

Research Article

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LADM procedure to find the analytical solutions of the nonlinear fractional dynamics of partial integro-differential equations

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Abstract: Generally, fractional partial integro-differential equations (FPIDEs) play a vital role in modeling various complex phenomena. Because of the several applications of FPIDEs in applied sciences, mathematicians have taken a keen interest in developing and utilizing the various techniques for its solutions. In this context, the exact and analytical solutions are not very easy to investigate the solution of FPIDEs. In this article, a novel analytical approach that is known as the Laplace adomian decomposition method is implemented to calculate the solutions of FPIDEs. We obtain the approximate solution of the nonlinear FPIDEs. The results are discussed using graphs and tables. The graphs and tables have shown the greater accuracy of the suggested method compared to the extended cubic-B spline method. The accuracy of the suggested method is higher at all fractional orders of the derivatives. A sufficient degree of accuracy is achieved with fewer calculations with a simple procedure. The presented method requires no parametrization or discretization and, therefore, can be extended for the solutions of other nonlinear FPIDEs and their systems.

Keywords: approximate solution, Laplace transformation, adomian decomposition method, Caputo derivative operator, analytical method

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1 Introduction

Fractional calculus (FC) has investigated the notions of differentiation and integration in the context of noninteger order. FC is a broader variant of classical calculus and is gaining popularity due to its applications in a variety of engineering and applied sciences. In the same context, fractional partial differential equations

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have become more popular because of their remarkable modeling qualities in various scientific fields. There are many scientific and technological events that are extremely well characterized by using fractional differential equations. The fractional derivative models are used to represent some important systems in nature, such as precise modeling of damping, relaxation, acoustic dissipation, nonFourier heat conduction, geophysics, fluid dynamics, creep, viscoelasticity, rheology, fractional COVID-19 model, and malaria [1–4]. Other physical phenomena such as robotic technology, genetic algorithms, and physics, besides the economy and finance, have been modeled accurately by using the concept of FC [5–8]. Some other important physical phenomena in nature have been modeled more accurately by using FC compared to ordinary calculus. In literature, the applications of FC can be found in modeling of earth quack nonlinear oscillation, time-fractional nonlinear water wave equation [9], the heat transfer model [10], Chaos theory, Cancer chemotherapy, Zener, the Airfoil, the fluid traffic, Poisson-Nerst Planck diffusion, tuberculosis, hepatitis B virus, hepatitis B disease model, electrodynamics, Pine wilt disease, diabetes, and other applications in various branches of research can be seen [5–8]. FC has attracted more attention from researchers in recent decades due to the abovementioned applications.

The mathematical models that contain integrals and fractional derivatives are known as integro-differential equations of fractional order. The idea of fractional partial integro-differential equations (FPIDEs) with fractional specifications is a key component of FC theory and implementations. These ideas are considered critical mathematical methods for describing and analyzing a wide range of real-world challenges in natural engineering, science, and technology [11–13]. The mathematical models of physical phenomena and their implementations in heat conduction, flow in fractured bio-materials, reactor dynamics, visco-elasticity, electricity swaption, convection-diffusion, grain growth, and population dynamics (for more detail, please see the studies by Guo *et al.* [15] and Akram *et al.* [14]). In this study, we will consider a nonlinear mathematical model of the form [14,15]:

$$D_t^\alpha v(\zeta, \tau) + v(\zeta, \tau) \frac{\partial}{\partial y} v(\zeta, \tau) = \int_0^\tau (\tau - q)^{\alpha-1} \frac{\partial^2}{\partial y^2} v(\zeta, \tau) dq + f(\zeta, \tau) \quad \alpha \in (0, 1],$$

where D_t^α is the fractional derivative in Caputo sense and $f(\zeta, \tau)$ is the source term, with initial and boundary conditions,

$$v(\zeta, 0) = k(\zeta), \quad v(0, \tau) = v(L, \tau) = 0.$$

Generally, the analytical and numerical investigations of FPIDEs are difficult compared to ordinary problems. However, the researchers have made successful efforts and obtained the solutions to FPIDEs. In this context, Mittal and Nigam have implemented the adomian decomposition procedure to solve the FPIDEs in the study by Mittal and Nigam [16]. To find the analytical solutions of the linear FPIDEs, Awawdeh and coworkers have utilized the homotopy analysis approximation; see the study by Jaradat *et al.* [17]. Hussain *et al.* used the variation iteration method in their study [18] to find the analytical solutions of FPIDEs. The nonlinear FPIDEs are solved by Eslahchi *et al.* by using the Jacobi technique, and the stability and convergence are analyzed in the study by Eslahchi *et al.* [19]. Rawashdeh [20] proposed a numerical approach to solve FPIDEs numerically by using a spline polynomial. Zhao *et al.* [21] illustrated a piecewise polynomial collocation procedure to tackle the solutions of FPIDEs having weakly singular kernels. Unhale and Kendre presented a collocation method to find the solutions of nonlinear FPIDEs by utilizing the shifted Legendre polynomials and the Chebyshev polynomials [22]. To find the solutions of linear FPIDEs, Arshed [23] used the B-spline technique. Avazzadeh *et al.* [24], developed a hybrid approach by combining Legendre wavelets and an operational matrix of fractional integration. Dehestani *et al.* [25] employed a numerical strategy based on Legendre-Laguerre and the collocation method. Khan and Arif [26] presented the solutions of higher-order FIDEs by using the Chebyshev wavelet method (CWM). Also, Mohyud-Din *et al.* used the CWM for nonlinear FIDEs [27]. Hosseini *et al.* [28,29] used the homotopy analysis Laplace transform method (HATM) to obtain the approximate solutions of the time-fractional Sharma-Tasso-Olver-Burgers equation and nonlinear time-fractional Cauchy reaction-diffusion equation. Yavuz and Özdemir used HATM [30] to obtain the solutions of the fractional Black-Scholes equations. The first kind's generalized fractional order of the Chebyshev functions has been introduced in the

studies by Parand and Delkhosh [31,32] to solve the linear and nonlinear Volterra-Fredholm integro-differential equations. Parand and Rad [33] used the collocation method based on Legendre polynomials for the solution of the Singularly Perturbed Volterra integro-differential and Volterra integral equations. Gokce and Gurbuz used a numerical scheme (pseudo-spectral method) for a one-dimensional neural field model [34]. The existence-uniqueness criteria of nonlinear fractional integro-differential equations of variable order with multi-term boundary conditions are discussed in the study by Refice et al. [35]. Multistage adomian decomposition method (ADM) is used in the study of Evirgen and Özdemir [36] for solving a class of nonlinear programming (NLP) problems over a dynamical system of nonlinear fractional partial differential equations.

The Laplace adomian decomposition method (LADM) is the combination of the Laplace transform (LT) and the decomposition procedure. LADM is an analytical technique that gives efficient approximate solutions for a class of nonlinear systems of partial differential equations and ordinary differential equations. In LADM, the LT is used to convert more complex problems into simpler forms, and then the decomposition procedure is used to obtain the complete solutions. The challenging moment for the researcher is how to deal with the nonlinear terms in a given system (LADM) that deals with the nonlinear system directly without any discretization or linearization. LADM approach was successfully used to find the approximate solution of different linear and nonlinear systems, for example, nonlinear differential delay equations, the heat equation, nonlinear dynamic systems, the wave equation, coupled nonlinear partial differential equations, linear and nonlinear integro-differential equations, and Airy's equation. LADM was successfully used to obtain the solution of Bratu and Duffing equations in the study by Syam and Hamdan [37]. Yavuz and Özdemir used the ADM to obtain the solution of fractional order generalized Black-Scholes equation [38]. Wazwaz et al. presented a remarkable work while using the adomian decomposition method [39–41].

