

Research Article

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On initial value problem for elliptic equation on the plane under Caputo derivative

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Abstract: In this article, we are interested to study the elliptic equation under the Caputo derivative. We obtain several regularity results for the mild solution based on various assumptions of the input data. In addition, we derive the lower bound of the mild solution in the appropriate space. The main tool of the analysis estimation for the mild solution is based on the bound of the Mittag-Leffler functions, combined with analysis in Hilbert scales space. Moreover, we provide a regularized solution for our problem using the Fourier truncation method. We also obtain the error estimate between the regularized solution and the mild solution. Our current article seems to be the first study to deal with elliptic equations with Caputo derivatives on the unbounded domain.

Keywords: fractional elliptic equation, Caputo derivative, Mittag-Leffler functions

MSC 2020: 60G15, 60G22, 60G52, 60G57

1 Introduction

In this article, we consider the following fractional elliptic equation on the plane:

$$\begin{cases} \frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} + \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} = G(x, y, t) & (x, y, t) \in \mathbb{R}^2 \times (0, T), \\ u(x, y, 0) = u_0(x, y) & (x, y) \in \mathbb{R}^2, \\ u_t(x, y, 0) = 0 & (x, y) \in \mathbb{R}^2, \end{cases} \quad (1)$$

where u_0 and G are defined later. The symbol $\frac{\partial^\alpha w(x, y, t)}{\partial t^\alpha}$ with the case $1 < \alpha < 2$ on the left-hand side of the first equation of (1) is defined as follows [1]:

$$\begin{cases} \frac{\partial^\alpha w(x, y, t)}{\partial t^\alpha} = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-r)^{1-\alpha} \frac{\partial^2 w}{\partial r^2}(x, y, r) dr, & \text{for } 1 < \alpha < 2, \\ \frac{\partial^\alpha w(x, y, t)}{\partial t^\alpha} = \frac{\partial^2 w}{\partial t^2}(x, y, t), & \text{for } \alpha = 2, \end{cases} \quad (2)$$

and Γ is the Gamma function.

If $\alpha = 2$, the first equation of problem (1) becomes the classical elliptic equation

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$$u_{tt} + \Delta u = G(x, y, t), \quad (3)$$

where

$$\Delta u = \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2}. \quad (4)$$

As you know, elliptic partial differential equations have applications in almost all areas of mathematics, from harmonic analysis to geometry to Lie theory, as well as numerous applications in physics. An elliptical partial differential equation (PDE) occurs when the laws of physics, such as conservation of energy or charge, are studied in a final or steady state, or are independent of time.

The number of studies on the elliptic equation and also similar equations is so abundant that we could list typical works, e.g., [2–12] and references therein.

As is known, some physical phenomena involving memory are difficult to describe by classical derivatives, so it is necessary to have some derivatives of nonlocal forms, such as the Caputo derivative. This is also the reason why problem (1) is called the Caputo elliptic equation. The direction of our study of our problem (1) lies in the branch concerned with partial differential equations with fractional derivatives. A fractional partial differential equation is a type of mathematical equation that describes the behavior of complex systems with memory and non-local effects. Some of the works related to derivatives of order can be listed, such as [13–29].

Let us list some interesting articles on fractional elliptic equations. Berdyshev et al. [30] studied the solvability of fractional elliptic equations with the Riemann-Liouville derivative. Turmetov and Nazarova [31] studied the Dirichlet and Neumann boundary-value problems for the Laplace equation with Caputo and Riemann-Liouville derivative. Turmetov [32] derived a question about the solvability of some boundary value problems for a non-homogenous poly-harmonic equation.

The study by Jin and Rundell [33] is one of the first introductory works on the fractional elliptic, which has the form of (1). Jin and Rundell [33] provided ill-posedness in the sense of Hadamard for the fractional elliptic equation without giving its approximate solution. In Hadamard's view, a problem is said to be well-posed if it satisfies all three conditions. The first condition is that the problem must have a solution. The second condition is that the solution to the problem is unique. The third condition is the continuous dependence of the solution on the input data. Motivated by the study of Jin and Rundell [33], Tuan et al. [34] studied the Cauchy problem for a semilinear fractional elliptic equation. In fact, they considered the following elliptic problem of Caputo derivative:

$$\begin{cases} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} + \Delta u = G(x, t, u(x, t)) & (x, t) \in \Omega \times (0, T), \\ u(x, 0) = u_0(x) & x \in \Omega, \\ u_t(x, 0) = 0 & x \in \Omega, \end{cases} \quad (5)$$

with the Dirichlet condition $u(x, t) = 0$, $(x, t) \in \partial\Omega \times (0, T)$. Here, $\Omega \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary $\partial\Omega$. Under some assumptions of the sought solution, they proposed the Fourier truncation method for approximating the problem. Some estimates of the logarithmic type between the sought solution and the regularized solution are established. Further development work of Tuan et al. [34] has been completed in detail in the study by Au et al. [35]. In a recent article by Tuan et al. [36], they studied the new method for solving problem (5) when the input data are noised in L^p space. For the random model for an elliptic equation with Caputo derivative, Tuan et al. [36] also proposed the Fourier truncation method for stabilizing the ill-posed problem. They established some convergence rates between the exact solution and the regularized solution.

To the best of our knowledge, there are not any results on the fractional elliptic equation of the form (1) on an unbounded domain $\mathbb{R}^n$. It is a fact that our model research on $\mathbb{R}^n$ is often more difficult and complicated than in the bounded domain. The main technique is to use the Fourier transform that combines the intercepted evaluations of the Mittag-Leffler functions. Processing evaluations on $\mathbb{R}^2$ and also some unbounded domain is a difficult task, thus we have difficulty choosing suitable function spaces for the input functions.

Some new results are obtained in this article, which are described in detail in the following sections. We obtain the regularity of the mild solution under various assumptions of the source function G and the initial

datum u_0 . In addition, we obtain the lower bound of the mild solution at the terminal time $t = T$. It is surprising that the lower bound results for mild solution are not so much for fractional PDEs of Caputo derivative. In addition, we also emphasize that the regularity of the mild solution is not mentioned in the series of works [36]. The second result is to provide a regularized solution when the source function $G = 0$. The key method is the truncation method, which has been learned through works such as [34,36–38]. We obtain the error estimate between the exact solution and the regularized solution. The third result is to investigate the approximation solution when $u_0 = 0$ and G have two various forms. Some complicated evaluation of the convergence rate between the regularized solution and the sought solution is introduced.

This article is organized as follows: in Section 2, we introduce some preliminary results in which are mentioned in some solution spaces; Section 3 introduces a regularity result for the mild solution to problem (1); and in Section 4, we introduce the truncation method and provide the error estimate between the regularized solution and the exact solution.

