. Comparing coefficients of the two series, we have $g_2=g_3=0$ and $f=-400z^{-4}$. But $f=-400z^{-4}$ does not satisfy the equation $f^3+(f'')^2\equiv 1$. Therefore, the equation $f^3+(f'')^2\equiv 1$ does not have any nonconstant solutions.

Case 9. $(n,m)=(3,3)$, $f^3+(f'')^3\equiv 1$.

It is easy to know that the equation does not have any meromorphic solution with at least one pole. If f has a pole, then the order of the pole for f'' is higher than f and thus the equation $f^3+(f'')^3\equiv 1$ cannot hold. Furthermore, the equation does not have any polynomial solution. If f is a polynomial, and $T(r,f)=\deg(f)\log r=p\log r$, then we must have $3p\log r=T(r,f^3)=\deg(f'')^3\log r=3(p-2)\log r=(3p-6)\log r$, which is a contradiction. Then, by Theorem 1.2, we know that $f^3+(f'')^3\equiv 1$ does not have any nonconstant entire solution. Therefore, the equation does not have any nonconstant meromorphic solution.

Case 10. $(n,m)=(4,2)$, $f^4+(f'')^2\equiv 1$.

The equation does not have any polynomial solutions, because if we assume that polynomial f with degree $p>0$ satisfies the equation, we have $T(r,f^4)=4T(r,f)=4\deg(f)\log r=4p\log r=T(r,(f'')^2)=2\deg(f'')\log r=2(p-2)\log r$, so $p=-2$, which is a contradiction to $p>0$. Furthermore, by Lemma 2.6, the equation does not have any transcendental entire solution. Now we only consider the meromorphic solution with a pole. The equation $f^4+(f'')^2\equiv 1$ is a second-order Briot-Bouquet equation, and by Lemma 2.7, all meromorphic solutions belong to the class W. If we assume that $f(z)$ is a meromorphic solution, and $f(z)$ has a movable pole z_0, then in a neighborhood of z_0, the Laurent series of w is in the form of $\sum_{k=-q}^{\infty}c_k(z-z_0)^k(c_{-q}\neq 0)$, noting that the $4q=2(q+2)$, then $q=2$. By simply computing, we know that the equation only admits the following Laurent series in a neighborhood of $z=0$: $f=\pm(6iz^{-2}+c_4z^4-\frac{1}{1296}z^6+\cdots)$, where c_4 is arbitrary.

By Lemma 2.8, we infer that the indeterminant relations of elliptic solution of $f^4+(f'')^2\equiv 1$ with a pole at $z=0$ is

$$f=\frac{(-1)^{j}c_{-j}}{(j-1)!}\frac{\mathrm{d}^{j-2}}{\mathrm{d}z^{j-2}}\wp(z)+c_0=\frac{(-1)^{2}c_{-2}}{(2-1)!}\wp(z)+c_0=\pm 6i\wp(z). \tag{36}$$

Noting that the Laurent series of equation (36) being $f=\pm(6iz^{-2}+3g_2iz^2/10+3g_3iz^4/14+g_2^2iz^6/200+\cdots)$. Comparing the coefficients of the two series, we know g_2 does not exist. Furthermore, the rational degeneracy of equation (36) is $f=\pm 6iz^{-2}+\text{const}$, but $f=\pm 6iz^{-2}+\text{const}$ does not satisfy the equation $f^4+(f'')^2\equiv 1$. Therefore, the equation does not have any nonconstant solution.

The proof is completed. □

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