In this article, the LADM is implemented to find the analytical solutions of some nonlinear FPIDEs [14,15]. The Caputo derivative operator is used to express fractional derivatives. The numerical simulations of LADM are done for two illustrative examples of FIDEs. The general LADM scheme is first developed to show the procedure of LADM for the generalized FPIDs. The LADM algorithm is constructed via maple software for various numerical problems. The LADM results are displayed via graphs and tables. The absolute error is measured associated with LADM by using tables. The LADM solutions are compared with the extended cubic-B spline method via tables. The comparison has shown the greater accuracy of the proposed schemes. Moreover, higher accuracy is achieved by considering a few terms of the LADM series solutions. The series form of solutions with components of higher rate of convergence are achieved easily. Furthermore, it is noted that the present method required less volume of calculations and, therefore, can be extended for the solutions of other FPIDEs and their systems.

2 Definitions and preliminaries concepts

In this section, we will discuss some basic definitions.

Definition 2.1. The Caputo operator of fractional order ξ for fractional derivatives is defined as follows [42,43]:

$$D_{\mathcal{J}}^{\xi} \mu(\zeta, \mathcal{J}) = \frac{1}{\Gamma(m - \xi)} \int_0^{\mathcal{J}} (\mathcal{J} - \mathcal{J}_0)^{m-\xi-1} \frac{\partial^m \mu(\zeta, \mathcal{J})}{\partial \mathcal{J}^m} d\mathcal{J}, \quad \text{if } m - 1 < \xi \leq m$$

if $\xi = m$, for $m \in \mathbb{N}$, then

$$D_{\mathcal{J}}^{\xi} \mu(\zeta, \mathcal{J}) = \frac{\partial^m \mu(\zeta, \mathcal{J})}{\partial \mathcal{J}^m}.$$

Definition 2.2. For a function $g(\tau)$, the LT is given as follows [11,43]:

$$G(s) = \mathcal{L}[g(\tau)] = \int_0^{\infty} e^{-st} g(\tau) d\tau.$$

Definition 2.3. The LT of the Caputo operator is defined as follows [42,43]:

$$\mathbb{L}(D_{\tau}^{\alpha} \mu(\tau)) = s^{\alpha} \mathbb{L}[\mu(\tau)] - \sum_{k=0}^{j-1} s^{\alpha-j-1} U^k(0^+), \quad j-1 < \alpha \leq j, \quad (1)$$

$$\mathbb{L}(D_{\tau}^{\alpha} \mu(\tau)) = s^{\alpha} U(s) - s^{\alpha-1} U(0), \quad 0 < \alpha \leq 1. \quad (2)$$

Definition 2.4. The Mittag-Leffler function is expressed as follows [44]:

$$E_{\alpha}(\zeta) = \sum_{j=0}^{\infty} \frac{\zeta^j}{\Gamma(\alpha j + 1)} \quad \alpha > 0, \zeta \in \mathbb{C},$$

Definition 2.5. The adomian polynomials to express nonlinear term in a given problem is defined as follows [45]:

$$Nu(\zeta, \tau) = \sum_{j=0}^{\infty} A_j, \quad (3)$$

where

$$A_j = \frac{1}{j!} \left[\frac{d^j}{d\lambda^j} \left\{ N \sum_{j=0}^{\infty} (\lambda^j u_j) \right\} \right]_{\lambda=0} \quad j = 0, 1, \dots \quad (4)$$

are called adomian polynomials.

Lemma 2.6. Let $f(\tau)$ and $g(\tau)$ are two piecewise continuous functions on $[0, \infty)$ and of exponential order, $F(s) = \mathbb{L}[f(\tau)]$, $G(s) = \mathbb{L}[g(\tau)]$, and λ , η , ρ , and ζ are constants. Then the properties (1–8) are satisfied; see previous studies [11,46–49]:

$$(1) \quad \mathbb{L}[\eta f(\tau) + \lambda g(\tau)] = \eta F(s) + \lambda G(s).$$

$$(2) \quad \mathbb{L}^{-1}[\eta F(s) + \lambda G(s)] = \eta f(\tau) + \lambda g(\tau).$$

$$(3) \quad \mathbb{L}[e^{\rho\tau} f(\tau)] = F(s - \rho).$$

$$(4) \quad \mathbb{L}[f(\zeta\tau)] = \frac{1}{\zeta} F\left(\frac{s}{\zeta}\right), \quad \zeta > 0.$$

$$(5) \quad \lim_{s \rightarrow \infty} sF(s) = f(0).$$

$$(6) \quad \mathbb{L}[J_{\tau}^{\beta} f(\tau)] = \frac{F(s)}{s^{\beta}}, \quad \beta > 0.$$

$$(7) \quad \mathbb{L}[D_{\tau}^{\delta} f(\tau)] = s^{\delta} F(s) - \sum_{k=0}^{m-1} s^{\delta-k-1} f^{(k)}(0), \quad m-1 < \delta \leq m.$$

$$(8) \quad \mathbb{L}[D_{\tau}^{\alpha} f(\tau)] = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{(n-k)\alpha-1} (D_{\tau}^{k\alpha} f)(0), \quad 0 < \alpha \leq 1.$$

Proof. The proof from (1) to (7) is trivial, and the proof of part (8) is given as follows.

For $n = 1$, the formula: $\mathbb{L}[D_{\tau}^{\alpha} f(\tau)] = s^{\alpha} F(s) - s^{\alpha-1} f(0)$ is true based on part (7).

For $n = 2$, we have

$$\mathbb{L}[D_{\tau}^{2\alpha} f(\tau)] = \mathbb{L}[D_{\tau}^{\alpha} (D_{\tau}^{\alpha} f(\tau))] = \mathbb{L}[D_{\tau}^{\alpha} h(\tau)],$$

where $H(s) = \mathbb{L}[D_{\tau}^{\alpha} f(\tau)] = s^{\alpha} F(s) - s^{\alpha-1} f(0)$. Then,

$$\begin{aligned} \mathbb{L}[D_{\tau}^{2\alpha} f(\tau)] &= s^{\alpha-1} (sH(s) - h(0)) = s^{\alpha} H(s) - s^{\alpha-1} (D_{\tau}^{\alpha} h)(0), \\ &= s^{2\alpha} F(s) - s^{2\alpha-1} f(0) - s^{\alpha-1} (D_{\tau}^{\alpha} f)(0). \end{aligned}$$

Let us assume that it is true for $n = r$, and we need to prove it for $n = r + 1$. Consider,

$$\mathbb{L}[D_\tau^{(r+1)\alpha}f(\tau)] = \mathbb{L}[D_\tau^\alpha(D_\tau^{r\alpha}f(\tau))] = \mathbb{L}[D_\tau^\alpha p(\tau)] = \mathbb{L}[J^{1-\alpha}p'(\tau)] = s^{\alpha-1}\mathbb{L}[z'(\tau)] = s^{\alpha-1}(sP(s) - p(0)),$$

where $p(\tau) = D_\tau^{r\alpha}f(\tau)$ and $P(s) = \mathbb{L}[p(\tau)] = \mathbb{L}[D_\tau^{r\alpha}f(\tau)] = s^{r\alpha}F(s) - \sum_{k=0}^{r-1} s^{(r-k)\alpha-1}(D_\tau^{k\alpha}f)(0)$. Thus,

$$\mathbb{L}[D_\tau^{(r+1)\alpha}f(\tau)] = s^{(r+1)\alpha}F(s) - \sum_{k=0}^r s^{(r+1-k)\alpha-1}(D_\tau^{k\alpha}f)(0).$$