2 Preliminary results

We start by introducing some notations and assumptions that are needed for our analysis in the next sections. Thankful to the interesting work of Au et al. [35], we can introduce some similar and modified spaces. The Sobolev space $H^k(\mathbb{R}^2)$ is defined as follows:

$$\begin{aligned} H^k(\mathbb{R}^2) &:= \left\{ f(x, y) : f \in L^2(\mathbb{R}^2), \int \int_{-\infty-\infty}^{+\infty+\infty} (1 + \xi^2 + \eta^2)^k |\widehat{f}(\xi, \eta)|^2 d\xi d\eta < \infty \right\} \\ \overline{H}^k(\mathbb{R}^2) &:= \left\{ f(x, y) : f \in L^2(\mathbb{R}^2), \int \int_{-\infty-\infty}^{+\infty+\infty} (\xi^2 + \eta^2)^k |\widehat{f}(\xi, \eta)|^2 d\xi d\eta < \infty \right\} \end{aligned} \quad (6)$$

and equipped with the following norm:

$$\begin{aligned} \|f\|_{H^k(\mathbb{R}^2)} &:= \left(\int \int_{-\infty-\infty}^{+\infty+\infty} (1 + \xi^2 + \eta^2)^k |\widehat{f}(\xi, \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \\ \|f\|_{\overline{H}^k(\mathbb{R}^2)} &:= \left(\int \int_{-\infty-\infty}^{+\infty+\infty} (\xi^2 + \eta^2)^k |\widehat{f}(\xi, \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}}. \end{aligned} \quad (7)$$

Here, $\widehat{f}(\xi, \eta)$ is the Fourier transform of $f(x, y)$ which is defined as follows:

$$\widehat{f}(\xi, \eta) = \frac{1}{2\pi} \int \int_{-\infty-\infty}^{+\infty+\infty} e^{-i\xi x - i\eta y} f(x, y) dx dy. \quad (8)$$

Note that there is no embedding between the two spaces $H^k(\mathbb{R}^2)$ and $\overline{H}^k(\mathbb{R}^2)$. Let us introduce the subspace $\mathbb{X}_{\theta, M}, \overline{\mathbb{X}}_{\theta, M}$ in $L^\infty(0, T; L^2(\mathbb{R}^2))$ as follows:

$$\begin{aligned} \mathbb{X}_{\theta, M} &:= \left\{ f \in L^2(\mathbb{R}^2), \int \int_{-\infty-\infty}^{+\infty+\infty} (1 + \xi^2 + \eta^2)^\theta \exp\left(2(\xi^2 + \eta^2)^{\frac{1}{\theta}} M\right) |\widehat{f}(\xi, \eta)|^2 d\xi d\eta < \infty \right\} \\ \overline{\mathbb{X}}_{\theta, M} &:= \left\{ f \in L^2(\mathbb{R}^2), \int \int_{-\infty-\infty}^{+\infty+\infty} (\xi^2 + \eta^2)^\theta \exp\left(2(\xi^2 + \eta^2)^{\frac{1}{\theta}} M\right) |\widehat{f}(\xi, \eta)|^2 d\xi d\eta < \infty \right\} \end{aligned} \quad (9)$$

and

$$L^\infty(0, T; \mathbb{X}_{\theta, M}) = \left\{ f \in L^\infty(0, T; L^2(\mathbb{R}^2)) : \int \int_{-\infty-\infty}^{+\infty+\infty} (1 + \xi^2 + \eta^2)^\theta \exp\left(2(\xi^2 + \eta^2)^{\frac{1}{\theta}} M\right) \sup_{0 \leq t \leq T} |\widehat{f}(\xi, \eta, t)|^2 d\xi d\eta < \infty \right\}, \quad (10)$$

and

$$L^\infty(0, T; \mathbb{X}_{\theta, M}) = \left\{ f \in L^\infty(0, T; L^2(\mathbb{R}^2)) : \int_{-\infty-\infty}^{+\infty+\infty} (\xi^2 + \eta^2)^\theta \exp\left(2(\xi^2 + \eta^2)^{\frac{1}{\alpha}} M\right) \sup_{0 \leq t \leq T} |\widehat{f}(\xi, \eta, t)|^2 d\xi d\eta < \infty \right\}. \quad (11)$$

The norm of $f \in \mathbb{X}_{\theta, M}$ is given as follows:

$$\|f\|_{\mathbb{X}_{\theta, M}} = \sqrt{\int_{-\infty-\infty}^{+\infty+\infty} (1 + \xi^2 + \eta^2)^\theta \exp\left((\xi^2 + \eta^2)^{\frac{1}{\alpha}} M\right) |\widehat{f}(\xi, \eta)|^2 d\xi d\eta}. \quad (12)$$

The norm of $f \in L^\infty(0, T; \mathbb{X}_{\theta, M})$ is given as follows:

$$\|f\|_{L^\infty(0, T; \mathbb{X}_{\theta, M})} = \sqrt{\int_{-\infty-\infty}^{+\infty+\infty} (1 + \xi^2 + \eta^2)^\theta \exp\left(2(\xi^2 + \eta^2)^{\frac{1}{\alpha}} M\right) \sup_{0 \leq t \leq T} |\widehat{f}(\xi, \eta, t)|^2 d\xi d\eta}. \quad (13)$$

Definition 2.1. The Mittag-Leffler function is defined as follows:

$$E_{\alpha, \theta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \theta)}, \quad z \in \mathbb{C}, \quad (14)$$

where $\alpha > 0$ and $\theta \in \mathbb{R}$ are arbitrary constants.

Let us give some following properties for Mittag-Leffler functions $E_{\alpha, 1}(z)$, $E_{\alpha, 2}(z)$, and $E_{\alpha, \alpha}(z)$.

Lemma 2.1. (See [39]) Let $0 < \alpha < 1$. There exist some constants $D_1, D_2, D_3, D_4 > 0$ such that

$$\frac{D_1}{\alpha} \exp(z^{1/\alpha}) \leq E_{\alpha, 1}(z) \leq \frac{D_2}{\alpha} \exp(z^{1/\alpha}), \quad z > 0, \quad (15)$$

and

$$\frac{D_3}{\alpha} \exp(z^{1/\alpha}) \leq z^{\frac{\alpha-1}{\alpha}} E_{\alpha, \alpha}(z) \leq \frac{D_4}{\alpha} \exp(z^{1/\alpha}), \quad z > 0. \quad (16)$$

Lemma 2.2. (See [39]) Let $z \in \mathbb{R}$. Then, we have

$$\frac{d}{dz} E_{\alpha, 1}(z) = \frac{E_{\alpha, \alpha}(z)}{\alpha} \quad (17)$$

and

$$\frac{d}{dz} E_{\alpha, 1}(\lambda z^\alpha) = \frac{1}{z} E_{\alpha, 0}(\lambda z^\alpha). \quad (18)$$

3 Regularity of the mild solution

In this section, we provide some regularity results for the mild solution to problem (1) under some suitable assumptions of u_0 and G .

Theorem 3.1. Let $u_0 \in \mathbb{X}_{\theta, T}$ and $G \in L^\infty(0, T; \mathbb{X}_{\theta, T})$. Then, we have

$$\|u(\cdot, t)\|_{H^\theta(\mathbb{R}^2)} \leq \|u_0\|_{\mathbb{X}_{\theta, T}} + \|G\|_{L^\infty(0, T; \mathbb{X}_{\theta, T})}. \quad (19)$$

In addition, if $G = 0$ and $u_0 \in \mathbb{X}_{\theta + \frac{2}{\alpha}, T}$, then $\frac{\partial u}{\partial t} \in L^\infty(0, T; \mathbb{X}_{\theta, T})$, and we have

$$\left\| \frac{\partial u}{\partial t} \right\|_{\mathbb{H}^\theta(\mathbb{R}^2)} \leq \frac{D_2}{\alpha} \|u_0\|_{\mathbb{X}_{\theta+\frac{2}{\alpha},T}}. \quad (20)$$

Proof. Applying Fourier transform (with respect to x and y) to (1), we obtain

$$\begin{cases} \frac{\partial^\alpha \hat{u}(\xi, \eta, t)}{\partial t^\alpha} = (\xi^2 + \eta^2) \hat{u}(\xi, \eta, t) + \hat{G}(\xi, \eta, t) & (\xi, \eta, t) \in \mathbb{R}^2 \times (0, T), \\ \hat{u}(\xi, \eta, 0) = \hat{u}_0(\xi, \eta) & (\xi, \eta) \in \mathbb{R}^2. \end{cases} \quad (21)$$

Using the Laplace transform with respect to t , we obtain the solution

$$\hat{u}(\xi, \eta, t) = E_{\alpha,1}((\xi^2 + \eta^2)t^\alpha) \hat{u}_0(\xi, \eta) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((\xi^2 + \eta^2)(t-s)^\alpha) \hat{G}(\xi, \eta, s) ds. \quad (22)$$

The inverse Fourier transform gives the following formula:

$$\begin{aligned} u(x, y, t) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \left[E_{\alpha,1}((\xi^2 + \eta^2)t^\alpha) \hat{u}_0(\xi, \eta) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((\xi^2 + \eta^2)(t-s)^\alpha) \hat{G}(\xi, \eta, s) ds \right] e^{ix\xi + iy\eta} d\xi d\eta \\ &= v(x, t) + w(x, t) \end{aligned} \quad (23)$$

where we denote

$$v(x, y, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} E_{\alpha,1}((\xi^2 + \eta^2)t^\alpha) \hat{u}_0(\xi, \eta) e^{ix\xi + iy\eta} d\xi d\eta, \quad (24)$$

and

$$w(x, y, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((\xi^2 + \eta^2)(t-s)^\alpha) \hat{G}(\xi, \eta, s) ds e^{ix\xi + iy\eta} d\xi d\eta. \quad (25)$$

The norm of the solution v in the Hilbert scale is equal to

$$\|v(\cdot, \cdot, t)\|_{\mathbb{H}^\theta(\mathbb{R}^2)}^2 = \int_{-\infty-\infty}^{+\infty+\infty} \int (1 + \xi^2 + \eta^2)^\theta [E_{\alpha,1}((\xi^2 + \eta^2)t^\alpha) \hat{u}_0(\xi, \eta)]^2 d\xi d\eta. \quad (26)$$

In view of Lemma (2.1), we obtain that

$$E_{\alpha,1}((\xi^2 + \eta^2)t^\alpha) \leq \frac{D_2}{\alpha} \exp((\xi^2 + \eta^2)^{\frac{1}{\alpha}} t) \leq \frac{D_2}{\alpha} \exp((\xi^2 + \eta^2)^{\frac{1}{\alpha}} T). \quad (27)$$

Combining equations (26) and (30), we find that

$$\|v(\cdot, \cdot, t)\|_{\mathbb{H}^\theta(\mathbb{R}^2)}^2 \leq \int_{-\infty-\infty}^{+\infty+\infty} \int (1 + \xi^2 + \eta^2)^\theta \exp(2(\xi^2 + \eta^2)^{\frac{1}{\alpha}} T) |\hat{u}_0(\xi, \eta)|^2 d\xi d\eta = \|u_0\|_{\mathbb{X}_{\theta,T}}^2. \quad (28)$$

The norm of the function w is bounded by

$$\|w(\cdot, \cdot, t)\|_{\mathbb{H}^\theta(\mathbb{R}^2)}^2 = \int_{-\infty-\infty}^{+\infty+\infty} \int (1 + \xi^2 + \eta^2)^\theta \left| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((\xi^2 + \eta^2)(t-s)^\alpha) \hat{G}(\xi, \eta, s) ds \right|^2 d\xi d\eta. \quad (29)$$

In view of Lemma (2.1), we obtain that

$$\begin{aligned} E_{\alpha,\alpha}((\xi^2 + \eta^2)(t-s)^\alpha) &\leq \frac{D_2}{\alpha} (\xi^2 + \eta^2)^{\frac{1-\alpha}{\alpha}} (t-s)^{\alpha-\frac{1}{\alpha}} \exp((\xi^2 + \eta^2)^{\frac{1}{\alpha}} (t-s)) \\ &= \frac{D_2}{\alpha} (\xi^2 + \eta^2)^{\frac{1-\alpha}{\alpha}} (t-s)^{1-\alpha} \exp((\xi^2 + \eta^2)^{\frac{1}{\alpha}} (t-s)). \end{aligned} \quad (30)$$

This estimate implies that the following bound:

$$\left| \int_0^t (t-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(t-s)^a) \widehat{G}(\xi, \eta, s) ds \right|^2 \leq (\xi^2 + \eta^2)^{\frac{2-2a}{a}} \left| \int_0^t \exp\left((\xi^2 + \eta^2)^{\frac{1}{a}}(t-s)\right) |\widehat{G}(\xi, \eta, s)| ds \right|^2. \quad (31)$$

Thus, using Hölder inequality, we obtain the following bound:

$$\begin{aligned} \|w(\cdot, \cdot, t)\|_{\mathbb{H}^\theta(\mathbb{R}^2)}^2 &= \int_{-\infty-\infty}^{+\infty+\infty} \int_{-\infty-\infty}^{+\infty+\infty} (1 + \xi^2 + \eta^2)^\theta \left| \int_0^t (t-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(t-s)^a) \widehat{G}(\xi, \eta, s) ds \right|^2 d\xi d\eta \\ &\leq \int_{-\infty-\infty}^{+\infty+\infty} \int_{-\infty-\infty}^{+\infty+\infty} (1 + \xi^2 + \eta^2)^{\theta + \frac{2-2a}{a}} \exp\left(2T(\xi^2 + \eta^2)^{\frac{1}{a}}\right) |\widehat{G}(\xi, \eta, s)|^2 d\xi d\eta ds \\ &\leq \|G\|_{L^\infty(0,T; \mathbb{X}_{\theta,T})}^2. \end{aligned} \quad (32)$$

Combining equations (42), (44), and (48), we derive that

$$\|u(\cdot, \cdot, t)\|_{\mathbb{H}^\theta(\mathbb{R}^2)} \leq \|v(\cdot, \cdot, t)\|_{\mathbb{H}^\theta(\mathbb{R}^2)} + \|w(\cdot, \cdot, t)\|_{\mathbb{H}^\theta(\mathbb{R}^2)} \leq \|u_0\|_{\mathbb{X}_{\theta,T}} + \|G\|_{L^\infty(0,T; \mathbb{X}_{\theta,T})}. \quad (33)$$

Our next aim is to show (20). If $G = 0$, then we have immediately that

$$u(x, y, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} E_{a,1}((\xi^2 + \eta^2)t^a) \widehat{u}_0(\xi, \eta) e^{ix\xi + iy\eta} d\xi d\eta. \quad (34)$$

By looking at Lemma (2.2), we can see that

$$\frac{d}{dt} E_{a,1}(\lambda t^a) = \frac{dE_{a,1}(\lambda t^a)}{d(\lambda t^a)} \frac{d(\lambda t^a)}{dt} = \lambda t^{a-1} E_{a,a}(\lambda t^a), \quad (35)$$

for $\lambda \in \mathbb{R}$. Set $\lambda = (\xi^2 + \eta^2)$, we obtain

$$\frac{d}{dt} E_{a,1}((\xi^2 + \eta^2)t^a) = (\xi^2 + \eta^2) t^{a-1} E_{a,a}((\xi^2 + \eta^2)t^a). \quad (36)$$