Thus, the proof is complete. $\square$

3 Laplace adomian decomposition method

3.1 LADM procedure for FPIDEs [50]

To understand the basic procedure of LADM, consider a nonlinear FPIDEs [14,15]

$$D_\tau^\alpha v(\zeta, \tau) + v(\zeta, \tau) \frac{\partial}{\partial y} v(\zeta, \tau) = \int_0^\tau (\tau - q)^{\alpha-1} \frac{\partial^2}{\partial y^2} v(\zeta, \tau) dq + f(\zeta, \tau) \quad \alpha \in (0, 1], \quad (5)$$

where D_τ^α is the fractional derivative in Caputo sense and $f(\zeta, \tau)$ is the source term, with initial and boundary conditions

$$v(\zeta, 0) = k(\zeta), \quad v(0, \tau) = v(L, \tau) = 0, \quad (6)$$

taking LT of equation (5), we obtain

$$\mathcal{L}\{D_\tau^\alpha v(\zeta, \tau)\} = \mathcal{L}\left\{\int_0^\tau (\tau - q)^{\alpha-1} \frac{\partial^2}{\partial y^2} v(\zeta, \tau) dq - v(\zeta, \tau) \frac{\partial}{\partial \zeta} v(\zeta, \tau) + f(\zeta, \tau)\right\}$$

using the differentiation property of LT, and we obtain

$$s^\alpha \mathcal{L}\{v(\zeta, \tau)\} - s^{\alpha-1}v(\zeta, 0) = \mathcal{L}\left\{\int_0^\tau (\tau - q)^{\alpha-1} \frac{\partial^2}{\partial \zeta^2} v(\zeta, \tau) dq - v(\zeta, \tau) \frac{\partial}{\partial \zeta} v(\zeta, \tau) + f(\zeta, \tau)\right\}$$

after simplification, we obtain

$$s^\alpha \mathcal{L}\{v(\zeta, \tau)\} = s^{\alpha-1}v(\zeta, 0) + \mathcal{L}\left\{\int_0^\tau (\tau - q)^{\alpha-1} \frac{\partial^2}{\partial \zeta^2} v(\zeta, \tau) dq - v(\zeta, \tau) \frac{\partial}{\partial \zeta} v(\zeta, \tau) + f(\zeta, \tau)\right\},$$

and further,

$$\mathcal{L}\{v(\zeta, \tau)\} = s^{-1}v(\zeta, 0) + \frac{1}{s^\alpha} \mathcal{L}\left\{\int_0^\tau (\tau - q)^{\alpha-1} \frac{\partial^2}{\partial \zeta^2} v(\zeta, \tau) dq - v(\zeta, \tau) \frac{\partial}{\partial \zeta} v(\zeta, \tau) + f(\zeta, \tau)\right\}, \quad (7)$$

taking inverse LT of equation (7), and we obtain

$$v(\zeta, \tau) = v(\zeta, 0) + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha} \mathcal{L}\left\{\int_0^\tau (\tau - q)^{\alpha-1} \frac{\partial^2}{\partial \zeta^2} v(\zeta, \tau) dq - v(\zeta, \tau) \frac{\partial}{\partial \zeta} v(\zeta, \tau) + f(\zeta, \tau)\right\}\right], \quad (8)$$

and decomposition solutions for variables $v(\zeta, \tau)$ and nonlinear term, can be written as follows:

$$v(\zeta, \tau) = \sum_{j=0}^{\infty} v_j(\zeta, \tau), \quad Nv(\zeta, \tau) = v(\zeta, \tau) \frac{\partial}{\partial \zeta} v(\zeta, \tau) = \sum_{j=0}^{\infty} \mathcal{A}_j.$$

After applying decomposition procedure, equation (8) can be written as follows:

$$\sum_{j=0}^{\infty} v_j(\zeta, \tau) = v(\zeta, 0) + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha} \mathcal{L}\left\{\int_0^\tau (\tau - q)^{\alpha-1} \frac{\partial^2}{\partial \zeta^2} \left(\sum_{j=0}^{\infty} v_j(\zeta, \tau)\right) dq - \sum_{j=0}^{\infty} \mathcal{A}_j + f(\zeta, \tau)\right\}\right]. \quad (9)$$

The recursive LADM algorithm for equation (9) become as follows:

$$v_0(\zeta, \tau) = v(\zeta, 0) + \mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L} \{ f(\zeta, \tau) \} \right] \quad (10)$$

and in general, for $j \geq 0$, it is given by

$$v_{j+1}(\zeta, \tau) = \mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L} \left[\int_0^\tau (\tau - q)^{\alpha-1} \frac{\partial^2}{\partial \zeta^2} v_j(\zeta, \tau) dq - \mathcal{A}_j \right] \right]. \quad (11)$$

3.2 Existence, uniqueness, and exactness

The existence, uniqueness, and exactness of the solutions can be seen in the study by Alrabaiah et al. [51].

3.3 Convergence of the ADM for initial value problems

The convergence of the ADM for the initial value problem can be seen in the studies by Abdelrazec et al. [52,53].

3.4 Limitation for some function

There are many functions for which the LT may be difficult or some time impossible to compute analytically. One example of a function is the Bessel function of the first kind of noninteger order. Other examples of functions for which the LT is difficult to compute analytically include the Gamma function, Beta function, and Zeta function, as well as functions with nonelementary expressions such as exponential integrals, hypergeometric functions, and elliptic integrals. The details for these function can be seen in previous studies [54–56]. However, the mentioned functions can be expanded by using the Taylor series. The Laplace or inverse transformation of the resultant Taylor polynomial can be easily computed with a sufficient degree of accuracy.

4 Numerical examples

Example 4.1. Consider the following nonlinear FPIDEs [14,15]

$$D_\tau^\alpha v(\zeta, \tau) + v(\zeta, \tau) \frac{\partial}{\partial \zeta} v(\zeta, \tau) = \int_0^\tau (\tau - q)^{\alpha-1} \frac{\partial^2}{\partial \zeta^2} v(\zeta, \tau) dq + f(\zeta, \tau) \quad \alpha \in (0, 1],$$

where D_τ^α is the fractional derivative in Caputo sense and

$$\begin{aligned} f(\zeta, \tau) = & \pi \left(\frac{\pi \tau^\alpha}{\alpha} - 4\tau^3 \cos(2\pi\zeta) \right) \sin(\pi\zeta) + \pi/2 - 12 \frac{\tau^{3-\alpha}}{\Gamma(4-\alpha)} - 2\pi\tau^3 \cos(\pi\zeta) \\ & - 48 \frac{\pi^2 \Gamma(\alpha) \tau^{3+\alpha}}{\Gamma(4+\alpha)} + 8\pi\tau^6 \cos(2\pi\zeta) \sin(2\pi\zeta). \end{aligned} \quad (12)$$

The initial condition is,

$$v(\zeta, 0) = \sin(\pi\zeta). \quad (13)$$

The exact solution at $\alpha = 1$ is,

$$v(\zeta, \tau) = \sin(\pi\zeta) - 2\tau^3 \sin(2\pi\zeta).$$

By using equations (12) and (13) in equation (10) and setting $\alpha = 1$, we obtain

$$v_0(\zeta, \tau) = \sin(\pi\zeta) \left\{ \begin{aligned} & \frac{32\pi \sin(\pi\zeta)\tau^7(\cos(\pi\zeta))^3}{7} - \frac{16 \cos(\pi\zeta)\pi\tau^7}{7} - \frac{16\pi^2\tau^5 \cos(\pi\zeta)}{5} \\ & - 3\pi t^4(\cos(\pi\zeta))^2 + \sin(\pi\zeta)\pi\tau^4 - 4 \cos(\pi\zeta)\tau^3 \\ & + 1/2\pi^2\tau^2 + \pi t \cos(\pi\zeta). \end{aligned} \right. \quad (14)$$