Combining equations (34) and (36) yields that

$$\frac{\partial u(x, y, t)}{\partial t} = \frac{1}{2\pi} t^{a-1} \int_{\mathbb{R}^2} (\xi^2 + \eta^2) E_{a,a}((\xi^2 + \eta^2)t^a) \widehat{u}_0(\xi, \eta) e^{ix\xi + iy\eta} d\xi d\eta. \quad (37)$$

In view of (30), we obtain that

$$t^{a-1} (\xi^2 + \eta^2) E_{a,a}((\xi^2 + \eta^2)t^a) \leq \frac{D_2}{a} (\xi^2 + \eta^2)^{\frac{1}{a}} \exp\left((\xi^2 + \eta^2)^{\frac{1}{a}} t\right). \quad (38)$$

Combining equations (37) and (38) and using Plancherel theorem, we derive that

$$\begin{aligned} \left\| \frac{\partial u(x, y, t)}{\partial t} \right\|_{\mathbb{H}^\theta(\mathbb{R}^2)}^2 &= \int_{-\infty-\infty}^{+\infty+\infty} \int_{-\infty-\infty}^{+\infty+\infty} (1 + \xi^2 + \eta^2)^\theta (t^{a-1} (\xi^2 + \eta^2) E_{a,a}((\xi^2 + \eta^2)t^a) \widehat{u}_0(\xi, \eta))^2 d\xi d\eta \\ &\leq \frac{D_2^2}{a^2} \int_{-\infty-\infty}^{+\infty+\infty} \int_{-\infty-\infty}^{+\infty+\infty} (1 + \xi^2 + \eta^2)^{\theta + \frac{2}{a}} \exp\left(2(\xi^2 + \eta^2)^{\frac{1}{a}} t\right) |\widehat{u}_0(\xi, \eta)|^2 d\xi d\eta. \end{aligned} \quad (39)$$

This estimate implies the desired result (20). The proof of the theorem is completed. $\square$

Theorem 3.2. Let $u_0 = 0$ and assume that $G \equiv g(x, y)$. If $g \in \mathbb{X}_{\theta-2,T}$, then the mild solution u belongs to $L^\infty(0, T; \overline{H}^\theta(\mathbb{R}^2))$, which satisfies that

$$\|u\|_{L^\infty(0,T; \overline{H}^\theta(\mathbb{R}^2))} \leq \frac{|D_2|}{a} \|G\|_{\mathbb{X}_{\theta-2,T}}. \quad (40)$$

If $g \in \overline{H}^{\frac{\theta-1}{2}+\frac{\mu}{a}}(\mathbb{R}^2)$, then we have the following lower bound:

$$\|u(\cdot, \cdot, T)\|_{\overline{H}^\theta(\mathbb{R}^2)} \geq \frac{|D_1|T^\mu}{\alpha} \|g\|_{\overline{H}^{\frac{\theta-1}{2}+\frac{\mu}{a}}(\mathbb{R}^2)}. \quad (41)$$

Proof. We have given proof of some estimations.

Step 1. Proof of (40). Since $u_0 = 0$, then the mild solution of problem (1) is defined as follows:

$$u(x, y, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_0^t (t-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(t-s)^a) ds \widehat{g}(\xi, \eta) e^{ix\xi + iy\eta} d\xi d\eta. \quad (42)$$

In view of equation (30), we obtain that

$$(t-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(t-s)^a) \leq \frac{D_2}{\alpha} (\xi^2 + \eta^2)^{\frac{1-a}{a}} \exp\left((\xi^2 + \eta^2)^{\frac{1}{a}}(t-s)\right). \quad (43)$$

This implies that

$$\begin{aligned} \int_0^t (t-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(t-s)^a) ds &\leq \frac{D_2}{\alpha} (\xi^2 + \eta^2)^{\frac{1-a}{a}} \int_0^t \exp\left((\xi^2 + \eta^2)^{\frac{1}{a}}(t-s)\right) ds \\ &= \frac{D_2}{\alpha} \frac{\exp\left((\xi^2 + \eta^2)^{\frac{1}{a}}t\right) - 1}{\xi^2 + \eta^2}. \end{aligned} \quad (44)$$

By Plancherel theorem, we follow from equations (42) and (44) that

$$\begin{aligned} \|u(\cdot, \cdot, t)\|_{\overline{H}^\theta(\mathbb{R}^2)}^2 &= \int_{-\infty-\infty}^{+\infty+\infty} \int_{-\infty-\infty}^{+\infty+\infty} (\xi^2 + \eta^2)^\theta \left| \int_0^t (t-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(t-s)^a) ds \right|^2 |\widehat{g}(\xi, \eta)|^2 d\xi d\eta \\ &\leq \frac{|D_2|^2}{\alpha^2} \int_{-\infty-\infty}^{+\infty+\infty} \int_{-\infty-\infty}^{+\infty+\infty} (\xi^2 + \eta^2)^\theta \left| \frac{\exp\left((\xi^2 + \eta^2)^{\frac{1}{a}}t\right) - 1}{\xi^2 + \eta^2} \right|^2 |\widehat{g}(\xi, \eta)|^2 d\xi d\eta \\ &\leq \frac{|D_2|^2}{\alpha^2} \int_{-\infty-\infty}^{+\infty+\infty} \int_{-\infty-\infty}^{+\infty+\infty} (\xi^2 + \eta^2)^{\theta-2} \exp\left(2(\xi^2 + \eta^2)^{\frac{1}{a}}t\right) |\widehat{g}(\xi, \eta)|^2 d\xi d\eta. \end{aligned} \quad (45)$$

Therefore, we have immediately that

$$\|u(\cdot, \cdot, t)\|_{\overline{H}^\theta(\mathbb{R}^2)} \leq \frac{|D_2|}{\alpha} \|g\|_{\overline{X}_{\theta-2,T}}, \quad (46)$$

which allows us to obtain the desired result (40).

Step 2. Proof of equation (41).

In view of equation (30), we obtain the following inequality:

$$(T-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(T-s)^a) \leq \frac{D_1}{\alpha} (\xi^2 + \eta^2)^{\frac{1-a}{a}} \exp\left((\xi^2 + \eta^2)^{\frac{1}{a}}(T-s)\right). \quad (47)$$

Thus, using the above estimate, we have immediately that

$$\begin{aligned} \int_0^T (T-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(T-s)^a) ds &\geq \frac{D_1}{\alpha} (\xi^2 + \eta^2)^{\frac{1-a}{a}} \int_0^T \exp\left((\xi^2 + \eta^2)^{\frac{1}{a}}(T-s)\right) ds \\ &= \frac{D_1}{\alpha} \frac{\exp\left((\xi^2 + \eta^2)^{\frac{1}{a}}T\right) - 1}{\xi^2 + \eta^2}. \end{aligned} \quad (48)$$

In view of Plancherel theorem and combining equations (42) and (44), we follow from equation (48) that

$$\begin{aligned} \|u(\cdot, \cdot, T)\|_{\bar{H}^\theta(\mathbb{R}^2)}^2 &= \int_{-\infty-\infty}^{+\infty+\infty} \int_{-\infty-\infty}^{+\infty+\infty} (\xi^2 + \eta^2)^\theta \left| \int_0^T (T-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(t-s)^a) ds \right|^2 |\widehat{g}(\xi, \eta)|^2 d\xi d\eta \\ &\geq \frac{|D_1|^2}{\alpha^2} \int_{-\infty-\infty}^{+\infty+\infty} \int_{-\infty-\infty}^{+\infty+\infty} (\xi^2 + \eta^2)^\theta \left| \frac{\exp\left((\xi^2 + \eta^2)^{\frac{1}{a}} T\right) - 1}{\xi^2 + \eta^2} \right|^2 |\widehat{g}(\xi, \eta)|^2 d\xi d\eta. \end{aligned} \quad (49)$$

For any $\mu > 0$, there exists a constant $C_\mu > 0$ such that the following inequality is true:

$$e^z - 1 \geq C_\mu z^\mu. \quad (50)$$

This inequality together with equation (49) yields that

$$\begin{aligned} \|u(\cdot, \cdot, T)\|_{\bar{H}^\theta(\mathbb{R}^2)}^2 &\geq \frac{|D_1|^2}{\alpha^2} \int_{-\infty-\infty}^{+\infty+\infty} \int_{-\infty-\infty}^{+\infty+\infty} (\xi^2 + \eta^2)^\theta \left| \frac{\exp\left((\xi^2 + \eta^2)^{\frac{1}{a}} T\right) - 1}{\xi^2 + \eta^2} \right|^2 |\widehat{g}(\xi, \eta)|^2 d\xi d\eta \\ &\geq \frac{|D_1|^2 T^{2\mu}}{\alpha^2} \int_{-\infty-\infty}^{+\infty+\infty} \int_{-\infty-\infty}^{+\infty+\infty} (\xi^2 + \eta^2)^{\theta-2} (\xi^2 + \eta^2)^{\frac{2\mu}{a}} |\widehat{g}(\xi, \eta)|^2 d\xi d\eta. \end{aligned} \quad (51)$$

Thus, we can derive the following estimate:

$$\|u(\cdot, \cdot, T)\|_{\bar{H}^\theta(\mathbb{R}^2)} \geq \frac{|D_1| T^\mu}{\alpha} \|g\|_{\bar{H}^{\frac{\theta-1}{2} + \frac{\mu}{a}}(\mathbb{R}^2)}. \quad (52)$$

The proof of equation (41) is completed. $\square$

Theorem 3.3. Let $u_0 = 0$ and assume that $G(x, y, t) \equiv \varphi(t)g(x, y)$. Let us assume that $|\varphi(t)| \leq Ct^m$ for any $m > -1$ and the function $g \in \bar{\mathbb{X}}_{\frac{2-2a-2\beta}{a}, T}$, where $0 < \beta < 1 + m$. Then, we obtain $u \in L^\infty(0, T; L^2(\mathbb{R}^2))$ and

$$\|u\|_{L^\infty(0, T; L^2(\mathbb{R}^2))} \leq T^{1-\beta+m} \|g\|_{\bar{\mathbb{X}}_{\frac{2-2a-2\beta}{a}, T}}. \quad (53)$$

If we further assume that $|\varphi(t)| \geq \tilde{C}t^\mu$, $\mu > -1$, and $g \in \bar{H}^{\frac{2-2a+2\sigma}{a}}(\mathbb{R}^2)$ for any $\sigma > 0$, then we obtain

$$\|u(\cdot, \cdot, T)\|_{L^2(\mathbb{R}^2)} \geq \frac{D_1 C_\sigma \sqrt{\tilde{C} B(1 + 2\sigma, 1 + \mu) T^{2\sigma + \mu + 1}}}{\alpha} \|g\|_{\bar{H}^{\frac{2-2a+2\sigma}{a}}(\mathbb{R}^2)}. \quad (54)$$

Proof. Since $u_0 = 0$ and $G(x, y, t) \equiv \varphi(t)$, we have that

$$u(x, y, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_0^t (t-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(t-s)^a) \varphi(s) ds \widehat{g}(\xi, \eta) e^{ix\xi + iy\eta} d\xi d\eta. \quad (55)$$

By Plancherel theorem, we obtain that

$$\|u(\cdot, \cdot, t)\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} \int_0^t (t-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(t-s)^a) \varphi(s) ds \left| \widehat{g}(\xi, \eta) \right|^2 d\xi d\eta. \quad (56)$$

Using equation (30) and in view of the inequality $e^{-z} \leq C_\beta z^{-\beta}$ for any $\beta > 0$, we can see that

$$\begin{aligned} (t-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(t-s)^a) \varphi(s) &\leq \frac{D_2}{\alpha} (\xi^2 + \eta^2)^{\frac{1-a}{a}} \exp\left((\xi^2 + \eta^2)^{\frac{1}{a}}(t-s)\right) \varphi(s) \\ &\leq \frac{D_2}{\alpha} (\xi^2 + \eta^2)^{\frac{1-a}{a}} \exp\left((\xi^2 + \eta^2)^{\frac{1}{a}} T\right) \exp\left(-(\xi^2 + \eta^2)^{\frac{1}{a}} s\right) \varphi(s) \\ &\leq \frac{D_2 C_\gamma}{\alpha} (\xi^2 + \eta^2)^{\frac{1-a-\beta}{a}} \exp\left((\xi^2 + \eta^2)^{\frac{1}{a}} T\right) s^{-\beta} \varphi(s). \end{aligned} \quad (57)$$

Therefore, since the condition $|\varphi(s)| \leq Cs^m$ for $m > -1$, we derive that

$$\begin{aligned} & \left| \int_0^t (t-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(t-s)^a) \varphi(s) ds \right| \\ & \leq \frac{D_2 C_\gamma}{a} (\xi^2 + \eta^2)^{\frac{1-a-\beta}{a}} \exp\left((\xi^2 + \eta^2)^{\frac{1}{a}} T\right) \left| \int_0^t s^{-\beta} \varphi(s) ds \right| \\ & \leq (\xi^2 + \eta^2)^{\frac{1-a-\beta}{a}} \exp\left((\xi^2 + \eta^2)^{\frac{1}{a}} T\right) t^{1-\beta+m}. \end{aligned} \quad (58)$$

Combining equations (56) and (58), we obtain that

$$\|u(\cdot, \cdot, t)\|_{L^2(\mathbb{R}^2)}^2 \leq T^{2-2\beta+2m} \int_{\mathbb{R}^2} (\xi^2 + \eta^2)^{\frac{2-2a-2\beta}{a}} \exp\left(2(\xi^2 + \eta^2)^{\frac{1}{a}} T\right) |\widehat{g}(\xi, \eta)|^2 d\xi d\eta = T^{2-2\beta+2m} \|g\|_{\mathbb{X}_{\frac{2-2a-2\beta}{a}, T}}^2, \quad (59)$$

which allows us to deduce equation (53). In view of equation (56), we obtain that

$$\|u(\cdot, \cdot, T)\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} \left| \int_0^T (T-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(T-s)^a) \varphi(s) ds \right|^2 |\widehat{g}(\xi, \eta)|^2 d\xi d\eta. \quad (60)$$

Using equation (30) and in view of the inequality $e^{-z} \leq C_\beta z^{-\beta}$ for any $\beta > 0$, we can see that

$$(T-s)^{a-1} E_{a,a}((\xi^2 + \eta^2)(T-s)^a) \varphi(s) \geq \frac{D_1}{a} (\xi^2 + \eta^2)^{\frac{1-a}{a}} \exp\left((\xi^2 + \eta^2)^{\frac{1}{a}} (T-s)\right) \varphi(s). \quad (61)$$

Using the inequality $e^z \geq C_\sigma z^\sigma$ for $\sigma > 0$, we derive that

$$\exp\left((\xi^2 + \eta^2)^{\frac{1}{a}} (T-s)\right) \geq C_\sigma (\xi^2 + \eta^2)^{\frac{\sigma}{a}} (T-s)^\sigma. \quad (62)$$