By substituting $j = 0$ and $t = 0.001$ in equation (11), we obtain

$$v_1(\zeta, \tau) = \sin(\pi\zeta) \left\{ \begin{aligned} & -1.727914413 \times 10^{-43}(\cos(\pi\zeta))^7 + 2.480502136 \times 10^{-34}(\cos(\pi\zeta))^6 \\ & -1.875140512 \times 10^{-25} \sin(\pi\zeta)(\cos(\pi\zeta))^5 + 7.751558954 \times 10^{-17}(\cos(\pi\zeta))^4 \\ & -0.00000002067079586 \sin(\pi\zeta)(\cos(\pi\zeta))^3 - 0.00001480443371(\cos(\pi\zeta))^2 \\ & -0.003141633997 \sin(\pi\zeta) \cos(\pi\zeta) - 3.141650533 \times 10^{-12}. \end{aligned} \right. \quad (15)$$

By substituting $j = 1$ and $t = 0.001$ in equation (11), we obtain

$$v_2(\zeta, \tau) = \sin(\pi\zeta) \left\{ \begin{aligned} & 8.9 \times 10^{-11}(\cos(\pi\zeta))^4 + 0.0000049(\cos(\pi\zeta))^2 - 7.3 \times 10^{-11} \\ & +1.355 \times 10^{-66}(\cos(\pi\zeta))^{11} + 3.51 \times 10^{-48}(\cos(\pi\zeta))^9 + 1.3 \\ & \times 10^{-30}(\cos(\pi\zeta))^7 + 8.1 \times 10^{-14}(\cos(\pi\zeta))^5 + 0.000000037(\cos(\pi\zeta))^3 \\ & +0.000000031 \cos(\pi\zeta) - 2.58 \times 10^{-39}(\cos(\pi\zeta))^8 - 4.17 \times \\ & 10^{-22}(\cos(\pi\zeta))^6 - 3.02 \times 10^{-57}(\cos(\pi\zeta))^{10}, \\ & \vdots \end{aligned} \right. \quad (16)$$

The general series form solution of LADM is given by

$$v(\zeta, \tau) = \sum_{j=0}^{\infty} v_j(\zeta, \tau) = v_0(\zeta, \tau) + v_1(\zeta, \tau) + v_2(\zeta, \tau) + v_3(\zeta, \tau) + \cdots \quad (17)$$

Substituting equations (14), (15), and (16) in equation (17) and setting $t = 0.001$, we obtain

$$v(\zeta, \tau) = \sin(\pi\zeta) \left\{ \begin{aligned} & 1.3558 \times 10^{-66}(\cos(\pi\zeta) + 361.17) \\ & ((\cos(\pi\zeta))^2 + 1204.5 \cos(\pi\zeta) + 410440.0) \\ & ((\cos(\pi\zeta))^2 - 471.46 \cos(\pi\zeta) + 82665.0) \\ & ((\cos(\pi\zeta))^2 - 219360000.0 \cos(\pi\zeta) + 3.3551 \times 10^{17}) \\ & ((\cos(\pi\zeta))^2 - 701460000.0 \cos(\pi\zeta) + 3.0917 \times 10^{17}) \\ & ((\cos(\pi\zeta))^2 - 1296100000.0 \cos(\pi\zeta) + 5.8028 \times 10^{17}), \end{aligned} \right. \quad (18)$$

which is the LADM analytical solution.

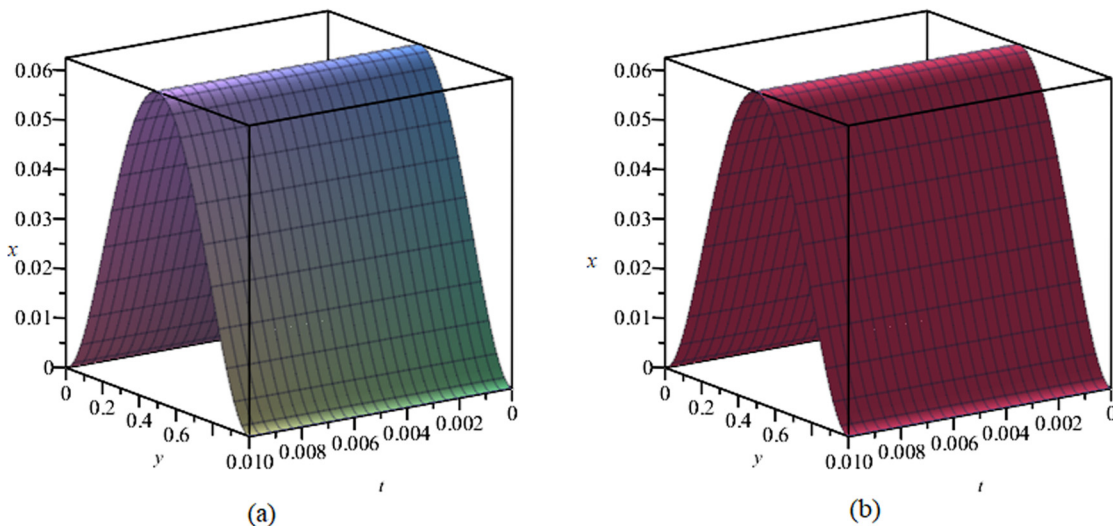


Figure 1: (a) 3D solution plot of (a) exact (b) LADM solutions at $\alpha = 1$ of Example 4.1.

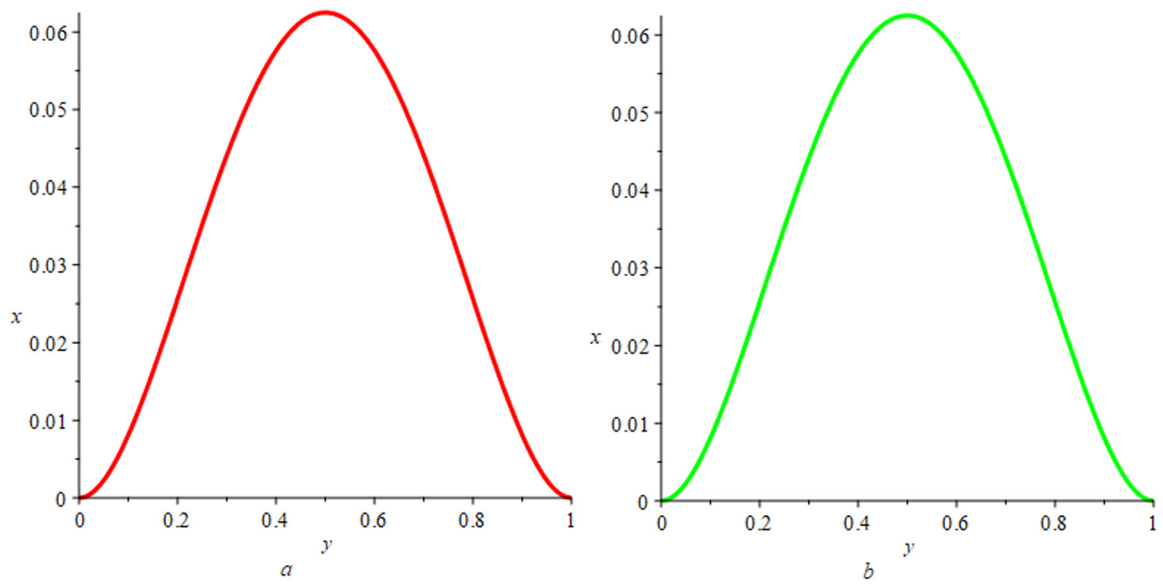


Figure 2: 2D solution plot of (a) exact (b) LADM solutions at $\alpha = 1$ of Example 4.1.

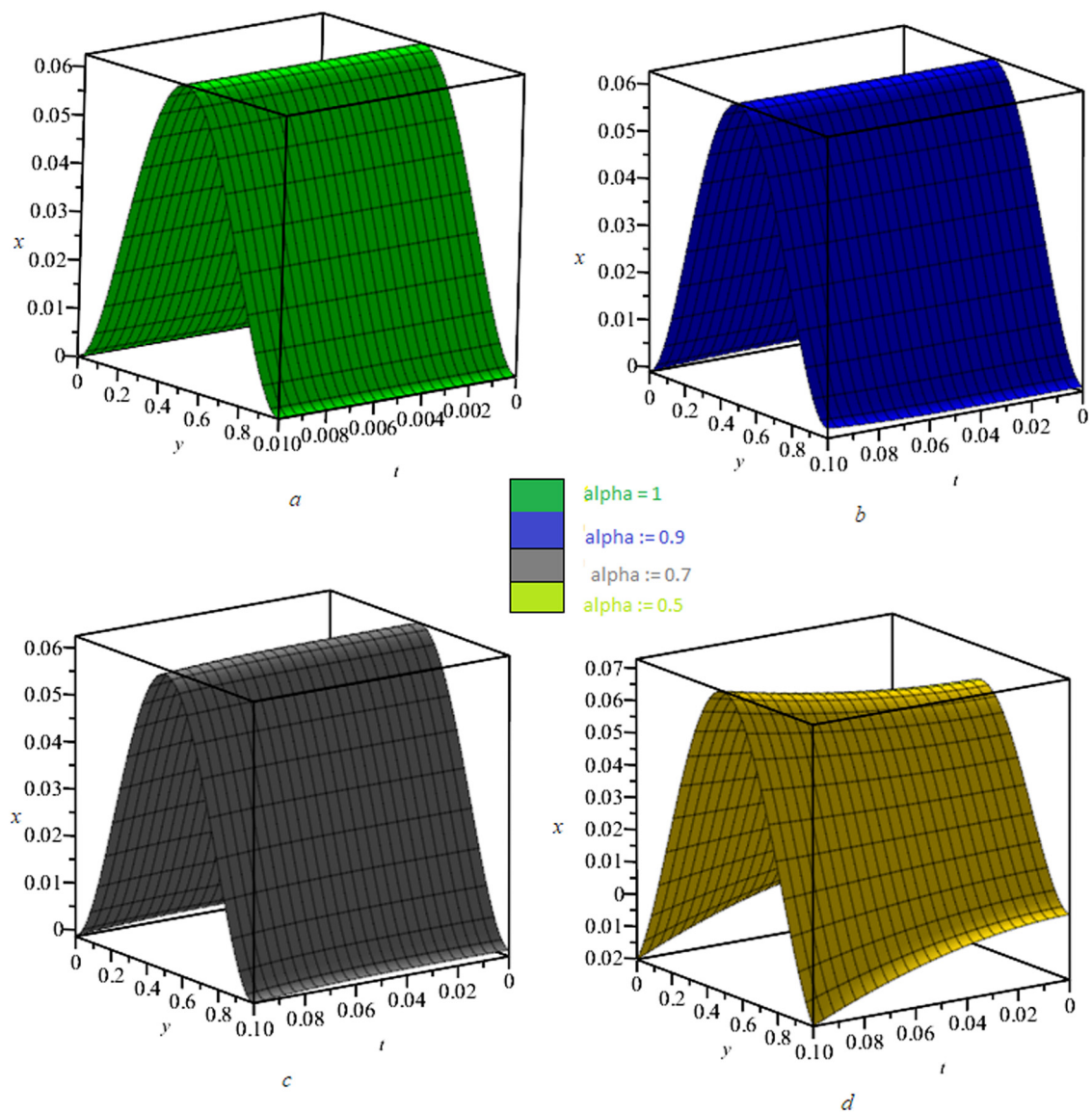


Figure 3: 3D LADM-solution graphs at (a) $\alpha = 1$, (b) $\alpha = 0.9$, (c) $\alpha = 0.7$, and (d) $\alpha = 0.5$, of Example 4.1.

Table 1: Absolute error associated with LADM at $\alpha = 1$ for Example 4.1

τ	ζ	Exact solution	Approximate solution	AE at $\alpha = 1$
0.001	0.0	0.0000000000	0.0000000020	0.0000000020
	0.1	0.3090157607	0.3090169928	0.0000012321
	0.2	0.5877843541	0.5877852517	0.0000008976
	0.3	0.8090182238	0.8090169950	0.0000012288
	0.4	0.9510603094	0.9510565179	0.0000037915
	0.5	1.0000049347	1.0000000020	0.0000049327
	0.6	0.9510603170	0.9510565179	0.0000037991
	0.7	0.8090182324	0.8090169950	0.0000012374
	0.8	0.5877843580	0.5877852517	0.0000008936
	0.9	0.3090157609	0.3090169928	0.0000012319
	1.0	0.0000000000	0.0000000020	0.0000000020

Table 2: Absolute error associated with LADM at different value of α for Example 4.1

ζ	AE at $\alpha = 0.5$	AE at $\alpha = 0.75$	AE at $\alpha = 0.95$	AE at $\alpha = 0.95$ [15]	AE at $\alpha = 1$
0.1	0.00135309533	0.00005808750	0.00000268972	0.0224217000	0.00000123209
0.2	0.00073324838	0.00004257371	0.00000195874	0.0344188000	0.00000089761
0.3	0.00245631124	0.00005759707	0.00000268347	0.0373676000	0.00000122886
0.4	0.00667218435	0.00017926924	0.00000827961	0.0388078000	0.00000379152
0.5	0.00902821200	0.00023452300	0.00001077400	0.0378141000	0.00000493274
0.6	0.00760203986	0.00018173383	0.00000829981	0.0246163000	0.00000379907
0.7	0.00331062959	0.00006003658	0.00000270362	0.0022242000	0.00000123740
0.8	0.00068678178	0.00004204829	0.00000195414	0.0255941000	0.00000089362
0.9	0.00173037152	0.00005872010	0.00000269479	0.0248123000	0.00000123190
1.0	0.00000000159	0.00000000159	0.00000000159	—	0.00000000200

Example 4.2. Consider the following nonlinear FPIDEs [14,15]

$$D_{\tau}^{\alpha} v(\zeta, \tau) + v(\zeta, \tau) \frac{\partial}{\partial \zeta} v(\zeta, \tau) = \int_0^{\tau} (\tau - q)^{\alpha-1} \frac{\partial^2}{\partial \zeta^2} v(\zeta, q) dq + f(\zeta, \tau) \quad \alpha \in (0, 1], \quad (19)$$

where D_{τ}^{α} is the fractional derivative in Caputo sense and

$$f(\zeta, \tau) = \frac{\Gamma(7/2) \tau^{5/2-\alpha} \zeta^2 (1-\zeta)^2}{\Gamma(7/2-\alpha)} - 2 \left(\frac{\tau^{\alpha}}{\alpha} + \frac{\Gamma(7/2) \Gamma(\alpha) \tau^{5/2+\alpha}}{\Gamma(7/2+\alpha)} \right) (6\zeta^2 - 6\zeta + 1) + 2(1 + \tau^{5/2})^2 (1-2\zeta) \zeta^3 (1-\zeta)^3. \quad (20)$$

The initial condition is

$$v(\zeta, 0) = \zeta^2 (1-\zeta)^2. \quad (21)$$

The exact solution at $\alpha = 1$ is,

$$v(\zeta, \tau) = (1 + \tau^{5/2}) \zeta^2 (1-\zeta)^2,$$

By using initial condition and source term in equation (10) and setting $\alpha = 1$, we obtain

$$v_0(\zeta, \tau) = \zeta^2 (1-\zeta)^2 + \tau^{5/2} \zeta^2 (-1 + \zeta)^2 - 1/9 (6\zeta^2 - 6\zeta + 1) (4\tau^{9/2} + 9\tau^2) + 1/21 \zeta^3 (2\zeta - 1) (-1 + \zeta)^3 (24\tau^{7/2} + 7\tau^6 + 42t). \quad (22)$$