From the three latter estimates and condition $|\varphi(t)| \geq \widetilde{C} t^\mu$, $\mu > -1$, we obtain that

$$\begin{aligned} \|u(\cdot, \cdot, T)\|_{L^2(\mathbb{R}^2)}^2 & \geq \frac{D_1^2 C_\sigma^2}{a^2} \left| \int_0^T (T-s)^{2\sigma} \varphi(s) ds \right| \int_{\mathbb{R}^2} (\xi^2 + \eta^2)^{\frac{2-2a+2\sigma}{a}} |\widehat{g}(\xi, \eta)|^2 d\xi d\eta \\ & \geq \frac{D_1^2 C_\sigma^2 \widetilde{C} B(1+2\sigma, 1+\mu) T^{2\sigma+\mu+1}}{a^2} \|g\|_{\mathbb{H}_{\frac{2-2a+2\sigma}{a}}(\mathbb{R}^2)}^2. \end{aligned} \quad (63)$$

The proof of our theorem is completed. $\square$

4 Regularization solutions and error estimates

Theorem 4.1. Let $G = 0$ and assume that u_0 is noised by $u_{0,\varepsilon}$ such that

$$\|u_{0,\varepsilon} - u_0\|_{L^2(\mathbb{R}^2)} \leq \varepsilon. \quad (64)$$

Let us give the following:

$$M_\varepsilon = \{(\eta, \xi) \in \mathbb{R}^2, \quad \eta^2 + \xi^2 \leq N_\varepsilon\}, \quad (65)$$

where N_ε satisfies that

$$\lim_{\varepsilon \rightarrow 0} N_\varepsilon = +\infty, \quad \lim_{\varepsilon \rightarrow 0} (1 + N_\varepsilon)^{k/2} \exp\left((N_\varepsilon)^{\frac{1}{a}} T\right) = 0. \quad (66)$$

Let us define a regularized solution

$$U_N^\varepsilon = \frac{1}{2\pi} \int_{M_\varepsilon} [E_{a,1}((\xi^2 + \eta^2)t^\alpha) \widehat{u_{0,\varepsilon}}(\xi, \eta)] e^{ix\xi + iy\eta} d\xi d\eta. \quad (67)$$

Let us assume that $u \in L^\infty(0, T; \mathbb{H}^{k+\beta}(\mathbb{R}^2))$ for $\beta > 0$ and $k \geq 0$. Then, the error $\|u(\cdot, \cdot, t) - U_N^\varepsilon(\cdot, \cdot, t)\|_{\mathbb{H}^k(\mathbb{R}^2)}$ is of order

$$\max \left[(1 + N_\varepsilon)^{k/2} \exp \left((N_\varepsilon)^{\frac{1}{a}} T \right), \quad (1 + N_\varepsilon)^{-\beta/2} \right]. \quad (68)$$

Remark 4.1. Let us choose $N_\varepsilon = \left(\frac{1-\sigma}{T} \log \left(\frac{1}{\varepsilon} \right) \right)^\alpha$ for any $0 < \sigma < 1$, then we deduce that

$$\|u(\cdot, \cdot, t) - U_N^\varepsilon(\cdot, \cdot, t)\|_{\mathbb{H}^k(\mathbb{R}^2)} \leq \max \left[\left| \log \left(\frac{1}{\varepsilon} \right) \right|^{\frac{\beta\sigma}{2}}, \quad \left[1 + \frac{1-\sigma}{T} \log \left(\frac{1}{\varepsilon} \right) \right]^{-\frac{\beta\alpha}{2}} \right]. \quad (69)$$

Proof. To simplify the proof, we put the following function:

$$V_N^\varepsilon = \frac{1}{2\pi} \int_{M_\varepsilon} [E_{a,1}((\xi^2 + \eta^2)t^\alpha) \widehat{u_0}(\xi, \eta)] e^{ix\xi + iy\eta} d\xi d\eta. \quad (70)$$

Using the Plancherel theorem, we obtain that

$$\|V_N^\varepsilon(\cdot, \cdot, t) - U_N^\varepsilon(\cdot, \cdot, t)\|_{\mathbb{H}^k(\mathbb{R}^2)}^2 = \int_{M_\varepsilon} (1 + \xi^2 + \eta^2)^k [E_{a,1}((\xi^2 + \eta^2)t^\alpha) (\widehat{u_{0,\varepsilon}}(\xi, \eta) - \widehat{u_0}(\xi, \eta))]^2 d\xi d\eta \quad (71)$$

Using equation (30), we obtain the following bound:

$$E_{a,1}((\xi^2 + \eta^2)t^\alpha) \leq \frac{D_2}{a} \exp \left((\xi^2 + \eta^2)^{\frac{1}{a}} T \right) \leq \frac{D_2}{a} \exp \left((N_\varepsilon)^{\frac{1}{a}} T \right), \quad (72)$$

for $(\eta, \xi) \in M_\varepsilon$. Combining equations (71) and (72), we derive that

$$\begin{aligned} \|V_N^\varepsilon(\cdot, \cdot, t) - U_N^\varepsilon(\cdot, \cdot, t)\|_{\mathbb{H}^k(\mathbb{R}^2)}^2 &\leq \left(\frac{D_2}{a} \right)^2 (1 + M_\varepsilon)^k \exp \left(2(N_\varepsilon)^{\frac{1}{a}} T \right) \int_{M_\varepsilon} (\widehat{u_{0,\varepsilon}}(\xi, \eta) - \widehat{u_0}(\xi, \eta))^2 d\xi d\eta \\ &\leq \left(\frac{D_2}{a} \right)^2 (1 + M_\varepsilon)^k \exp \left(2(N_\varepsilon)^{\frac{1}{a}} T \right) \|u_{0,\varepsilon} - u_0\|_{L^2(\mathbb{R}^2)}^2 \\ &\leq \left(\frac{D_2}{a} \right)^2 (1 + N_\varepsilon)^k \exp \left(2(N_\varepsilon)^{\frac{1}{a}} T \right) \varepsilon^2, \end{aligned} \quad (73)$$

where in the last estimate, we used equation (78). Hence, we obtain that

$$\|V_N^\varepsilon(\cdot, \cdot, t) - U_N^\varepsilon(\cdot, \cdot, t)\|_{\mathbb{H}^k(\mathbb{R}^2)} \leq \left(\frac{D_2}{a} \right) (1 + N_\varepsilon)^{k/2} \exp \left((N_\varepsilon)^{\frac{1}{a}} T \right) \varepsilon. \quad (74)$$

Our next goal is to prove the error $\|u(\cdot, \cdot, t) - V_N^\varepsilon(\cdot, \cdot, t)\|_{\mathbb{H}^k(\mathbb{R}^2)}$. Indeed, we obtain

$$\begin{aligned} \|u(\cdot, \cdot, t) - V_N^\varepsilon(\cdot, \cdot, t)\|_{\mathbb{H}^k(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}^2 \setminus M_\varepsilon} (1 + \xi^2 + \eta^2)^k [E_{a,1}((\xi^2 + \eta^2)t^\alpha) (\widehat{u_0}(\xi, \eta))]^2 d\xi d\eta \\ &= \int_{\mathbb{R}^2 \setminus M_\varepsilon} (1 + \xi^2 + \eta^2)^{-\beta} (1 + \xi^2 + \eta^2)^{k+\beta} |\widehat{u}(\xi, \eta, t)|^2 d\xi d\eta. \end{aligned} \quad (75)$$

It is easy to verify that if $(\xi, \eta) \in \mathbb{R}^2 \setminus M_\varepsilon$, then

$$(1 + \xi^2 + \eta^2)^{-\beta} \leq (1 + N_\varepsilon)^{-\beta}.$$

This inequality together with equation (95) yields that

$$\|u(\cdot, \cdot, t) - V_N^\varepsilon(\cdot, \cdot, t)\|_{H^k(\mathbb{R}^2)} \leq (1 + N_\varepsilon)^{-\beta/2} \|u(\cdot, \cdot, t)\|_{H^{k+\beta}(\mathbb{R}^2)} \leq (1 + N_\varepsilon)^{-\beta/2} \|u\|_{L^\infty(0, T; H^{k+\beta}(\mathbb{R}^2))}. \quad (76)$$

Thus, from some above observations, we obtain that

$$\begin{aligned} \|u(\cdot, \cdot, t) - U_N^\varepsilon(\cdot, \cdot, t)\|_{H^k(\mathbb{R}^2)} &\leq \|u(\cdot, \cdot, t) - V_N^\varepsilon(\cdot, \cdot, t)\|_{H^k(\mathbb{R}^2)} + \|V_N^\varepsilon(\cdot, \cdot, t) - U_N^\varepsilon(\cdot, \cdot, t)\|_{H^k(\mathbb{R}^2)} \\ &\leq \left(\frac{D_2}{\alpha}\right) (1 + N_\varepsilon)^{k/2} \exp\left((N_\varepsilon)^{\frac{1}{\alpha}} T\right) \varepsilon + (1 + N_\varepsilon)^{-\beta/2} \|u\|_{L^\infty(0, T; H^{k+\beta}(\mathbb{R}^2))}. \end{aligned} \quad (77)$$

Theorem 4.2. Let $u_0 = 0$ and assume that $G(x, y, t) \equiv g(x, y)$. Let us assume that g is noised by g_ε such that

$$\|g_\varepsilon - g\|_{L^2(\mathbb{R}^2)} \leq \varepsilon. \quad (78)$$

Let us give the following set:

$$\mathcal{M}_\varepsilon = \{(\eta, \xi) \in \mathbb{R}^2, \eta^2 + \xi^2 \leq N_\varepsilon\}, \quad (79)$$

where N_ε is defined later. Let us define a regularized solution

$$W_{N_\varepsilon}^\varepsilon(x, y, t) = \frac{1}{2\pi} \int_{\mathcal{M}_\varepsilon} \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}((\xi^2 + \eta^2)(t-s)^\alpha) ds \left| \widehat{g}_\varepsilon(\xi, \eta) e^{ix\xi + iy\eta} d\xi d\eta. \quad (80)$$

If $g_\varepsilon \in \bar{H}^{\frac{\theta-2}{2}}(\mathbb{R}^2)$, then we obtain

$$\|W_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t)\|_{\bar{H}^\theta(\mathbb{R}^2)} \leq \frac{|D_2|}{\alpha} \exp\left((N_\varepsilon)^{\frac{1}{\alpha}} T\right) \|g_\varepsilon\|_{\bar{H}^{\frac{\theta-2}{2}}(\mathbb{R}^2)}, \quad (81)$$

and

$$\|W_{N_\varepsilon}^\varepsilon(\cdot, \cdot, T)\|_{\bar{H}^\theta(\mathbb{R}^2)}^2 \geq \frac{C_\mu T^\mu D_1}{\alpha M_\varepsilon} \int_{M_\varepsilon} (\xi^2 + \eta^2)^{\theta + \frac{2\mu}{\alpha}} |\widehat{g}_\varepsilon(\xi, \eta)|^2 d\xi d\eta. \quad (82)$$

Let us assume that $u \in L^\infty(0, T; \bar{H}^{\theta+\beta}(\mathbb{R}^2))$ for $\theta \geq 2$ and $\beta > 0$. Then, we have

$$\|u(\cdot, \cdot, t) - W_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t)\|_{\bar{H}^\theta(\mathbb{R}^2)} \leq \max\left((N_\varepsilon)^{\frac{\theta}{2}-1} \exp\left((N_\varepsilon)^{\frac{1}{\alpha}} T\right) \varepsilon, (N_\varepsilon)^{-\beta/2}\right). \quad (83)$$

Here, N_ε as above is a positive real number, which satisfies that

$$\lim_{\varepsilon \rightarrow 0} N_\varepsilon = +\infty, \quad \lim_{\varepsilon \rightarrow 0} (N_\varepsilon)^{\frac{\theta}{2}-1} \exp\left((N_\varepsilon)^{\frac{1}{\alpha}} T\right) \varepsilon = 0. \quad (84)$$

Remark 4.2. We can choose N_ε as in Remark (4.1).

Proof. By Plancherel theorem, we follow from equations (80) and (44) that

$$\begin{aligned} \|W_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t)\|_{\bar{H}^\theta(\mathbb{R}^2)}^2 &= \int_{\mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^\theta \left| \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}((\xi^2 + \eta^2)(t-s)^\alpha) ds \right|^2 |\widehat{g}_\varepsilon(\xi, \eta)|^2 d\xi d\eta \\ &\leq \frac{|D_2|^2}{\alpha^2} \int_{\mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^\theta \left| \frac{\exp\left((\xi^2 + \eta^2)^{\frac{1}{\alpha}} t\right) - 1}{\xi^2 + \eta^2} \right|^2 |\widehat{g}_\varepsilon(\xi, \eta)|^2 d\xi d\eta \\ &\leq \frac{|D_2|^2}{\alpha^2} \int_{\mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^{\theta-2} \exp\left(2(\xi^2 + \eta^2)^{\frac{1}{\alpha}} T\right) |\widehat{g}_\varepsilon(\xi, \eta)|^2 d\xi d\eta. \end{aligned} \quad (85)$$

If $(\xi, \eta) \in N_\varepsilon$, then we have the following inequality:

$$\exp\left(2(\xi^2 + \eta^2)^{\frac{1}{\alpha}}T\right) \leq \exp\left(2(N_\varepsilon)^{\frac{1}{\alpha}}T\right).$$

This equality combined with equation (87) allows us to deduce that

$$\begin{aligned} \|W_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t)\|_{\bar{H}^\theta(\mathbb{R}^2)} &\leq \frac{|D_2|}{\alpha} \exp\left((N_\varepsilon)^{\frac{1}{\alpha}}T\right) \sqrt{\int_{\mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^{\theta-2} |\widehat{g}_\varepsilon(\xi, \eta)|^2 d\xi d\eta} \\ &\leq \frac{|D_2|}{\alpha} \exp\left((N_\varepsilon)^{\frac{1}{\alpha}}T\right) \|g_\varepsilon\|_{\bar{H}^{\frac{\theta-2}{2}}(\mathbb{R}^2)}. \end{aligned} \quad (86)$$

Let us show equation (82). By Plancherel theorem, we follow from equations (80) and (48) that

$$\begin{aligned} \|W_{N_\varepsilon}^\varepsilon(\cdot, \cdot, T)\|_{\bar{H}^\theta(\mathbb{R}^2)}^2 &= \int_{\mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^\theta \left[\int_0^T (T-s)^{\alpha-1} E_{\alpha, \alpha}((\xi^2 + \eta^2)(T-s)^\alpha) ds \right]^2 |\widehat{g}_\varepsilon(\xi, \eta)|^2 d\xi d\eta \\ &\geq \frac{|D_1|^2}{\alpha^2} \int_{\mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^\theta \left[\frac{\exp\left((\xi^2 + \eta^2)^{\frac{1}{\alpha}}T\right) - 1}{\xi^2 + \eta^2} \right]^2 |\widehat{g}_\varepsilon(\xi, \eta)|^2 d\xi d\eta \\ &\geq \frac{|D_1|^2}{\alpha^2 N_\varepsilon^2} \int_{\mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^\theta \left[\exp\left((\xi^2 + \eta^2)^{\frac{1}{\alpha}}T\right) - 1 \right]^2 |\widehat{g}_\varepsilon(\xi, \eta)|^2 d\xi d\eta. \end{aligned} \quad (87)$$