By substituting $j = 0$ in equation (11), and setting $\alpha = 1$, we obtain

$$\begin{aligned}
 v_1(\zeta, \tau) = & \left[6\tau^2\zeta^2 - 6\tau^2\zeta - 2/3\tau^6\zeta^7 + 7/3\tau^6\zeta^6 - 3\tau^6\zeta^5 + 5/3\tau^6\zeta^4 - 1/3\tau^6\zeta^3 \right. \\
 & - 4t\zeta^7 + 14t\zeta^6 - 18t\zeta^5 + 10t\zeta^4 - 2t\zeta^3 + 8/3\tau^{9/2}\zeta^2 - 8/3\tau^{9/2}\zeta \\
 & - \frac{16\tau^{7/2}\zeta^7}{7} + 8\tau^{7/2}\zeta^6 - \frac{72\tau^{7/2}\zeta^5}{7} + \frac{40\tau^{7/2}\zeta^4}{7} - \frac{8\tau^{7/2}\zeta^3}{7} + \frac{256\tau^{13/2}\zeta^2}{273} \\
 & + \frac{39,752\tau^{11/2}\zeta^3}{693} - \frac{740\tau^{11/2}\zeta^4}{11} - \frac{4,336\tau^{11/2}\zeta^2}{231} - \frac{280\tau^{19/2}\zeta^6}{57} + \frac{29,952\tau^{11/2}\zeta^8}{77} \\
 & + \frac{28\tau^{19/2}\zeta^5}{19} - \frac{968\tau^{9/2}\zeta^{10}}{63} - \frac{11,008\tau^{11/2}\zeta^7}{77} + \frac{4,840\tau^{9/2}\zeta^9}{63} + \frac{320\tau^{11/2}\zeta^6}{11} \\
 & - \frac{1,100\tau^{9/2}\zeta^8}{7} + \frac{1,880\tau^{11/2}\zeta^5}{77} + \frac{3,520\tau^{9/2}\zeta^7}{21} + \frac{108\tau^5\zeta^2}{5} - \frac{48\tau^5\zeta}{5} + \frac{62\tau^7\zeta^5}{7} \\
 & - \frac{155\tau^7\zeta^4}{147} + 42\tau^2\zeta^5 - 5\tau^2\zeta^4 - \frac{2,332\tau^9\zeta^3}{567} + \frac{53\tau^9\zeta^2}{189} - \frac{32\tau^{13/2}}{39} - \frac{682\tau^7\zeta^{10}}{147} \\
 & + \frac{7,566\tau^8\zeta^8}{49} + \frac{3,410\tau^7\zeta^9}{147} - \frac{8,342\tau^8\zeta^7}{147} - \frac{2,325\tau^7\zeta^8}{49} + \frac{485\tau^8\zeta^6}{42} + \frac{2,480\tau^7\zeta^7}{49} \\
 & + \frac{221\tau^8\zeta^5}{49} - \frac{620\tau^7\zeta^6}{21} - \frac{55\tau^8\zeta^4}{4} - 22t^2\zeta^{10} + \frac{211\tau^8\zeta^3}{18} + 624\tau^3\zeta^8 + 110\tau^2\zeta^9 \\
 & - \frac{23\tau^8\zeta^2}{6} - \frac{688\tau^3\zeta^7}{3} - 225\tau^2\zeta^8 + \frac{140\tau^3\zeta^6}{3} + 240\tau^2\zeta^7 + 64\tau^3\zeta^5 - 140\tau^2\zeta^6 \\
 & - 22\tau^4\zeta^3 - 170\tau^3\zeta^4 + 3/2\tau^4\zeta^2 + \frac{436\tau^3\zeta^3}{3} - 48\tau^3\zeta^2 + \frac{1,228\zeta\tau^{11/2}}{693} + 6/5t^5 \\
 & - \frac{72\tau^5\zeta^3}{5} + 4/9\tau^{9/2} + \tau^2 + \frac{88\tau^{9/2}\zeta^5}{3} - \frac{220\tau^{9/2}\zeta^4}{63} - 1/39\zeta^5\tau^{13} + 14/3\tau^3\zeta \\
 & + \frac{32\tau^{15/2}}{45} + \frac{16\tau^{10}}{135} - 3\tau^4 - \frac{11,264\tau^{13/2}\zeta^3}{819} - \frac{880\tau^{9/2}\zeta^6}{9} - \frac{150\tau^{19/2}\zeta^8}{19} - \frac{32\tau^{21/2}\zeta^5}{147} \\
 & + \frac{160\tau^{19/2}\zeta^7}{19} + \frac{1,664\tau^{21/2}\zeta^8}{49} - \frac{44\tau^{19/2}\zeta^{10}}{57} - \frac{5,504\tau^{21/2}\zeta^7}{441} + \frac{220\tau^{19/2}\zeta^9}{57} + \frac{160\tau^{21/2}\zeta^6}{63} \\
 & + 54\tau^4\zeta^8 - 216\tau^4\zeta^7 + 343\tau^4\zeta^6 - 273\tau^4\zeta^5 + \frac{225\tau^4\zeta^4}{2} + \frac{212\tau^9\zeta^8}{21} - \frac{848\tau^9\zeta^7}{21} \\
 & + \frac{5,194\tau^9\zeta^6}{81} - \frac{1,378\tau^9\zeta^5}{27} + \frac{1,325\tau^9\zeta^4}{63} - \frac{28\tau^{13}\zeta^{13}}{117} + \frac{14\tau^{13}\zeta^{12}}{9} - \frac{170\tau^{13}\zeta^{11}}{39} + \frac{803\tau^{13}\zeta^{10}}{117} \\
 & - \frac{775\tau^{13}\zeta^9}{117} + 4\tau^{13}\zeta^8 - \frac{194\tau^8\zeta^{13}}{21} - \frac{172\tau^{13}\zeta^7}{117} + \frac{1,261\tau^8\zeta^{12}}{21} + \frac{35\tau^{13}\zeta^6}{117} - \frac{8,245\tau^8\zeta^{11}}{49} \\
 & + \frac{77,891\tau^8\zeta^{10}}{294} - \frac{75,175\tau^8\zeta^9}{294} - \frac{112\tau^3\zeta^{13}}{3} + \frac{728\tau^3\zeta^{12}}{3} - 680\tau^3\zeta^{11} - \frac{64\tau^{10}\zeta^3}{45} + \frac{3,212\tau^3\zeta^{10}}{3} \\
 & + \frac{32\tau^{10}\zeta^2}{15} - \frac{3,100\tau^3\zeta^9}{3} - \frac{128\tau^{10}\zeta}{135} + \frac{13\tau^8\zeta}{36} + \frac{3,072\tau^{13/2}\zeta^8}{91} - \frac{12,288\tau^{13/2}\zeta^7}{91} + \frac{25,088\tau^{13/2}\zeta^6}{117} \\
 & - \frac{512\tau^{13/2}\zeta^5}{3} + \frac{6,400\tau^{13/2}\zeta^4}{91} + \frac{32\tau^{23/2}\zeta^8}{23} - \frac{128\tau^{23/2}\zeta^7}{23} + \frac{5,488\tau^{23/2}\zeta^6}{621} - \frac{1,456\tau^{23/2}\zeta^5}{207} \\
 & + \frac{200\tau^{23/2}\zeta^4}{69} - \frac{352\tau^{23/2}\zeta^3}{621} + \frac{8\tau^{23/2}\zeta^2}{207} - \frac{10\tau^{19/2}\zeta^4}{57} - \frac{128\tau^{21/2}\zeta^{13}}{63} + \frac{832\tau^{21/2}\zeta^{12}}{63} \\
 & - \frac{5,440\tau^{21/2}\zeta^{11}}{147} + \frac{25,696\tau^{21/2}\zeta^{10}}{441} - \frac{24,800\tau^{21/2}\zeta^9}{441} - \frac{256\tau^{11/2}\zeta^{13}}{11} + \frac{1,664\tau^{11/2}\zeta^{12}}{11} \\
 & - \frac{32,640\tau^{11/2}\zeta^{11}}{77} + \frac{4,672\tau^{11/2}\zeta^{10}}{7} - \frac{49,600\tau^{11/2}\zeta^9}{77} - \frac{128\tau^{15/2}\zeta^3}{15} + \frac{64\tau^{15/2}\zeta^2}{5} - \frac{256\tau^{15/2}\zeta}{45} \Big].
 \end{aligned} \tag{23}$$