Moreover, in view of the inequality $e^z - 1 \geq C_\mu z^\mu$, we obtain the following estimate:

$$\int_{\mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^\theta \left[\exp\left((\xi^2 + \eta^2)^{\frac{1}{\alpha}}T\right) - 1 \right]^2 |\widehat{g}_\varepsilon(\xi, \eta)|^2 d\xi d\eta \geq |C_\mu|^2 T^{2\mu} \int_{N_\varepsilon} (\xi^2 + \eta^2)^{\theta + \frac{2\mu}{\alpha}} |\widehat{g}_\varepsilon(\xi, \eta)|^2 d\xi d\eta. \quad (88)$$

Combining equations (87) and (88), we derive the following lower bound:

$$\|W_{N_\varepsilon}^\varepsilon(\cdot, \cdot, T)\|_{\bar{H}^\theta(\mathbb{R}^2)}^2 \geq \frac{C_\mu T^\mu D_1}{\alpha N_\varepsilon} \int_{\mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^{\theta + \frac{2\mu}{\alpha}} |\widehat{g}_\varepsilon(\xi, \eta)|^2 d\xi d\eta \quad (89)$$

Our next aim is to prove equation (83). Let us set the following function:

$$V_{N_\varepsilon}^\varepsilon(x, y, t) = \frac{1}{2\pi} \int_{\mathcal{M}_\varepsilon} \left[\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}((\xi^2 + \eta^2)(t-s)^\alpha) ds \right] \widehat{g}(\xi, \eta) e^{ix\xi + iy\eta} d\xi d\eta. \quad (90)$$

Using the Plancherel theorem, we have the following equality:

$$\|V_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t) - W_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t)\|_{\bar{H}^\theta(\mathbb{R}^2)}^2 = \int_{\mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^\theta \left[\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}((\xi^2 + \eta^2)(t-s)^\alpha) ds \right]^2 (\widehat{g}_\varepsilon(\xi, \eta) - \widehat{g}(\xi, \eta))^2 d\xi d\eta. \quad (91)$$

Combining equations (91) and (44), we obtain

$$\begin{aligned} &\|V_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t) - W_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t)\|_{\bar{H}^\theta(\mathbb{R}^2)}^2 \\ &\leq \frac{|D_1|^2}{\alpha^2} \int_{\mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^\theta \left[\frac{\exp\left((\xi^2 + \eta^2)^{\frac{1}{\alpha}}t\right) - 1}{\xi^2 + \eta^2} \right]^2 (\widehat{g}_\varepsilon(\xi, \eta) - \widehat{g}(\xi, \eta))^2 d\xi d\eta \\ &\leq \frac{|D_1|^2}{\alpha^2} \int_{\mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^{\theta-2} \exp\left(2(\xi^2 + \eta^2)^{\frac{1}{\alpha}}T\right) |\widehat{g}_\varepsilon(\xi, \eta) - \widehat{g}(\xi, \eta)|^2 d\xi d\eta. \end{aligned} \quad (92)$$

Since the condition $\theta \geq 2$ and $(\xi, \eta \in N_\varepsilon)$, we know the fact that

$$(\xi^2 + \eta^2)^{\theta-2} \exp\left(2(\xi^2 + \eta^2)^{\frac{1}{\alpha}} T\right) \leq (N_\varepsilon)^{\theta-2} \exp\left(2(N_\varepsilon)^{\frac{1}{\alpha}} T\right). \quad (93)$$

This inequality together with equation (92) allows us to obtain that

$$\begin{aligned} \|V_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t) - W_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t)\|_{\bar{H}^\theta(\mathbb{R}^2)} &\leq \frac{D_1}{\alpha} (M_\varepsilon)^{\frac{\theta}{2}-1} \exp\left((M_\varepsilon)^{\frac{1}{\alpha}} T\right) \sqrt{\int_{\mathcal{M}_\varepsilon} |\widehat{g}_\varepsilon(\xi, \eta) - \widehat{g}(\xi, \eta)|^2 d\xi d\eta} \\ &\leq \frac{D_1}{\alpha} (N_\varepsilon)^{\frac{\theta}{2}-1} \exp\left((N_\varepsilon)^{\frac{1}{\alpha}} T\right) \|g_\varepsilon - g\|_{L^2(\mathbb{R}^2)} \\ &\leq \frac{D_1}{\alpha} (N_\varepsilon)^{\frac{\theta}{2}-1} \exp\left((N_\varepsilon)^{\frac{1}{\alpha}} T\right) \varepsilon. \end{aligned} \quad (94)$$

Next, it is obvious to see that

$$\begin{aligned} \|u(\cdot, \cdot, t) - V_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t)\|_{\bar{H}^\theta(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}^2 \setminus \mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^\theta \left[\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}((\xi^2 + \eta^2)(t-s)^\alpha) ds \right]^2 |\widehat{g}(\xi, \eta)|^2 d\xi d\eta \\ &= \int_{\mathbb{R}^2 \setminus \mathcal{M}_\varepsilon} (\xi^2 + \eta^2)^{-\beta} (\xi^2 + \eta^2)^{\theta+\beta} |\widehat{u}(\xi, \eta, t)|^2 d\xi d\eta. \end{aligned} \quad (95)$$

It is easy to verify that if $(\xi, \eta) \in \mathbb{R}^2 \setminus \mathcal{M}_\varepsilon$, then $(\xi^2 + \eta^2)^{-\beta} \leq (N_\varepsilon)^{-\beta}$. This inequality together with equation (95) yields that

$$\|u(\cdot, \cdot, t) - V_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t)\|_{\bar{H}^\theta(\mathbb{R}^2)} \leq (N_\varepsilon)^{-\beta/2} \|u(\cdot, \cdot, t)\|_{\bar{H}^{\theta+\beta}(\mathbb{R}^2)} \leq (N_\varepsilon)^{-\beta/2} \|u\|_{L^\infty(0, T; \bar{H}^{\theta+\beta}(\mathbb{R}^2))}. \quad (96)$$

Combining equations (94) and (96), we derive that

$$\begin{aligned} \|u(\cdot, \cdot, t) - W_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t)\|_{\bar{H}^\theta(\mathbb{R}^2)} &\leq \|V_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t) - W_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t)\|_{\bar{H}^\theta(\mathbb{R}^2)} + \|u(\cdot, \cdot, t) - V_{N_\varepsilon}^\varepsilon(\cdot, \cdot, t)\|_{\bar{H}^\theta(\mathbb{R}^2)} \\ &\leq \frac{D_1}{\alpha} (N_\varepsilon)^{\frac{\theta}{2}-1} \exp\left((N_\varepsilon)^{\frac{1}{\alpha}} T\right) \varepsilon + (N_\varepsilon)^{-\beta/2} \|u\|_{L^\infty(0, T; \bar{H}^{\theta+\beta}(\mathbb{R}^2))}. \end{aligned} \quad (97)$$

The proof is completed. $\square$

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