In same manner, the remaining terms of v_j for $(j > 2)$ can be calculated easily by using LADM. In general, solution of LADM is given by

$$v(\zeta, \tau) = \sum_{j=0}^{\infty} v_j(\zeta, \tau) = v_0(\zeta, \tau) + v_1(\zeta, \tau) + v_2(\zeta, \tau) + v_3(\zeta, \tau) + \dots \quad (24)$$

Putting equations (22) and (23) in equation (24), and setting $t = 0.001$, we obtain

$$v(\zeta, \tau) = \begin{aligned} & -1.674545510 \times 10^{-14} + 0.000000002709883322\zeta^{15} - 0.00000001029446274\zeta^{14} \\ & - 0.00000005287620897\zeta^{13} + 0.0000004558966121\zeta^{12} + 0.00000000666821556\zeta \\ & - 0.000001333328227\zeta^{11} + 3.296845507 \times 10^{-14}\zeta^{21} + 0.00004200596\zeta^5 \\ & + 1.374456220 \times 10^{-11}\zeta^{18} - 2.474922739 \times 10^{-15}\zeta^{22} - 6.234400617 \times 10^{-11}\zeta^{17} \\ & - 0.0000000002107409613\zeta^{16} - 0.00001987508000\zeta^{10} + 0.0001079400927\zeta^9 \\ & - 0.0002237541655\zeta^8 + 0.9999950012\zeta^4 - 2.000000038\zeta^3 + 0.000239543151\zeta^7 \\ & + 1.000000024\zeta^2 - 0.00013990914\zeta^6 + 3.235555650 \times 10^{-17}\zeta^{24} - 1.879893389 \\ & \times 10^{-16}\zeta^{23} - 1.721803518 \times 10^{-13}\zeta^{20} - 1.029720189 \times 10^{-12}\zeta^{19} - 2.588444520 \\ & \times 10^{-18}\zeta^{25} + \dots, \end{aligned} \quad (25)$$

which is the LADM analytical solution.

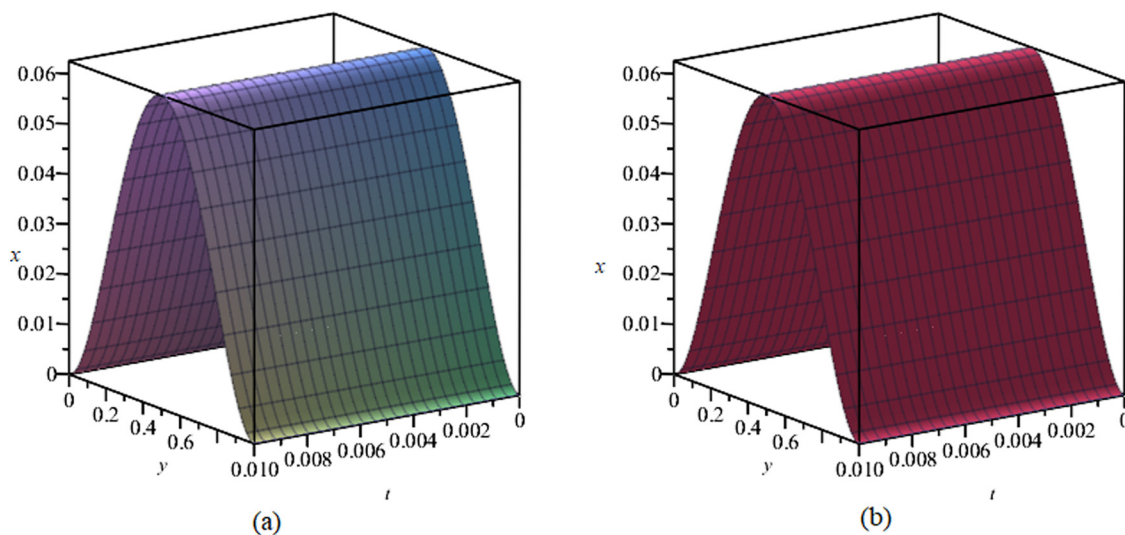


Figure 4: (a) 3D exact-solution and (b) 2D approximate-solution graph of Example 4.2.

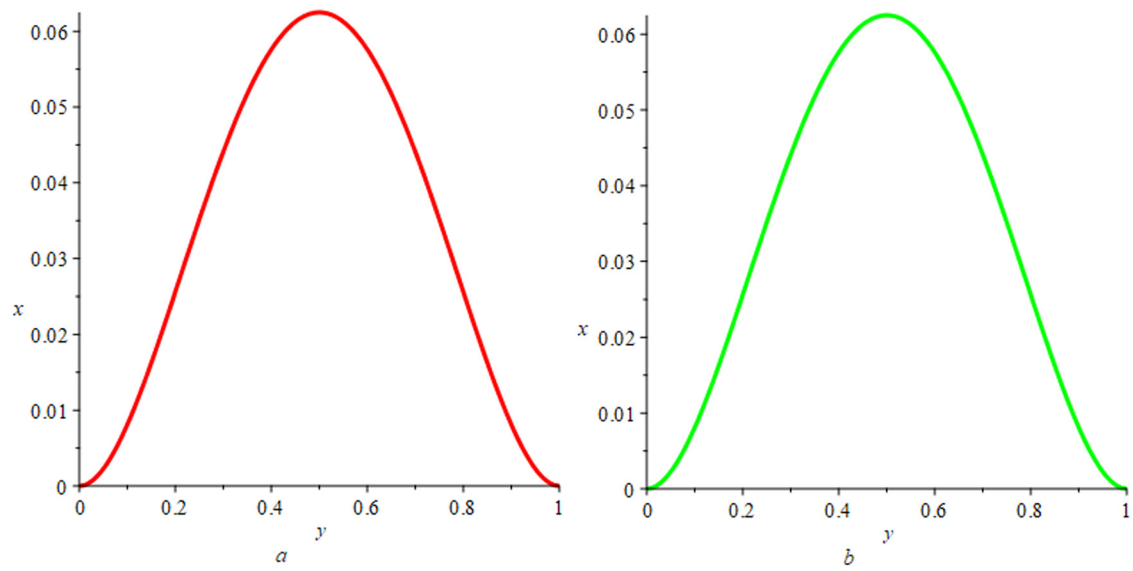


Figure 5: (a) 2D exact-solution and (b) 2D approximate-solution graph of Example 4.2.

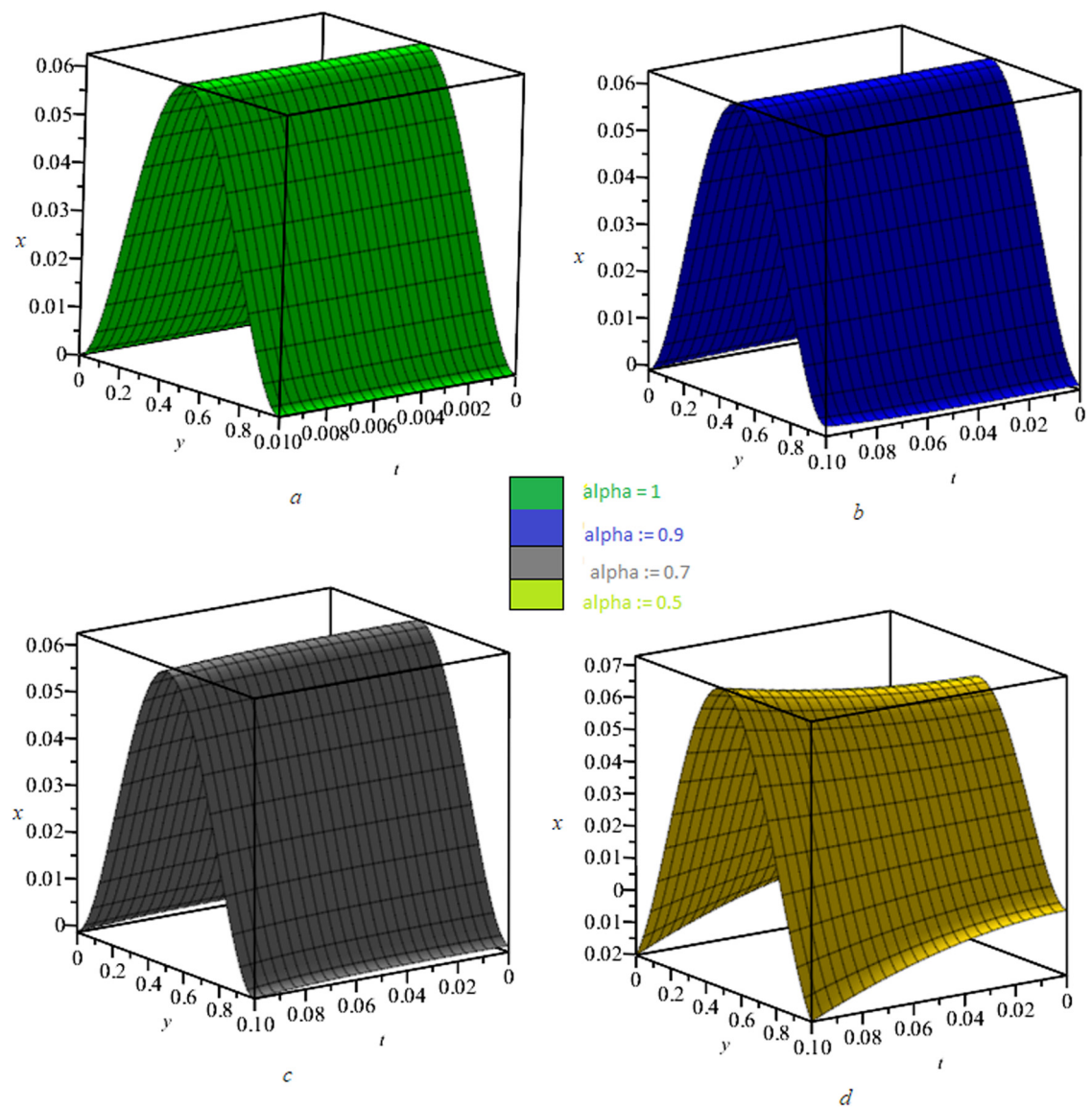


Figure 6: 3D LADM-solution graphs at (a) $\alpha = 1$, (b) $\alpha = 0.9$, (c) $\alpha = 0.7$, and (d) $\alpha = 0.5$, of Example 4.2.

Table 3: AE at different time level and spaces with fractional order $\alpha = 1$ of Example 4.2

τ	ζ	Exact solution	Approximate solution	AE at $\alpha = 1$
0.001	0.2	0.02560000028	0.02560000082	0.00000000054000
	0.4	0.05760000299	0.05760000184	0.00000000115000
	0.6	0.05760000305	0.05760000184	0.00000000121000
	0.8	0.02560000034	0.02560000082	0.00000000048000
	1.0	0.00000000000	0.00000000000	0.00000000000002
0.003	0.2	0.02560000732	0.02560001262	0.00000000530000
	0.4	0.05760003851	0.05760002840	0.00000001011000
	0.6	0.05760003977	0.05760002840	0.00000001137000
	0.8	0.02560000898	0.02560001262	0.00000000364000
	1.0	0.00000000000	0.00000000000	0.00000000000466
0.005	0.2	0.02560001748	0.02560004562	0.00000002778000
	0.4	0.05760012042	0.05760010284	0.00000001858000
	0.6	0.05760013065	0.05760010842	0.00000002781000
	0.8	0.02560002554	0.02560004526	0.00000002072000
	1.0	0.00000000005	0.00000000000	0.00000000005250
0.007	0.2	0.02560004779	0.02560010496	0.00000005717000
	0.4	0.05760026899	0.05760023616	0.00000003283000
	0.6	0.05760029427	0.05760023616	0.00000005811000
	0.8	0.02560006715	0.02560010496	0.00000003781000
	1.0	0.00000000028	0.00000000000	0.00000000028232
0.009	0.2	0.02560009776	0.02560019671	0.00000009895000
	0.4	0.05760049099	0.05760044258	0.00000004841000
	0.6	0.05760054460	0.05760044259	0.00000010201000
	0.8	0.02560013883	0.02560019671	0.00000005788000
	1.0	0.00000000099	0.00000000000	0.0000000099186

Table 4: AE at different fractional order α of Example 4.2

ζ	AE at $\alpha = 0.5$	AE at $\alpha = 0.75$	AE at $\alpha = 0.95$	AE at $\alpha = 0.95$ [15]
0.1	0.00000307130	0.00000000497	0.00000000040	0.0000426000
0.2	0.00000706078	0.00000006049	0.00000000219	0.0011812000
0.3	0.00000998019	0.00000006047	0.00000000174	0.0023206000
0.4	0.00000500327	0.00000002539	0.00000000194	0.0026410000
0.5	0.00000344896	0.00000009277	0.00000000450	0.0017125000
0.6	0.00000846348	0.00000006319	0.00000000242	0.0003752000
0.7	0.00000773908	0.00000000941	0.00000000084	0.0029206000
0.8	0.00000363026	0.00000003137	0.00000000105	0.0046428000
0.9	0.00000186667	0.00000001419	0.00000000072	0.0046428000
1.0	0.00001746190	0.00000000019	0.00000000146	—

5 Results and discussion

Figures 1 and 2 represent the 2D and 3D comparison plots of exact and LADM solutions of Example 4.1, while Figure 3 is a 3D solution graph at different fractional order of Example 4.1. Similarly, Figures 4 and 5 are the 2D and 3D comparison plots of exact and LADM solutions of Example 4.2, while Figure 6 is a 3D solution graph at different fractional orders of Example 4.2. In Table 1, the absolute error associated with LADM at $\alpha = 1$ is discussed for Example 4.1. Table 2 represents the absolute errors associated with LADM at different fractional order α of Example 4.1. Similarly, Table 3 shows the absolute error (AE) at different time level and spaces with fractional order $\alpha = 1$ of Example 4.2. Table 4 consists of AE associated with LADM at different fractional orders α of Example 4.1. The aforementioned graphs and tables have confirmed the higher degree of accuracy

as compared to the extended cubic B-Spline. The obtained fractional-order solutions are well organized, and their graphs are in well agreement with the integer-order solutions.

6 Conclusion

In this article, an efficient and straightforward procedure of LADM was implemented for the solutions of FPIDEs. The solutions are obtained with the Caputo operator. It had been very rare in the literature to investigate the solutions of FPIDEs, but in this article, the task has been done successfully via LADM. The LADM solutions were displayed by using graphs and tables. The graphs and tables have shown that the proposed method has a higher degree of accuracy. Furthermore, the LADM results were compared with the solutions of other existing techniques that show the superiority and efficiency of the current technique. For convergence analysis, uniqueness, and exactness, the related references have been cited in this article. The LADM procedure is found to be straightforward, which required fewer calculations. Thus, it would be suggested to apply the present method for the solutions of other nonlinear FPIDEs and their systems.

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Ethical approval: The conducted research is not related to either human or animals use.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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